# THERE ONCE WAS A CLASSICAL THEORY...

Introductory Classical Mechanics,
with Problems and Solutions

David Morin

…Of which quantum disciples were leery.
They said, “Why spend so long
On a theory that’s wrong?”
Well, it works for your everyday query!

# Contents

# Preface

This textbook has grown out of the first-semester honors freshman physics course that has been taught at Harvard University during recent years. The book is essentially two books in one. Roughly half of it follows the form of a normal textbook, consisting of text, along with exercises suitable for homework assignments. The other half takes the form of a problem book, with all sorts of problems (with solutions) of varying degrees of difficulty. If you've been searching for a supply of practice problems to work on, this should keep you busy for a while.

A brief outline of the book is as follows. Chapter 1 covers statics. Most of this will probably look familiar, but you'll find some fun problems. In Chapter 2, we learn about forces and how to apply $F = ma$. There's a bit of math here needed for solving some simple differential equations. Chapter 3 deals with oscillations and coupled oscillators. Again, there's a fair amount of math needed for solving linear differential equations, but there's no way to avoid it. Chapter 4 deals with conservation of energy and momentum. You've probably seen much of this before, but again, it has lots of neat problems.

In Chapter 5, we introduce the Lagrangian method, which will undoubtedly be new to you. It looks rather formidable at first, but it's really not all that rough. There are difficult concepts at the heart of the subject, but the nice thing is that the technique is easy to apply. The situation here analogous to taking a derivative in calculus; there are substantive concepts on which the theory rests, but the act of taking a derivative is fairly straightforward.

Chapter 6 deals with central forces, Kepler's Laws, and such things. Chapter 7 covers the easier type of angular momentum situations, ones where the direction of the angular momentum is fixed. Chapter 8 covers the more difficult type, ones where the direction changes. Gyroscopes, spinning tops, and other fun and perplexing objects fall into this category. Chapter 9 deals with accelerated frames of reference and fictitious forces.

Chapters 10 through 13 cover relativity. Chapter 10 deals with relativistic kinematics – abstract particles flying through space and time. Chapter 11 covers relativistic dynamics – energy, momentum, force, etc. Chapter 12 introduces the important concept of "4-vectors." The material in this chapter could alternatively be put in the previous two, but for various reasons I thought it best to create a separate chapter for it. Chapter 13 covers a few topics from general relativity. It's not possible for one chapter to do this subject justice, of course, so we'll just look at some basic (but still very interesting) examples.

The appendices contain various useful things. Indeed, Appendices B and C, which cover dimensional analysis and limiting cases, are the first parts of this book you should read.

Throughout the book, I have included many "remarks." These are written in a slightly smaller font than the surrounding text. They begin with a small-capital "REMARK" and end with a shamrock (♣). The purpose of these remarks is to say something that needs to be said, without disrupting the overall flow of the argument. In some sense these are "extra" thoughts, although they are invariably useful in understanding what is going on. They are usually more informal than the rest of the text, and I reserve the right to occasionally use them to babble about things I find interesting, but which you may find a bit tangential. For the most part, however, the remarks address issues and questions that arise naturally in the course of the discussion.

At the end of the solutions to many problems, the obvious thing to do is to check limiting cases.[1] I have written these in a smaller font, but I have not always bothered to start them with a "REMARK" and end them with a "♣", because they are not "extra" thoughts. Checking limiting cases of your answer is something you should *always* do.

For your reading pleasure (I hope), I have included many limericks scattered throughout the text. I suppose that they might be viewed as educational, but they certainly don't represent any deep insight I have on the teaching of physics. I have written them solely for the purpose of lightening things up. Some are funny. Some are stupid. But at least they're all physically accurate (give or take).

A word on the problems. Some are easy, but many are very difficult. I think you'll find them quite interesting, but don't get discouraged if you have trouble solving them. Some are designed to be brooded over for hours. Or days, or weeks, or months (as I can attest to). I have chosen to write them up for two reasons: (1) Students invariably want extra practice problems, with solutions, to work on, and (2) I find them rather fun.

The problems are marked with a number of asterisks. Harder problems earn more asterisks, on a scale from zero to four. You may, of course, disagree with my judgment of difficulty, but I think that an arbitrary weighting scheme is better than none at all. As a rough idea of what I mean by the number of stars: one-star problems are solid problems that require some thought, and four-star problems are really really really difficult. Try a few and you'll see what I mean.

Just to warn you, even if you understand the material in the text backwards and forwards, the four-star (and many of the three-star) problems will still be extremely challenging. But that's how it should be. My goal was to create an unreachable upper bound on the number (and difficulty) of problems, because it would be an unfortunate circumstance, indeed, if you were left twiddling your thumbs, having run out of problems to solve. I hope I have succeeded.

For the problems you choose to work on, be careful not to look at the solution too soon. There is nothing wrong with putting a problem aside for a while and

[1]This topic is discussed in Appendix C.

coming back to it later. Indeed, this is probably the best way to approach things. If you head to the solution at the first sign of not being able to solve a problem, then you have wasted the problem.

REMARK: This gives me an opportunity for my first remark (and first limerick, too). One thing many people don't realize is that you need to know more than the correct way(s) to do a problem; you also need to be familiar with many *incorrect* ways of doing it. Otherwise, when you come upon a new problem, there may be a number of decent-looking approaches to take, and you won't be able to immediately weed out the poor ones. Struggling a bit with a problem invariably leads you down some wrong paths, and this is an essential part of learning. To understand something, you not only have to know what's right about the right things; you also have to know what's wrong about the wrong things. Learning takes a serious amount of effort, many wrong turns, and a lot of sweat. Alas, there are no short-cuts to understanding physics.

The ad said, For one little fee,
You can skip all that course-work ennui.
So send your tuition,
For boundless fruition!
Get your mail-order physics degree! ♣

One last note: the problems with included solutions are called "Problems." The problems without included solutions are called "Exercises." There is no fundamental difference between the two, except for the existence of written-up solutions.

I hope you enjoy the book!

— David Morin

# Chapter 1

# Statics



*Before reading any of the text in this book, you should read Appendices B and C. The material discussed there (dimensional analysis, checking limiting cases, etc.) is extremely important. It's fairly safe to say that an understanding of these topics is absolutely necessary for an understanding of physics. And they make the subject a lot more fun, too!*

For many of you, the material in this first chapter will be mainly review. As such, the text here will be relatively short. This is an "extra" chapter. Its main purpose is that it provides me with an excuse to give you some nice statics problems. Try as many as you like, but don't go overboard; more important and relevant material will soon be at hand.

## 1.1 Balancing forces

A "static" situation is one where all the objects are motionless. If an object remains motionless, then $F = ma$ tells us that the total force acting on it must be zero. (The converse is not true, of course. The total force on an object is also zero if it moves with constant nonzero velocity. But we'll deal only with statics problems here). The whole goal in a statics problem is to find out what the various forces have to be so that there is zero net force acting on each object (and zero net torque, too, but that's the topic of the next section). Since a force is a vector, this goal involves breaking the force up into its components. You can pick cartesian coordinates, polar coordinates, or another set. It is usually clear from the problem which system will make your calculations easiest. Once you pick a system, you simply have to demand that the total force in each direction is zero.

There are many different types of forces in the world, most of which are large-scale effects of complicated things going on at smaller scales. For example, the tension in a rope comes from the chemical bonds that hold the molecules in the rope together (and these chemical forces are just electrical forces). In doing a mechanics problem involving a rope, there is certainly no need to analyze all the details of the forces taking place at the molecular scale. You simply call the force in the rope a "tension" and get on with the problem. Four types of forces come up repeatedly:

### Tension

Tension is the general name for a force that a rope, stick, etc., exerts when it is pulled on. Every piece of the rope feels a tension force in both directions, except the end point, which feels a tension on one side and a force on the other side from whatever object is attached to the end.

In some cases, the tension may vary along the rope. The "Rope wrapped around a pole" example at the end of this section is a good illustration of this. In other cases, the tension must be the same everywhere. For example, in a hanging massless rope, or in a massless rope hanging over a frictionless pulley, the tension must be the same at all points, because otherwise there would be a net force on at least one tiny piece, and then $F = ma$ would yield an infinite acceleration for this tiny piece.

### Normal force

This is the force perpendicular to a surface that the surface applies to an object. The total force applied by a surface is usually a combination of the normal force and the friction force (see below). But for frictionless surfaces such as greasy ones or ice, only the normal force exists. The normal force comes about because the surface actually compresses a tiny bit and acts like a very rigid spring. The surface gets squashed until the restoring force equals the force the object applies.

REMARK: For the most part, the only difference between a "tension" and a "normal force" is the direction of the force. Both situations can be modeled by a spring. In the case of a tension, the spring (a rope, a stick, or whatever) is stretched, and the force on the given object is directed toward the spring. In the case of a normal force, the spring is compressed, and the force on the given object is directed away from the spring. Things like sticks can provide both normal forces and tensions. But a rope, for example, has a hard time providing a normal force.

In practice, in the case of elongated objects such as sticks, a compressive force is usually called a "compressive tension," or a "negative tension," instead of a normal force. So by these definitions, a tension can point either way. At any rate, it's just semantics. If you use any of these descriptions for a compressed stick, people will know what you mean. ♣

### Friction

Friction is the force parallel to a surface that a surface applies to an object. Some surfaces, such as sandpaper, have a great deal of friction. Some, such as greasy ones, have essentially no friction. There are two types of friction, called "kinetic" friction and "static" friction.

Kinetic friction (which we won't cover in this chapter) deals with two objects moving relative to each other. It is usually a good approximation to say that the kinetic friction between two objects is proportional to the normal force between them. The constant of proportionality is called $\mu_k$ (the "coefficient of kinetic friction"), where $\mu_k$ depends on the two surfaces involved. Thus, $F = \mu_k N$, where $N$

is the normal force. The direction of the force is opposite to the motion.

Static friction deals with two objects at rest relative to each other. In the static case, we have $F \le \mu_s N$ (where $\mu_s$ is the "coefficient of static friction"). Note the inequality sign. All we can say prior to solving a problem is that the static friction force has a *maximum* value equal to $F_{\text{max}} = \mu_s N$. In a given problem, it is most likely less than this. For example, if a block of large mass $M$ sits on a surface with coefficient of friction $\mu_s$, and you give the block a tiny push to the right (tiny enough so that it doesn't move), then the friction force is of course not equal to $\mu_s N = \mu_s Mg$ to the left. Such a force would send the block sailing off to the left. The true friction force is simply equal and opposite to the tiny force you apply. What the coefficient $\mu_s$ tells us is that if you apply a force larger than $\mu_s Mg$ (the maximum friction force on a horizontal table), then the block will end up moving to the right.

### Gravity

Consider two point objects, with masses $M$ and $m$, separated by a distance $R$. Newton's gravitational force law says that the force between these objects is attractive and has magnitude $F = GMm/R^2$, where $G = 6.67 \cdot 10^{-11}$ $\text{m}^3/(\text{kg} \cdot \text{s}^2)$. As we will show in Chapter 4, the same law applies to spheres. That is, a sphere may be treated like a point mass located at its center. Therefore, an object on the surface of the earth feels a gravitational force equal to

$$F = m\left(\frac{GM}{R^2}\right) \equiv mg, \tag{1.1}$$

where $M$ is the mass of the earth, and $R$ is its radius. This equation defines $g$. Plugging in the numerical values, we obtain (as you can check) $g \approx 9.8$ $\text{m/s}^2$. Every object on the surface of the earth feels a force of $mg$ downward. If the object is not accelerating, then there must also be other forces present (normal forces, etc.) to make the total force equal to zero.

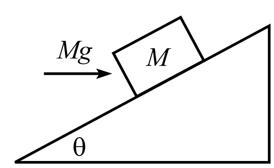


**Figure 1.1**

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**Example (Block on a plane):** A block of mass $M$ rests on a fixed plane inclined at angle $\theta$. You apply a horizontal force of $Mg$ on the block, as shown in Fig. 1.1.

(a) Assume that the friction force between the block and the plane is large enough to keep the block at rest. What are the normal and friction forces (call them $N$ and $F_f$) that the plane exerts on the block?

(b) Let the coefficient of static friction be $\mu$. For what range of angles $\theta$ will the block remain still?

**Solution:**

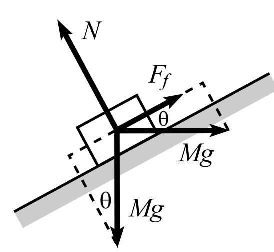


**Figure 1.2**

(a) We will break the forces up into components parallel and perpendicular to the plane. (The horizontal and vertical components would also work, but the calculation would be a little longer.) The forces are $N$, $F_f$, the applied $Mg$, and the weight $Mg$, as shown in Fig. 1.2. Balancing the forces parallel and perpendicular to the plane gives, respectively (with upward along the plane taken to be positive),

$$\begin{aligned} F_f &= Mg\sin\theta - Mg\cos\theta, \qquad \text{and} \\ N &= Mg\cos\theta + Mg\sin\theta. \end{aligned} \tag{1.2}$$

REMARKS: Note that if $\tan\theta > 1$, then $F_f$ is positive (that is, it points up the plane). And if $\tan\theta < 1$, then $F_f$ is negative (that is, it points down the plane). There is no need to worry about which way it points when drawing the diagram. Just pick a direction to be positive, and if $F_f$ comes out to be negative (as it does in the above figure because $\theta < 45°$), so be it.

$F_f$ ranges from $-Mg$ to $Mg$, as $\theta$ ranges from 0 to $\pi/2$ (convince yourself that these limiting values make sense). As an exercise, you can show that $N$ is maximum when $\tan\theta = 1$, in which case $N = \sqrt{2}Mg$ and $F_f = 0$. ♣

(b) The coefficient $\mu$ tells us that $|F_f| \le \mu N$. Using eqs. (1.2), this inequality becomes

$$Mg|\sin\theta - \cos\theta| \le \mu Mg(\cos\theta + \sin\theta). \tag{1.3}$$

The absolute value here signifies that we must consider two cases:

- If $\tan\theta \ge 1$, then eq. (1.3) becomes

$$\sin\theta - \cos\theta \le \mu(\cos\theta + \sin\theta) \qquad \Longrightarrow \qquad \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{1.4}$$

- If $\tan\theta \le 1$, then eq. (1.3) becomes

$$-\sin\theta + \cos\theta \le \mu(\cos\theta + \sin\theta) \qquad \Longrightarrow \qquad \tan\theta \ge \frac{1-\mu}{1+\mu}. \tag{1.5}$$

Putting these two ranges for $\theta$ together, we have

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{1.6}$$

REMARKS: For very small $\mu$, these bounds both approach 1, which means that $\theta$ must be very close to 45°. This makes sense. If there is very little friction, then the components along the plane of the horizontal and vertical $Mg$ forces must nearly cancel; hence, $\theta \approx 45°$. A special value for $\mu$ is 1, because from eq. (1.6), we see that $\mu = 1$ is the cutoff value that allows $\theta$ to reach 0 and $\pi/2$. If $\mu \ge 1$, then any tilt of the plane is allowed. ♣

Let's now do an example involving a rope in which the tension varies with position. We'll need to consider differential pieces of the rope to solve this problem.

**Example (Rope wrapped around a pole):** A rope wraps an angle $\theta$ around a pole. You grab one end and pull with a tension $T_0$. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is $\mu$, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

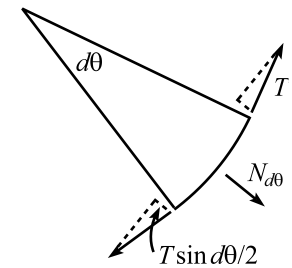


**Figure 1.3**

**Solution:** Consider a small piece of the rope that subtends an angle $d\theta$. Let the tension in this piece be $T$ (which will vary slightly over the small length). As shown in Fig. 1.3, the pole exerts a small outward normal force, $N_{d\theta}$, on the piece. This normal force exists to balance the inward components of the tensions at the ends. These inward components have magnitude $T\sin(d\theta/2)$. Therefore, $N_{d\theta} = 2T\sin(d\theta/2)$. The small-angle approximation, $\sin x \approx x$, then allows us to write this as $N_{d\theta} = T\,d\theta$.

The friction force on the little piece of rope satisfies $F_{d\theta} \leq \mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the piece. In other words, the tension, as a function of $\theta$, satisfies

$$\begin{aligned} T(\theta + d\theta) &\leq T(\theta) + \mu T\,d\theta \\ \Longrightarrow \quad dT &\leq \mu T\,d\theta \\ \Longrightarrow \quad \int \frac{dT}{T} &\leq \int \mu\,d\theta \\ \Longrightarrow \quad \ln T &\leq \mu\theta + C \\ \Longrightarrow \quad T &\leq T_0 e^{\mu\theta}, \end{aligned} \tag{1.7}$$

where we have used the fact that $T = T_0$ when $\theta = 0$.

The exponential behavior here is quite strong (as exponential behaviors tend to be). If we let $\mu = 1$, then just a quarter turn around the pole produces a factor of $e^{\pi/2} \approx 5$. One full revolution yields a factor of $e^{2\pi} \approx 530$, and two full revolutions yield a factor of $e^{4\pi} \approx 300{,}000$. Needless to say, the limiting factor in such a case is not your strength, but rather the structural integrity of the pole around which the rope winds.

## 1.2 Balancing torques

In addition to balancing forces in a statics problem, we must also balance torques. We'll have much more to say about torque in Chapters 7 and 8, but we'll need one important fact here.

Consider the situation in Fig. 1.4, where three forces are applied perpendicularly to a stick, which is assumed to remain motionless. $F_1$ and $F_2$ are the forces at the ends, and $F_3$ is the force in the interior. We have, of course, $F_3 = F_1 + F_2$, because the stick is at rest.

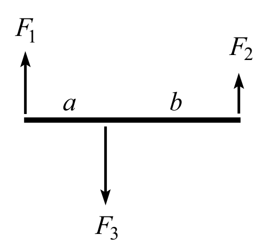


**Figure 1.4**

**Claim 1.1** *If the system is motionless, then $F_3 a = F_2(a + b)$. In other words, the torques (force times distance) around the left end cancel. And you can show that they cancel around any other point, too.*

We'll prove this claim in Chapter 7 by using angular momentum, but let's give a short proof here.

**Proof:** We'll make one reasonable assumption, namely, that the correct relationship between the forces and distances is of the form,

$$F_3 f(a) = F_2 f(a + b), \tag{1.8}$$

where $f(x)$ is a function to be determined.[1] Applying this assumption with the roles of "left" and "right" reversed in Fig. 1.4, we have

$$F_3 f(b) = F_1 f(a+b) \tag{1.9}$$

Adding the two preceding equations, and using $F_3 = F_1 + F_2$, gives

$$f(a) + f(b) = f(a+b). \tag{1.10}$$

This equation implies that $f(nx) = nf(x)$ for any $x$ and for any rational number $n$, as you can show. Therefore, assuming $f(x)$ is continuous, it must be the linear function, $f(x) = Ax$, as we wanted to show. The constant $A$ is irrelevant, because it cancels in eq. (1.8).[2] ∎

Note that dividing eq. (1.8) by eq. (1.9) gives $F_1 f(a) = F_2 f(b)$, and hence $F_1 a = F_2 b$, which says that the torques cancel around the point where $F_3$ is applied. You can show that the torques cancel around any arbitrary pivot point.

When adding up all the torques in a given physical setup, it is of course required that you use the same pivot point when calculating each torque.

In the case where the forces aren't perpendicular to the stick, the claim applies to the components of the forces perpendicular to the stick. This makes sense, because the components parallel to the stick have no effect on the rotation of the stick around the pivot point. Therefore, referring to the figures shown below, the equality of the torques can be written as

$$F_a a \sin\theta_a = F_b b \sin\theta_b. \tag{1.11}$$

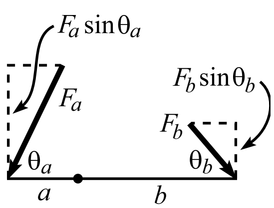


**Figure 1.5**

This equation can be viewed in two ways:

- $(F_a \sin\theta_a)a = (F_b \sin\theta_b)b$. In other words, we effectively have smaller forces acting on the given "lever-arms" (see Fig. 1.5).
- $F_a(a\sin\theta_a) = F_b(b\sin\theta_b)$. In other words, we effectively have the given forces acting on smaller "lever-arms" (see Fig. 1.6).

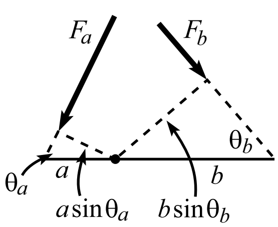


**Figure 1.6**

Claim 1.1 shows that even if you apply just a tiny force, you can balance the torque due to a very large force, provided that you make your lever-arm sufficiently long. This fact led a well-known mathematician of long ago to claim that he could move the earth if given a long enough lever-arm.

One morning while eating my Wheaties,
I felt the earth move 'neath my feeties.
The cause for alarm
Was a long lever-arm,
At the end of which grinned Archimedes!

[1]What we're doing here is simply assuming linearity in $F$. That is, two forces of $F$ applied at a point should be the same as a force of $2F$ applied at that point. You can't really argue with that.

[2]Another proof of this claim is given in Problem 12.

One handy fact that comes up often is that the gravitational torque on a stick of mass $M$ is the same as the gravitational torque due to a point-mass $M$ located at the center of the stick. The truth of this statement relies on the fact that torque is a linear function of the distance to the pivot point (see Exercise 7). More generally, the gravitational torque on an object of mass $M$ may be treated simply as the gravitational torque due to a force $Mg$ located at the center of mass.

We'll have much more to say about torque in Chapters 7 and 8, but for now we'll simply use the fact that in a statics problem, the torques around any given point must balance.

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**Example (Leaning ladder):** A ladder leans against a frictionless wall. If the coefficient of friction with the ground is $\mu$, what is the smallest angle the ladder can make with the ground and not slip?

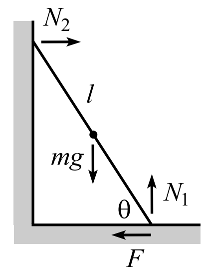


**Figure 1.7**

**Solution:** Let the ladder have mass $m$ and length $\ell$. As shown in Fig. 1.7, we have three unknown forces: the friction force, $F$, and the normal forces, $N_1$ and $N_2$. And we fortunately have three equations that will allow us to solve for these three forces: $\Sigma F_{\text{vert}} = 0$, $\Sigma F_{\text{horiz}} = 0$, and $\Sigma\tau = 0$.

Looking at the vertical forces, we see that $N_1 = mg$. And then looking at the horizontal forces, we see that $N_2 = F$. So we have quickly reduced the unknowns from three to one.

We will now use $\Sigma\tau = 0$ to find $N_2$ (or $F$). But first we must pick the "pivot" point around which we will calculate the torques. Any stationary point will work fine, but certain choices make the calculations easier than others. The best choice for the pivot is generally the point at which the most forces act, because then the $\Sigma\tau = 0$ equation will have the smallest number of terms in it (because a force provides no torque around the point where it acts, since the lever-arm is zero).

In this problem, there are two forces acting at the bottom end of the ladder, so this is the best choice for the pivot.[3] Balancing the torques due to gravity and $N_2$, we have

$$N_2\ell\sin\theta = mg(\ell/2)\cos\theta \qquad \Longrightarrow \qquad N_2 = \frac{mg}{2\tan\theta}\,. \tag{1.12}$$

This is also the value of the friction force $F$. The condition $F \le \mu N_1 = \mu mg$ therefore becomes

$$\frac{mg}{2\tan\theta} \le \mu mg \qquad \Longrightarrow \qquad \tan\theta \ge \frac{1}{2\mu}\,. \tag{1.13}$$

Remarks: The factor of 1/2 in this answer comes from the fact that the ladder behaves like a point mass located halfway up. As an exercise, you can show that the answer for the analogous problem, but now with a massless ladder and a person standing a fraction $f$ of the way up, is $\tan\theta \ge f/\mu$.

Note that the total force exerted on the ladder by the floor points up at an angle given by $\tan\beta = N_1/F = (mg)/(mg/2\tan\theta) = 2\tan\theta$. We see that this force does *not* point along the ladder. There is simply no reason why it should. But there *is* a nice reason why it should point upward with twice the slope of the ladder. This is the direction that causes the lines of the three forces on the ladder to be concurrent, as shown in Fig. 1.8.

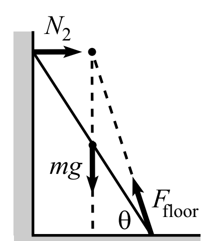


**Figure 1.8**

[3] But you should verify that other choices for the pivot, for example, the middle or top of the ladder, give the same result.

This concurrency is a neat little theorem for statics problems involving three forces. The proof is simple. If the three lines weren't concurrent, then one force would produce a nonzero torque around the intersection point of the other two lines of force.[4] ♣

Statics problems often involve a number of decisions. If there are various parts to the system, then you must decide which subsystems you want to balance the forces and torques on. And furthermore, you must decide which point to use as the origin for calculating the torques. There are invariably many choices that will give you the information you need, but some will make your calculations much cleaner than others (Exercise 11 is a good example of this). The only way to know how to choose wisely is to start solving problems, so you may as well tackle some...

[4]The one exception to this reasoning is where no two of the lines intersect; that is, where all three lines are parallel. Equilibrium is certainly possible in such a scenario, as we saw in Claim 1.1. Of course, you can hang onto the concurrency theorem in this case if you consider the parallel lines to meet at infinity.

## 1.3 Exercises

*Section 1.1 Balancing forces*

1. **Pulling a block** $*$

   A person pulls on a block with a force $F$, at an angle $\theta$ with respect to the horizontal. The coefficient of friction between the block and the ground is $\mu$. For what $\theta$ is the $F$ required to make the block slip a minimum?

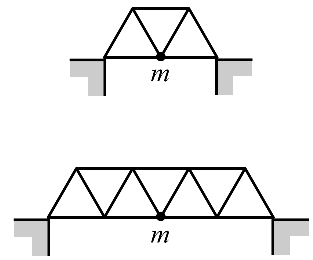

**Figure 1.9**

2. **Bridges** $**$

   (a) Consider the first bridge in Fig. 1.9, made of three equilateral triangles of beams. Assume that the seven beams are massless and that the connection between any two of them is a hinge. If a car of mass $m$ is located at the middle of the bridge, find the forces (and specify tension or compression) in the beams. Assume that the supports provide no horizontal forces on the bridge.

   (b) Same question, but now with the second bridge in Fig. 1.9, made of seven equilateral triangles.

   (c) Same question, but now with the general case of $4n-1$ equilateral triangles.

3. **Keeping the book up** $*$

   The task of Problem 4 is to find the minimum force required to keep a book up. What is the maximum allowable force? Is there a special angle that arises? Given $\mu$, make a rough plot of the allowed values of $F$ for $-\pi/2 < \theta < \pi/2$.

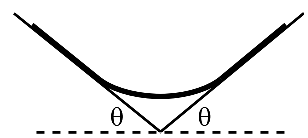

**Figure 1.10**

4. **Rope between inclines** $**$

   A rope rests on two platforms that are both inclined at an angle $\theta$ (which you are free to pick), as shown in Fig. 1.10. The rope has uniform mass density, and its coefficient of friction with the platforms is 1. The system has left-right symmetry. What is the largest possible fraction of the rope that does not touch the platforms? What angle $\theta$ allows this maximum value?

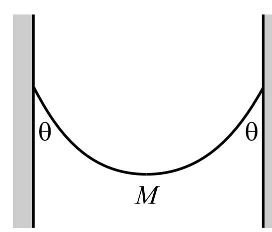

**Figure 1.11**

5. **Hanging chain** $**$

   A chain of mass $M$ hangs between two walls, with its ends at the same height. The chain makes an angle of $\theta$ with each wall, as shown in Fig. 1.11. Find the tension in the chain at the lowest point. Solve this by:

   (a) Considering the forces on half of the chain. (This is the quick way.)

   (b) Using the fact that the height of a hanging chain is given by $y(x) = (1/\alpha)\cosh(\alpha x)$, and considering the vertical forces on an infinitesimal piece at the bottom. (This is the long way.)

*Section 1.2: Balancing torques*

6. **Direction of the force** ∗

   A stick is connected to other parts of a system by hinges at its ends. Show that if the stick is massless, then the forces it feels at the hinges are directed along the stick; but if the stick has mass, then the forces need not point along the stick.

7. **Gravitational torque** ∗

   A horizontal stick of mass $M$ and length $L$ is pivoted at one end. Integrate the gravitational torque along the stick (relative to the pivot), and show that the result is the same as the torque due to a mass $M$ located at the center of the stick.

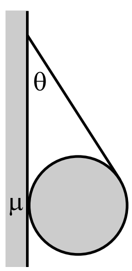


**Figure 1.12**

8. **Tetherball** ∗

   A ball is held up by a string, as shown in Fig. 1.12, with the string tangent to the ball. If the angle between the string and the wall is $\theta$, what is the minimum coefficient of static friction between the ball and the wall, if the ball is not to fall?

9. **Ladder on a corner** ∗

   A ladder of mass $M$ and length $L$ leans against a frictionless wall, with a quarter of its length hanging over a corner, as shown in Fig. 1.13. Assuming that there is sufficient friction at the corner to keep the ladder at rest, what is the total force that the corner exerts on the ladder?

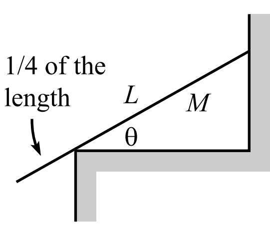


**Figure 1.13**

10. **Stick on a corner** ∗

    You hold one end of a stick of mass $M$ and length $L$. A quarter of the way up the stick, it rests on a frictionless corner of a table, as shown in Fig. 1.14. The stick makes an angle $\theta$ with the horizontal. What is the magnitude of the force your hand must apply, to keep the stick in this position? For what angle is the vertical component of your force equal to zero?

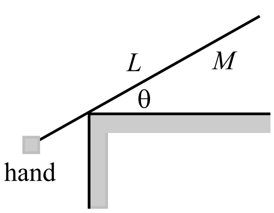


**Figure 1.14**

11. **Two sticks** ∗∗

    Two sticks, each of mass $m$ and length $\ell$, are connected by a hinge at their top ends. They each make an angle $\theta$ with the vertical. A massless string connects the bottom of the left stick to the right stick, perpendicularly, as shown in Fig. 1.15. The whole setup stands on a frictionless table.

    (a) What is the tension in the string?

    (b) What force does the left stick exert on the right stick at the hinge? *Hint:* No messy calculations required!

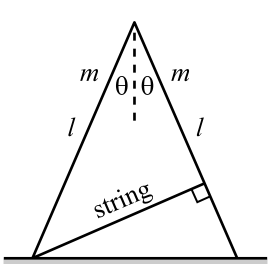


**Figure 1.15**

12. **Two sticks and a wall** $**$

Two sticks are connected, with hinges, to each other and to a wall. The bottom stick is horizontal and has length $L$, and the sticks make an angle of $\theta$ with each other, as shown in Fig. 1.16. If both sticks have the same mass per unit length, $\rho$, find the horizontal and vertical components of the force that the wall exerts on the top hinge, and show that the magnitude goes to infinity for both $\theta \to 0$ and $\theta \to \pi/2$.[5]

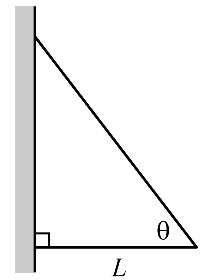


**Figure 1.16**

13. **Stick on a circle** $**$

Using the result from Problem 16 for the setup shown in Fig. 1.17, show that if the system is to remain at rest, then the coefficient of friction:

(a) between the stick and the circle must satisfy

$$\mu \geq \frac{\sin\theta}{(1+\cos\theta)} . \tag{1.14}$$

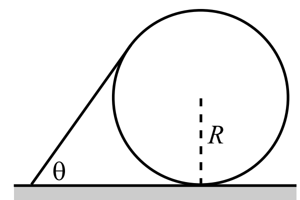


**Figure 1.17**

(b) between the stick and the ground must satisfy[6]

$$\mu \geq \frac{\sin\theta\cos\theta}{(1+\cos\theta)(2-\cos\theta)} . \tag{1.15}$$

[5]The force must therefore achieve a minimum at some intermediate angle. If you want to go through the algebra, you can show that this minimum occurs when $\cos\theta = \sqrt{3} - 1$, which gives $\theta \approx 43°$.

[6]If you want to go through the algebra, you can show that the maximum of the right-hand side occurs when $\cos\theta = \sqrt{3} - 1$, which gives $\theta \approx 43°$. (Yes, I did just cut and paste this from the previous footnote. But it's still correct!) This is the angle for which the stick is most likely to slip on the ground.

## 1.4 Problems

*Section 1.1: Balancing forces*

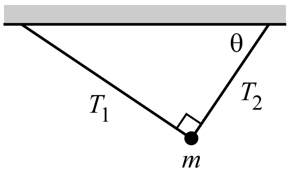


**Figure 1.18**

1. **Hanging mass**

   A mass $m$, held up by two strings, hangs from a ceiling, as shown in Fig. 1.18. The strings form a right angle. In terms of the angle $\theta$ shown, what is the tension in each string?

2. **Block on a plane**

   A block sits on a plane that is inclined at an angle $\theta$. Assume that the friction force is large enough to keep the block at rest. What are the horizontal components of the friction and normal forces acting on the block? For what $\theta$ are these horizontal components maximum?

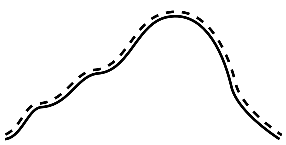

**Figure 1.19**

3. **Motionless chain** $*$

   A frictionless planar curve is in the shape of a function which has its endpoints at the same height but is otherwise arbitrary. A chain of uniform mass per unit length rests on the curve from end to end, as shown in Fig. 1.19. Show, by considering the net force of gravity along the curve, that the chain will not move.

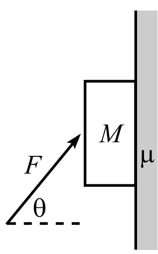


**Figure 1.20**

4. **Keeping the book up** $*$

   A book of mass $M$ is positioned against a vertical wall. The coefficient of friction between the book and the wall is $\mu$. You wish to keep the book from falling by pushing on it with a force $F$ applied at an angle $\theta$ with respect to the horizontal ($-\pi/2 < \theta < \pi/2$), as shown in Fig. 1.20. For a given $\theta$, what is the minimum $F$ required? What is the limiting value of $\theta$, below which there does not exist an $F$ that will keep the book up?

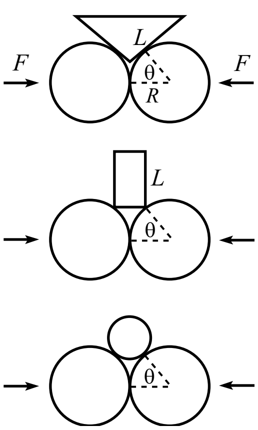


**Figure 1.21**

5. **Objects between circles** $**$

   Each of the following planar objects is placed, as shown in Fig. 1.21, between two frictionless circles of radius $R$. The mass density of each object is $\sigma$, and the radii to the points of contact make an angle $\theta$ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what $\theta$ is this force maximum or minimum?

   (a) An isosceles triangle with common side length $L$.

   (b) A rectangle with height $L$.

   (c) A circle.

6. **Hanging rope**

   A rope with length $L$ and mass density $\rho$ per unit length is suspended vertically from one end. Find the tension as a function of height along the rope.

7. **Rope on a plane** $*$

   A rope with length $L$ and mass density $\rho$ per unit length lies on a plane inclined at angle $\theta$ (see Fig. 1.22). The top end is nailed to the plane, and the coefficient of friction between the rope and plane is $\mu$. What are the possible values for the tension at the top of the rope?

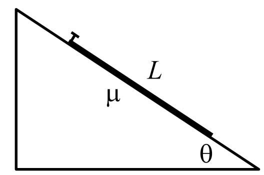


**Figure 1.22**

8. **Supporting a disk** $**$

   (a) A disk of mass $M$ and radius $R$ is held up by a massless string, as shown in Fig. 1.23. The surface of the disk is frictionless. What is the tension in the string? What is the normal force per unit length the string applies to the disk?

   (b) Let there now be friction between the disk and the string, with coefficient $\mu$. What is the smallest possible tension in the string at its lowest point?

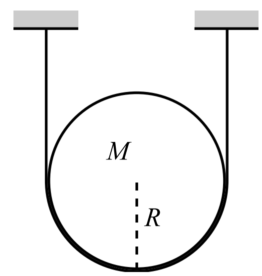


**Figure 1.23**

9. **Hanging chain** $****$

   (a) A chain with uniform mass density per unit length hangs between two given points on two walls. Find the shape of the chain. Aside from an arbitrary additive constant, the function describing the shape should contain one unknown constant.

   (b) The unknown constant in your answer depends on the horizontal distance $d$ between the walls, the vertical distance $\lambda$ between the support points, and the length $\ell$ of the chain (see Fig. 1.24). Find an equation involving these given quantities that determines the unknown constant.

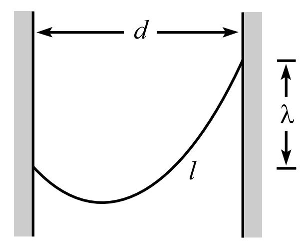


**Figure 1.24**

10. **Hanging gently** $**$

    A chain with uniform mass density per unit length hangs between two supports located at the same height, a distance $2d$ apart (see Fig. 1.25). What should the length of the chain be so that the magnitude of the force at the supports is minimized? You may use the fact that a hanging chain takes the form, $y(x) = (1/\alpha)\cosh(\alpha x)$. You will eventually need to solve an equation numerically.

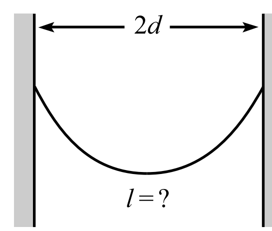


**Figure 1.25**

11. **Mountain Climber** $****$

    A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. Assume that the climber is of negligible height, so that the rope lies along the mountain, as shown in Fig. 1.26.

    At the bottom of the mountain are two stores. One sells "cheap" lassos (made of a segment of rope tied to a loop of *fixed* length). The other sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's

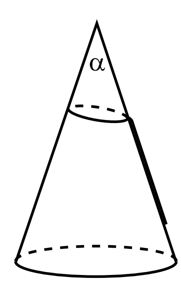


**Figure 1.26**

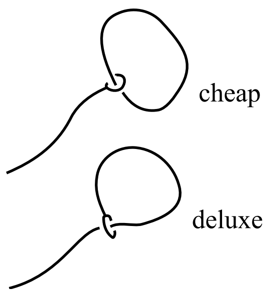

**Figure 1.27**

length may change without any friction of the rope with itself). See Fig. 1.27. When viewed from the side, the conical mountain has an angle $\alpha$ at its peak. For what angles $\alpha$ can the climber climb up along the mountain if he uses:

(a) a "cheap" lasso?

(b) a "deluxe" lasso?

*Section 1.2: Balancing torques*

12. **Equality of torques** $**$

This problem gives another way of demonstrating Claim 1.1, using an inductive argument. We'll get you started, and then you can do the general case.

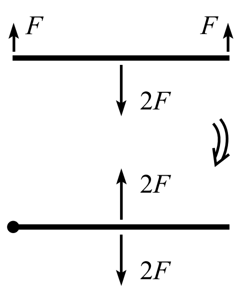

**Figure 1.28**

Consider the situation where forces $F$ are applied upward at the ends of a stick of length $\ell$, and a force $2F$ is applied downward at the midpoint (see Fig. 1.28). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). If we wish, we may consider the stick to have a pivot at the left end. If we then erase the force $F$ on the right end and replace it with a force $2F$ at the middle, then the two $2F$ forces in the middle will cancel, so the stick will remain at rest.[7] Therefore, we see that a force $F$ applied at a distance $\ell$ from a pivot is equivalent to a force $2F$ applied at a distance $\ell/2$ from the pivot, in the sense that they both have the same effect in cancelling out the rotational effect of the downwards $2F$ force.

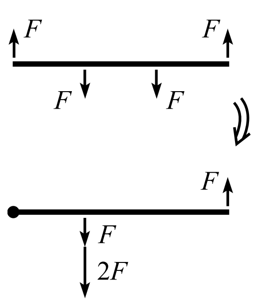

**Figure 1.29**

Now consider the situation where forces $F$ are applied upward at the ends, and forces $F$ are applied downward at the $\ell/3$ and $2\ell/3$ marks (see Fig. 1.29). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. From the above paragraph, the force $F$ at $2\ell/3$ is equivalent to a force $2F$ at $\ell/3$. Making this replacement, we now have a total force of $3F$ at the $\ell/3$ mark. Therefore, we see that a force $F$ applied at a distance $\ell$ is equivalent to a force $3F$ applied at a distance $\ell/3$.

Your task is to now use induction to show that a force $F$ applied at a distance $\ell$ is equivalent to a force $nF$ applied at a distance $\ell/n$, and to then argue why this demonstrates Claim 1.1.

13. **Find the force** $*$

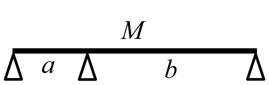

**Figure 1.30**

A stick of mass $M$ is held up by supports at each end, with each support providing a force of $Mg/2$. Now put another support somewhere in the middle, say, at a distance $a$ from one support and $b$ from the other; see Fig. 1.30. What forces do the three supports now provide? Can you solve this?

[7] There will now be a different force applied at the pivot, namely zero, but the purpose of the pivot is to simply apply whatever force is necessary to keep the left end motionless.

14. **Leaning sticks** $*$

One stick leans on another as shown in Fig. 1.31. A right angle is formed where they meet, and the right stick makes an angle $\theta$ with the horizontal. The left stick extends infinitesimally beyond the end of the right stick. The coefficient of friction between the two sticks is $\mu$. The sticks have the same mass density per unit length and are both hinged at the ground. What is the minimum angle $\theta$ for which the sticks do not fall?

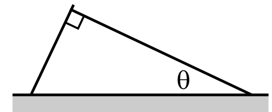


**Figure 1.31**

15. **Supporting a ladder** $*$

A ladder of length $L$ and mass $M$ has its bottom end attached to the ground by a pivot. It makes an angle $\theta$ with the horizontal, and is held up by a massless stick of length $\ell$ which is also attached to the ground by a pivot (see Fig. 1.32). The ladder and the stick are perpendicular to each other. Find the force that the stick exerts on the ladder.

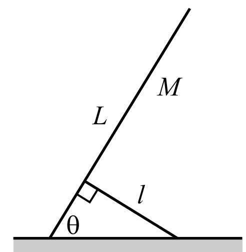


**Figure 1.32**

16. **Stick on a circle** $**$

A stick of mass density $\rho$ per unit length rests on a circle of radius $R$ (see Fig. 1.33). The stick makes an angle $\theta$ with the horizontal and is tangent to the circle at its upper end. Friction exists at all points of contact, and assume that it is large enough to keep the system at rest. Find the friction force between the ground and the circle.

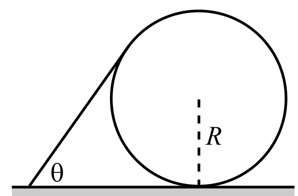


**Figure 1.33**

17. **Leaning sticks and circles** $***$

A large number of sticks (with mass density $\rho$ per unit length) and circles (with radius $R$) lean on each other, as shown in Fig. 1.34. Each stick makes an angle $\theta$ with the horizontal and is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is *frictionless* (unlike in the previous problem). In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? (Assume that the last circle leans against a wall, to keep it from moving.)

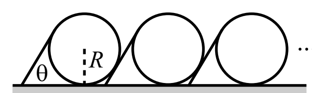


**Figure 1.34**

18. **Balancing the stick** $**$

Given a semi-infinite stick (that is, one that goes off to infinity in one direction), determine how its density should depend on position so that it has the following property: If the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support that is located a distance $\ell$ from the end (see Fig. 1.35).

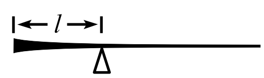


**Figure 1.35**

19. **The spool** $**$

A spool consists of an axle of radius $r$ and an outside circle of radius $R$ which rolls on the ground. A thread is wrapped around the axle and is pulled with tension $T$, at an angle $\theta$ with the horizontal (see Fig. 1.36).

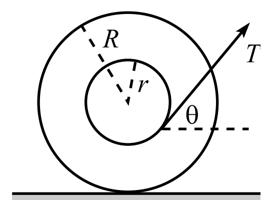


**Figure 1.36**

(a) Given $R$ and $r$, what should $\theta$ be so that the spool does not move? Assume that the friction between the spool and the ground is large enough so that the spool doesn't slip.

(b) Given $R$, $r$, and the coefficient of friction $\mu$ between the spool and the ground, what is the largest value of $T$ for which the spool remains at rest?

(c) Given $R$ and $\mu$, what should $r$ be so that you can make the spool slip with as small a $T$ as possible? That is, what should $r$ be so that the upper bound on $T$ from part (b) is as small as possible? What is the resulting value of $T$?

## 1.5 Solutions

1. **Hanging mass**

   Balancing the horizontal and vertical force components on the mass gives, respectively (see Fig. 1.37),

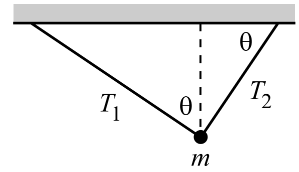


Figure 1.37

$$\begin{aligned} T_1 \sin\theta &= T_2 \cos\theta, \\ T_1 \cos\theta + T_2 \sin\theta &= mg. \end{aligned} \tag{1.16}$$

   Solving for $T_1$ in the first equation, and substituting into the second equation, gives

$$T_1 = mg\cos\theta, \qquad \text{and} \qquad T_2 = mg\sin\theta. \tag{1.17}$$

   As a double-check, these have the correct limits when $\theta \to 0$ or $\theta \to \pi/2$.

2. **Block on a plane**

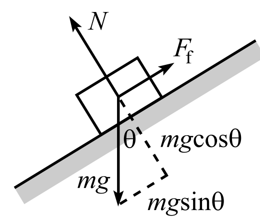


Figure 1.38

   Balancing the forces shown in Fig. 1.38, wee see that $F = mg\sin\theta$ and $N = mg\cos\theta$. The horizontal components of these are $F\cos\theta = mg\sin\theta\cos\theta$ (to the right), and $N\sin\theta = mg\cos\theta\sin\theta$ (to the left). These are equal, as they must be, because the net horizontal force on the block is zero. To maximize the value of $mg\sin\theta\cos\theta$, we can either take the derivative, or we can write it as $(mg/2)\sin 2\theta$, from which it is clear that the maximum occurs at $\theta = \pi/4$. The maximum value is $mg/2$.

3. **Motionless chain**

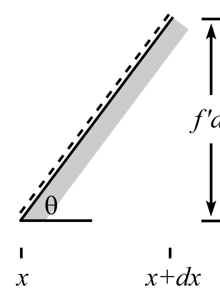


Figure 1.39

   Let the curve be described by the function $f(x)$, and let it run from $x = a$ to $x = b$. Consider a little piece of the chain between $x$ and $x + dx$ (see Fig. 1.39). The length of this piece is $\sqrt{1+f'^2}\,dx$, and so its mass is $\rho\sqrt{1+f'^2}\,dx$, where $\rho$ is the mass per unit length. The component of the gravitational acceleration along the curve is $-g\sin\theta = -gf'/\sqrt{1+f'^2}$, with positive corresponding to moving along the curve from $a$ to $b$. The total force along the curve is therefore

$$\begin{aligned} F &= \int_a^b (-g\sin\theta)\,dm \\ &= \int_a^b \left(\frac{-gf'}{\sqrt{1+f'^2}}\right)\left(\rho\sqrt{1+f'^2}\,dx\right) \\ &= -\rho g \int_a^b f'\,dx \\ &= -g\rho\big(f(a) - f(b)\big) \\ &= 0. \end{aligned} \tag{1.18}$$

4. **Keeping the book up**

   The normal force from the wall is $F\cos\theta$, so the friction force holding the book up is at most $\mu F\cos\theta$. The other vertical forces on the book are the gravitational force, which is $-Mg$, and the vertical component of $F$, which is $F\sin\theta$. If the book is to stay up, we must have

$$\mu F\cos\theta + F\sin\theta - Mg \ge 0. \tag{1.19}$$

   Therefore, $F$ must satisfy

$$F \ge \frac{Mg}{\mu\cos\theta + \sin\theta}\,. \tag{1.20}$$

There is no possible $F$ that satisfies this condition if the right-hand side is infinite. This occurs when

$$\tan\theta = -\mu. \tag{1.21}$$

If $\theta$ is more negative than this, then it is impossible to keep the book up, no matter how hard you push.

5. **Objects between circles**

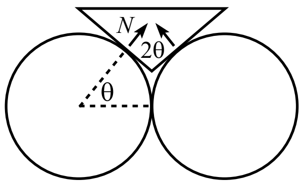


**Figure 1.40**

(a) Let $N$ be the normal force between the circles and the triangle. The goal in this problem is to find the horizontal component of $N$, that is, $N\cos\theta$.

From Fig. 1.40, we see that the upward force on the triangle from the normal forces is $2N\sin\theta$. This must equal the weight of the triangle, which is $g\sigma$ times the area. Since the bottom angle of the isosceles triangle is $2\theta$, the top side has length $2L\sin\theta$, and the altitude to this side is $L\cos\theta$. So the area of the triangle is $L^2\sin\theta\cos\theta$. The mass is therefore $\sigma L^2\sin\theta\cos\theta$. Equating the weight with the upward component of the normal forces gives $N = (g\sigma L^2/2)\cos\theta$. The horizontal component of $N$ is therefore

$$N\cos\theta = \frac{g\sigma L^2\cos^2\theta}{2}. \tag{1.22}$$

This equals zero when $\theta = \pi/2$, and it increases as $\theta$ decreases, even though the triangle is getting smaller. It has the interesting property of approaching the finite number $g\sigma L^2/2$, as $\theta \to 0$.

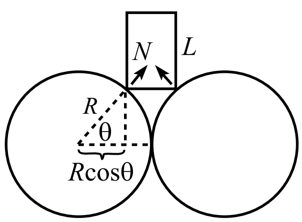


**Figure 1.41**

(b) In Fig. 1.41, the base of the rectangle has length $2R(1-\cos\theta)$. Its mass is therefore $\sigma 2RL(1-\cos\theta)$. Equating the weight with the upward component of the normal forces, $2N\sin\theta$, gives $N = g\sigma RL(1-\cos\theta)/\sin\theta$. The horizontal component of $N$ is therefore

$$N\cos\theta = \frac{g\sigma RL(1-\cos\theta)\cos\theta}{\sin\theta}. \tag{1.23}$$

This equals zero for both $\theta = \pi/2$ and $\theta = 0$ (because $1-\cos\theta \approx \theta^2/2$ goes to zero faster than $\sin\theta \approx \theta$, for small $\theta$). Taking the derivative to find where it reaches a maximum, we obtain (using $\sin^2\theta = 1-\cos^2\theta$),

$$\cos^3\theta - 2\cos\theta + 1 = 0. \tag{1.24}$$

Fortunately, there is an easy root of this cubic equation, namely $\cos\theta = 1$, which we know is not the maximum. Dividing through by the factor $(\cos\theta - 1)$ gives

$$\cos^2\theta + \cos\theta - 1 = 0. \tag{1.25}$$

The roots of this quadratic equation are

$$\cos\theta = \frac{-1\pm\sqrt{5}}{2}. \tag{1.26}$$

We must choose the plus sign, because we need $|\cos\theta| \le 1$. So our answer is $\cos\theta = 0.618$, which interestingly is the golden ratio. The angle $\theta$ is $\approx 51.8°$.

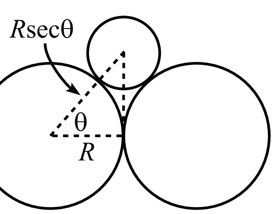


**Figure 1.42**

(c) In Fig. 1.42, the length of the hypotenuse shown is $R\sec\theta$, so the radius of the top circle is $R(\sec\theta - 1)$. Its mass is therefore $\sigma\pi R^2(\sec\theta-1)^2$. Equating the weight with the upward component of the normal forces, $2N\sin\theta$, gives $N = g\sigma\pi R^2(\sec\theta - 1)^2/(2\sin\theta)$. The horizontal component of $N$ is therefore

$$N\cos\theta = \frac{g\sigma\pi R^2\cos\theta}{2\sin\theta}\left(\frac{1}{\cos\theta} - 1\right)^2. \tag{1.27}$$

This equals zero when $\theta = 0$ (using $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$, for small $\theta$). For $\theta \to \pi/2$, it behaves like $1/\cos\theta$, which goes to infinity. In this limit, $N$ points almost vertically, but its magnitude is so large that the horizontal component still approaches infinity.

6. **Hanging rope**

Let $T(y)$ be the tension as a function of height. Consider a small piece of the rope between $y$ and $y + dy$ ($0 \le y \le L$). The forces on this piece are $T(y + dy)$ upward, $T(y)$ downward, and the weight $\rho g\,dy$ downward. Since the rope is at rest, we have $T(y + dy) = T(y) + \rho g\,dy$. Expanding this to first order in $dy$ gives $T'(y) = \rho g$. The tension in the bottom of the rope is zero, so integrating from $y = 0$ up to a position $y$ gives

$$T(y) = \rho g y. \tag{1.28}$$

As a double-check, at the top end we have $T(L) = \rho g L$, which is the weight of the entire rope, as it should be.

Alternatively, you can simply write down the answer, $T(y) = \rho g y$, by noting that the tension at a given point in the rope is what supports the weight of all the rope below it.

7. **Rope on a plane**

The component of the gravitational force along the plane is $(\rho L)g\sin\theta$, and the maximum value of the friction force is $\mu N = \mu(\rho L)g\cos\theta$. Therefore, you might think that the tension at the top of the rope is $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$. However, this is not necessarily the value. The tension at the top depends on how the rope is placed on the plane.

If, for example, the rope is placed on the plane without being stretched, the friction force will point upwards, and the tension at the top will indeed equal $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$. Or it will equal zero if $\mu\rho Lg\cos\theta > \rho Lg\sin\theta$, in which case the friction force need not achieve its maximum value.

If, on the other hand, the rope is placed on the plane after being stretched (or equivalently, it is dragged up along the plane and then nailed down), then the friction force will point downwards, and the tension at the top will equal $\rho Lg\sin\theta + \mu\rho Lg\cos\theta$.

Another special case occurs when the rope is placed on a frictionless plane, and then the coefficient of friction is "turned on" to $\mu$. The friction force will still be zero. Changing the plane from ice to sandpaper (somehow without moving the rope) won't suddenly cause there to be a friction force. Therefore, the tension at the top will equal $\rho Lg\sin\theta$.

In general, depending on how the rope is placed on the plane, the tension at the top can take any value from a maximum of $\rho Lg\sin\theta + \mu\rho Lg\cos\theta$, down to a minimum of $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$ (or zero, whichever is larger). If the rope were replaced by a stick (which could support a compressive force), then the tension could achieve negative values down to $\rho Lg\sin\theta - \mu\rho Lg\cos\theta$, if this happens to be negative.

8. **Supporting a disk**

(a) The gravitational force downward on the disk is $Mg$, and the force upward is $2T$. These forces must balance, so

$$T = \frac{Mg}{2}. \tag{1.29}$$

We can find the normal force per unit length that the string applies to the disk in two ways.

**First method:** Let $N\,d\theta$ be the normal force on an arc of the disk that subtends an angle $d\theta$. Such an arc has length $R\,d\theta$, so $N/R$ is the desired normal force per unit arclength. The tension in the string is constant because the string is massless, so $N$ is constant, independent of $\theta$. The upward component of the normal force is $N\,d\theta\cos\theta$, where $\theta$ is measured from the vertical (that is, $-\pi/2 \le \theta \le \pi/2$ here). Since the total upward force is $Mg$, we must have

$$\int_{-\pi/2}^{\pi/2} N\cos\theta\, d\theta = Mg. \tag{1.30}$$

The integral equals $2N$, so we find $N = Mg/2$. The normal force per unit length, $N/R$, is then $Mg/2R$.

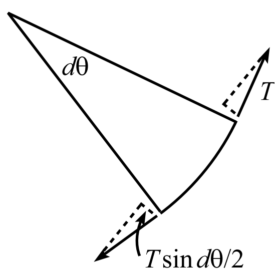


**Figure 1.43**

**Second method:** Consider the normal force, $N\,d\theta$, on a small arc of the disk that subtends and angle $d\theta$. The tension forces on each end of the corresponding small piece of string almost cancel, but they don't exactly, because they point in slightly different directions. Their non-zero sum is what produces the normal force on the disk. From Fig. 1.43, we see that the two forces have a sum of $2T\sin(d\theta/2)$, directed inward. Since $d\theta$ is small, we can use $\sin x \approx x$ to approximate this as $T\,d\theta$. Therefore, $N\,d\theta = T\,d\theta$, and so $N = T$. The normal force per unit arclength, $N/R$, then equals $T/R$. Using $T = Mg/2$ from eq. (1.29), we arrive at $N/R = Mg/2R$.

(b) Let $T(\theta)$ be the tension, as a function of $\theta$, for $-\pi/2 \le \theta \le \pi/2$. $T$ will depend on $\theta$ now, because there is a tangential friction force. Most of the work for this problem was already done in the example at the end of Section 1.1. We will simply invoke the second line of eq. (1.7), which says that[8]

$$dT \le \mu T\, d\theta. \tag{1.31}$$

Separating variables and integrating from the bottom of the rope up to an angle $\theta$ gives $\ln\big((T(\theta)/T(0)\big) \le \mu\theta$. Exponentiating this gives

$$T(\theta) \le T(0)e^{\mu\theta}. \tag{1.32}$$

Letting $\theta = \pi/2$, and using $T(\pi/2) = Mg/2$, we have $Mg/2 \le T(0)e^{\mu\pi/2}$. We therefore see that the tension at the bottom point must satisfy

$$T(0) \ge \frac{Mg}{2}e^{-\mu\pi/2}. \tag{1.33}$$

[8]This holds for $\theta > 0$. There would be a minus sign on the right-hand side if $\theta < 0$. But since the tension is symmetric around $\theta = 0$ in the case we're concerned with, we'll just deal with $\theta > 0$.

This minimum value of $T(0)$ goes to $Mg/2$ as $\mu \to 0$, as it should. And it goes to zero as $\mu \to \infty$, as it should (imagine a very sticky surface, so that the friction force from the rope near $\theta = \pi/2$ accounts for essentially all the weight). But interestingly, it doesn't exactly equal zero, no matter now large $\mu$ is.

9. **Hanging chain**

(a) Let the chain be described by the function $y(x)$, and let the tension be described by the function $T(x)$. Consider a small piece of the chain, with endpoints at $x$ and $x + dx$, as shown in Fig. 1.44. Let the tension at $x$ pull downward at an angle $\theta_1$ with respect to the horizontal, and let the tension at $x + dx$ pull upward at an angle $\theta_2$ with respect to the horizontal. Balancing the horizontal and vertical forces on the small piece of chain gives

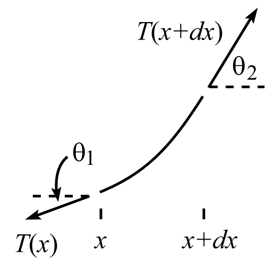


**Figure 1.44**

$$\begin{aligned} T(x+dx)\cos\theta_2 &= T(x)\cos\theta_1, \\ T(x+dx)\sin\theta_2 &= T(x)\sin\theta_1 + \frac{g\rho\, dx}{\cos\theta_1}, \end{aligned} \tag{1.34}$$

where $\rho$ is the mass per unit length. The second term on the right-hand side is the weight of the small piece, because $dx/\cos\theta_1$ (or $dx/\cos\theta_2$, which is essentially the same) is its length. We must now somehow solve these two differential equations for the two unknown functions, $y(x)$ and $T(x)$. There are various ways to do this. Here is one method, broken down into three steps.

First step: Squaring and adding eqs. (1.34) gives

$$\big(T(x+dx)\big)^2 = \big(T(x)\big)^2 + 2T(x)g\rho\tan\theta_1\, dx + \mathcal{O}(dx^2). \tag{1.35}$$

Writing $T(x+dx) \approx T(x) + T'(x)\,dx$, and using $\tan\theta_1 = dy/dx \equiv y'$, we can simplify eq. (1.35) to (neglecting second-order terms in $dx$)

$$T' = g\rho y'. \tag{1.36}$$

Therefore,

$$T = g\rho y + c_1, \tag{1.37}$$

where $c_1$ is a constant of integration.

Second step: Let's see what we can extract from the first equation in eqs. (1.34). Using

$$\cos\theta_1 = \frac{1}{\sqrt{1+(y'(x))^2}}, \qquad \text{and} \qquad \cos\theta_2 = \frac{1}{\sqrt{1+(y'(x+dx))^2}}, \tag{1.38}$$

and expanding things to first order in $dx$, the first of eqs. (1.34) becomes

$$\frac{T + T'dx}{\sqrt{1+(y'+y''dx)^2}} = \frac{T}{\sqrt{1+y'^2}}. \tag{1.39}$$

All of the functions here are evaluated at $x$, which we won't bother writing. Expanding the first square root gives (to first order in $dx$)

$$\frac{T + T'dx}{\sqrt{1+y'^2}}\left(1 - \frac{y'y''dx}{1+y'^2}\right) = \frac{T}{\sqrt{1+y'^2}}. \tag{1.40}$$

To first order in $dx$ this yields

$$\frac{T'}{T} = \frac{y'y''}{1+y'^2}\,. \tag{1.41}$$

Integrating both sides gives

$$\ln T + c_2 = \frac{1}{2}\ln(1+y'^2), \tag{1.42}$$

where $c_2$ is a constant of integration. Exponentiating then gives

$$c_3^2 T^2 = 1 + y'^2, \tag{1.43}$$

where $c_3 \equiv e^{c_2}$.

Third step: We will now combine eq. (1.43) with eq. (1.37) to solve for $y(x)$. Eliminating $T$ gives $c_3^2(g\rho y + c_1)^2 = 1 + y'^2$. We can rewrite this is the somewhat nicer form,

$$1 + y'^2 = \alpha^2(y+h)^2, \tag{1.44}$$

where $\alpha \equiv c_3 g\rho$, and $h = c_1/g\rho$. At this point we can cleverly guess (motivated by the fact that $1 + \sinh^2 z = \cosh^2 z$) that the solution for $y$ is given by

$$y(x) + h = \frac{1}{\alpha}\cosh\alpha(x+a). \tag{1.45}$$

Or, we can separate variables to obtain

$$dx = \frac{dy}{\sqrt{\alpha^2(y+h)^2 - 1}}\,, \tag{1.46}$$

and then use the fact that the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, to obtain the same result.

The shape of the chain is therefore a hyperbolic cosine function. The constant $h$ isn't too important, because it simply depends on where we pick the $y = 0$ height. Furthermore, we can eliminate the need for the constant $a$ if we pick $x = 0$ to be where the lowest point of the chain is (or where it would be, in the case where the slope is always nonzero). In this case, using eq. (1.45), we see that $y'(0) = 0$ implies $a = 0$, as desired. We then have (ignoring the constant $h$) the nice simple result,

$$y(x) = \frac{1}{\alpha}\cosh(\alpha x). \tag{1.47}$$

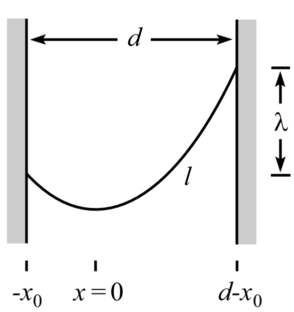


**Figure 1.45**

(b) The constant $\alpha$ can be determined from the locations of the endpoints and the length of the chain. As stated in the problem, the position of the chain may be described by giving (1) the horizontal distance $d$ between the two endpoints, (2) the vertical distance $\lambda$ between the two endpoints, and (3) the length $\ell$ of the chain, as shown in Fig. 1.45. Note that it is not obvious what the horizontal distances between the ends and the minimum point (which we have chosen as the $x = 0$ point) are. If $\lambda = 0$, then these distances are simply $d/2$. But otherwise, they are not so clear.

If we let the left endpoint be located at $x = -x_0$, then the right endpoint is located at $x = d - x_0$. We now have two unknowns, $x_0$ and $\alpha$. Our two conditions are[9]

$$y(d-x_0) - y(-x_0) = \lambda, \tag{1.48}$$

[9] We will take the right end to be higher than the left end, without loss of generality.

along with the condition that the length equals $\ell$, which takes the form (using eq. (1.47))

$$\begin{aligned} \ell &= \int_{-x_0}^{d-x_0} \sqrt{1+y'^2}\,dx \\ &= \frac{1}{\alpha}\sinh(\alpha x)\Big|_{-x_0}^{d-x_0}, \end{aligned} \tag{1.49}$$

where we have used $(d/dz)\cosh z = \sinh z$, and $1+\sinh^2 z = \cosh^2 z$. Writing out eqs. (1.48) and (1.49) explicitly, we have

$$\begin{aligned} \cosh\big(\alpha(d-x_0)\big) - \cosh(-\alpha x_0) &= \alpha\lambda, \\ \sinh\big(\alpha(d-x_0)\big) - \sinh(-\alpha x_0) &= \alpha\ell. \end{aligned} \tag{1.50}$$

If we take the difference of the squares of these two equations, and use the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x\cosh y - \sinh x \sinh y = \cosh(x-y)$, we obtain

$$2 - 2\cosh(\alpha d) = \alpha^2(\lambda^2 - \ell^2). \tag{1.51}$$

This is the desired equation that determines $\alpha$. Given $d$, $\lambda$, and $\ell$, we can numerically solve for $\alpha$. Using a "half-angle" formula, you can show that eq. (1.51) may also be written as

$$2\sinh(\alpha d/2) = \alpha\sqrt{\ell^2 - \lambda^2}. \tag{1.52}$$

Remark: Let's check a couple limits. If $\lambda = 0$ and $\ell = d$ (that is, the chain forms a horizontal straight line), then eq. (1.52) becomes $2\sinh(\alpha d/2) = \alpha d$. The solution to this is $\alpha = 0$, which does indeed correspond to a horizontal straight line, because for small $\alpha$, eq. (1.47) behaves like $\alpha x^2/2$ (up to an additive constant), which varies slowly with $x$ for small $\alpha$. Another limit is where $\ell$ is much larger than both $d$ and $\lambda$. In this case, eq. (1.52) becomes $2\sinh(\alpha d/2) \approx \alpha\ell$. The solution to this is a very large $\alpha$, which corresponds to a "droopy" chain, because eq. (1.47) varies rapidly with $x$ for large $\alpha$. ♣

10. **Hanging gently**

We must first find the mass of the chain by calculating its length. Then we must determine the slope of the chain at the supports, so we can find the components of the force there.

Using the given information, $y(x) = (1/\alpha)\cosh(\alpha x)$, the slope of the chain as a function of $x$ is

$$y' = \frac{d}{dx}\left(\frac{1}{\alpha}\cosh(\alpha x)\right) = \sinh(\alpha x). \tag{1.53}$$

The total length is therefore (using $1+\sinh^2 z = \cosh^2 z$)

$$\begin{aligned} \ell &= \int_{-d}^{d} \sqrt{1+y'^2}\,dx \\ &= \int_{-d}^{d} \cosh(\alpha x) \\ &= \frac{2}{\alpha}\sinh(\alpha d). \end{aligned} \tag{1.54}$$

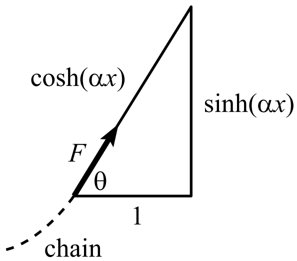


**Figure 1.46**

The weight of the rope is $W = \rho \ell g$, where $\rho$ is the mass per unit length. Each support applies a vertical force of $W/2$. This must equal $F \sin\theta$, where $F$ is the total force at each support, and $\theta$ is the angle it makes with the horizontal. Since $\tan\theta = y'(d) = \sinh(\alpha d)$, we see from Fig. 1.46 that $\sin\theta = \tanh(\alpha d)$. Therefore,

$$\begin{aligned} F &= \frac{1}{\sin\theta}\left(\frac{W}{2}\right) \\ &= \frac{1}{\tanh(\alpha d)}\left(\frac{\rho g \sinh(\alpha d)}{\alpha}\right) \\ &= \frac{\rho g}{\alpha}\cosh(\alpha d). \end{aligned} \tag{1.55}$$

Taking the derivative of this (as a function of $\alpha$), and setting the result equal to zero to find the minimum, gives

$$\tanh(\alpha d) = \frac{1}{\alpha d}\,. \tag{1.56}$$

This must be solved numerically. The result is

$$\alpha d \approx 1.1997 \equiv \eta. \tag{1.57}$$

We therefore have $\alpha = \eta/d$, and so the shape of the chain that requires the minimum $F$ is

$$y(x) \approx \frac{d}{\eta}\cosh\left(\frac{\eta x}{d}\right). \tag{1.58}$$

From eqs. (1.54) and (1.57), the length of the chain is

$$\ell = \frac{2d}{\eta}\sinh(\eta) \approx (2.52)d. \tag{1.59}$$

To get an idea of what the chain looks like, we can calculate the ratio of the height, $h$, to the width, $2d$.

$$\begin{aligned} \frac{h}{2d} &= \frac{y(d) - y(0)}{2d} \\ &= \frac{\cosh(\eta) - 1}{2\eta} \\ &\approx 0.338. \end{aligned} \tag{1.60}$$

We can also calculate the angle of the rope at the supports, using $\tan\theta = \sinh(\alpha d)$. This gives $\tan\theta = \sinh\eta$, and so $\theta \approx 56.5°$.

Remark: We can also ask what shape the chain should take in order to minimize the horizontal or vertical component of $F$.

The vertical component, $F_y$, is simply half the weight, so we want the shortest possible chain, namely a horizontal one (which requires an infinite $F$.) This corresponds to $\alpha = 0$.

The horizontal component, $F_x$, equals $F\cos\theta$. From Fig. 1.46, we see that $\cos\theta = 1/\cosh(\alpha d)$. Therefore, eq. (1.55) gives $F_x = \rho g/\alpha$. This goes to zero as $\alpha \to \infty$, which corresponds to a chain of infinite length, that is, a very "droopy" chain. ♣

11. **Mountain Climber**

(a) We will take advantage of the fact that a cone is "flat", in the sense that we can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, which is a sector of a circle, $S$ (see Fig. 1.47). If the cone is very sharp, then $S$ will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then $S$ will look like a pie with a piece taken out of it.

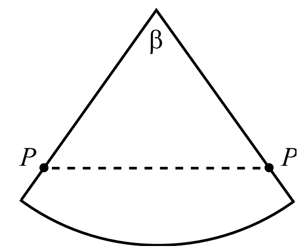

**Figure 1.47**

Points on the straight-line boundaries of the sector $S$ are identified with each other. Let $P$ be the location of the lasso's knot. Then $P$ appears on each straight-line boundary, at equal distances from the tip of $S$. Let $\beta$ be the angle of the sector $S$.

The key to this problem is to realize that the path of the lasso's loop must be a straight line on $S$, as shown by the dotted line in Fig. 1.47. (The rope will take the shortest distance between two points because there is no friction. And rolling the cone onto a plane does not change distances.) A straight line between the two identified points $P$ is possible if and only if the sector $S$ is smaller than a semicircle. The condition for a climbable mountain is therefore $\beta < 180°$.

What is this condition, in terms of the angle of the peak, $\alpha$? Let $C$ denote a cross-sectional circle of the mountain, a distance $d$ (measured along the cone) from the top.[10] A semicircular $S$ implies that the circumference of $C$ equals $\pi d$. This then implies that the radius of $C$ equals $d/2$. Therefore,

$$\sin(\alpha/2) < \frac{d/2}{d} = \frac{1}{2} \qquad \Longrightarrow \qquad \alpha < 60°. \tag{1.61}$$

This is the condition under which the mountain is climbable. In short, having $\alpha < 60°$ guarantees that there is a loop around the cone with shorter length than the distance straight to the peak and back.

Remark: When viewed from the side, the rope will appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies that the climbable condition is $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop. But the loop must certainly have a kink in it where the knot is, because there must exist a vertical component to the tension there, to hold the climber up. If we had posed the problem with a planar, triangular mountain, then the condition would have been $\alpha < 90°$.

(b) Use the same strategy as in part (a). Roll the cone onto a plane. If the mountain is very steep, then the climber's position can fall by means of the loop growing larger. If the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector $S$ in a plane, this condition requires that if we move $P$ a distance $\ell$ up (or down) along the mountain, the distance between the identified points $P$ must decrease (or increase) by $\ell$. In Fig. 1.47, we must therefore have an equilateral triangle, so $\beta = 60°$.

[10] We are considering such a circle for geometrical convenience. It is *not* the path of the lasso; see the remark below.

What peak-angle $\alpha$ does this correspond to? As in part (a), let $C$ be a cross-sectional circle of the mountain, a distance $d$ (measured along the cone) from the top. Then $\beta = 60°$ implies that the circumference of $C$ equals $(\pi/3)d$. This then implies that the radius of $C$ equals $d/6$. Therefore,

$$\sin(\alpha/2) = \frac{d/6}{d} = \frac{1}{6} \qquad \Longrightarrow \qquad \alpha \approx 19°. \tag{1.62}$$

This is the condition under which the mountain is climbable. We see that there is exactly one angle for which the climber can climb up along the mountain. The cheap lasso is therefore much more useful than the fancy deluxe lasso (assuming, of course, that you want to use it for climbing mountains, and not, say, for rounding up cattle).

REMARK: Another way to see the $\beta = 60°$ result is to note that the three directions of rope emanating from the knot must all have the same tension, because the deluxe lasso is one continuous piece of rope. They must therefore have 120° angles between themselves (to provide zero net force on the massless knot). This implies that $\beta = 60°$ in Fig. 1.47.

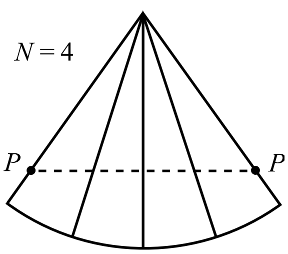

**Figure 1.48**

FURTHER REMARKS: For each type of lasso, we can also ask the question: For what angles can the mountain be climbed if the lasso is looped $N$ times around the top of the mountain? The solution here is similar to that above.

For the "cheap" lasso of part (a), roll the cone $N$ times onto a plane, as shown in Fig. 1.48 for $N = 4$. The resulting figure, $S_N$, is a sector of a circle divided into $N$ equal sectors, each representing a copy of the cone. As above, $S_N$ must be smaller than a semicircle. The circumference of the circle $C$ (defined above) must therefore be less than $\pi d/N$. Hence, the radius of $C$ must be less than $d/2N$. Thus,

$$\sin(\alpha/2) < \frac{d/2N}{d} = \frac{1}{2N} \qquad \Longrightarrow \qquad \alpha < 2\sin^{-1}\left(\frac{1}{2N}\right). \tag{1.63}$$

For the "deluxe" lasso of part (b), again roll the cone $N$ times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. The circumference of $C$ must therefore be $\pi d/3N$, and so its radius must be $d/6N$. Therefore,

$$\sin(\alpha/2) = \frac{d/6N}{d} = \frac{1}{6N} \qquad \Longrightarrow \qquad \alpha = 2\sin^{-1}\left(\frac{1}{6N}\right). \tag{1.64}$$

12. **Equality of torques**

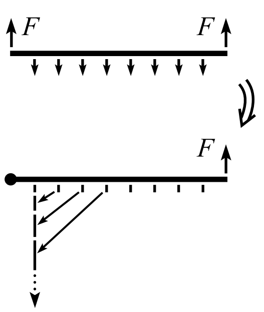

**Figure 1.49**

The proof by induction is as follows. Assume that we have shown that a force $F$ applied at a distance $d$ is equivalent to a force $kF$ applied at a distance $d/k$, for all integers $k$ up to $n-1$. We now want to show that the statement holds for $k = n$.

Consider the situation in Fig. 1.49. Forces $F$ are applied at the ends of a stick, and forces $2F/(n-1)$ are applied at the $j\ell/n$ marks (for $1 \le j \le n-1$). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. Replacing the interior forces by their equivalent ones at the $\ell/n$ mark (see Fig. 1.49) gives a total force there equal to

$$\frac{2F}{n-1}\Big(1+2+3+\cdots+(n-1)\Big) = \frac{2F}{n-1}\left(\frac{n(n-1)}{2}\right) = nF. \tag{1.65}$$

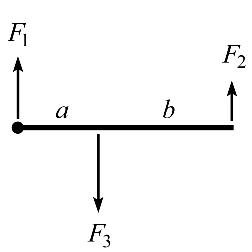

**Figure 1.50**

We therefore see that a force $F$ applied at a distance $\ell$ is equivalent to a force $nF$ applied at a distance $\ell/n$, as was to be shown.

We can now show that Claim 1.1 holds, for arbitrary distances $a$ and $b$ (see Fig. 1.50).

Consider the stick to be pivoted at its left end, and let $\epsilon$ be a tiny distance (small compared to $a$). Then a force $F_3$ at a distance $a$ is equivalent to a force $F_3(a/\epsilon)$ at a distance $\epsilon$.[11] But a force $F_3(a/\epsilon)$ at a distance $\epsilon$ is equivalent to a force $F_3(a/\epsilon)(\epsilon/(a+b)) = F_3a/(a+b)$ at a distance $(a+b)$. This equivalent force at the distance $(a+b)$ must cancel the force $F_2$ there, because the stick is motionless. Therefore, we have $F_3a/(a+b) = F_2$, which proves the claim.

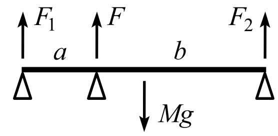

**Figure 1.51**

13. **Find the force**

In Fig. 1.51, let the supports at the ends exert forces $F_1$ and $F_2$, and let the support in the interior exert a force $F$. Then

$$F_1 + F_2 + F = Mg. \tag{1.66}$$

Balancing torques around the left and right ends gives, respectively,

$$\begin{aligned} Fa + F_2(a+b) &= Mg\frac{a+b}{2}, \\ Fb + F_1(a+b) &= Mg\frac{a+b}{2}, \end{aligned} \tag{1.67}$$

where we have used the fact that the stick can be treated as a point mass at its center. Note that the equation for balancing the torques around the center of mass is redundant; it is obtained by taking the difference of the two previous equations and then dividing by 2. And balancing torques around the middle pivot also takes the form of a linear combination of these equations, as you can show.

It appears as though we have three equations and three unknowns, but we really have only two equations, because the sum of eqs. (1.67) gives eq. (1.66). Therefore, since we have two equations and three unknowns, the system is underdetermined. Solving eqs. (1.67) for $F_1$ and $F_2$ in terms of $F$, we see that any forces of the form

$$(F_1, F, F_2) = \left(\frac{Mg}{2} - \frac{Fb}{a+b},\ F,\ \frac{Mg}{2} - \frac{Fa}{a+b}\right) \tag{1.68}$$

are possible. In retrospect, it makes sense that the forces are not determined. By changing the height of the new support an infinitesimal distance, we can make $F$ be anything from 0 up to $Mg(a+b)/2b$, which is when the stick comes off the left support (assuming $b \geq a$).

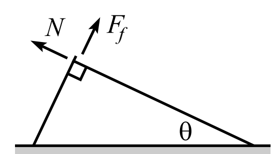

**Figure 1.52**

14. **Leaning sticks**

Let $M_l$ be the mass of the left stick, and let $M_r$ be the mass of the right stick. Then $M_l/M_r = \tan\theta$ (see Fig. 1.52). Let $N$ and $F_f$ be the normal and friction forces between the sticks. $F_f$ has a maximum value of $\mu N$. Balancing the torques on the left stick (around the contact point with the ground) gives

$$N = \frac{M_l g}{2}\sin\theta. \tag{1.69}$$

Balancing the torques on the right stick (around the contact point with the ground) gives

$$F_f = \frac{M_r g}{2}\cos\theta. \tag{1.70}$$

[11] Technically, we can use the reasoning in the previous paragraph to say this only if $a/\epsilon$ is an integer, but since $a/\epsilon$ is very large, we can simply pick the closest integer to it, and there will be negligible error.

The condition $F_f \le \mu N$ becomes

$$M_r \cos\theta \le \mu M_l \sin\theta. \tag{1.71}$$

Using $M_l/M_r = \tan\theta$, this becomes

$$\tan^2\theta \ge \frac{1}{\mu}\,. \tag{1.72}$$

This is the condition for the sticks not to fall. This answer checks in the two extremes: In the limit $\mu \to 0$, we see that $\theta$ must be very close to $\pi/2$, which makes sense. And in the limit $\mu \to \infty$ (that is, very sticky sticks), we see that $\theta$ can be very small, which also makes sense.

15. **Supporting a ladder**

Let $F$ be the desired force. Note that $F$ must be directed along the stick, because otherwise there would be a net torque on the (massless) stick relative to the pivot at its right end. This would contradict the fact that it is at rest.

Look at torques on the ladder around the pivot at its bottom. The gravitational force provides a torque of $Mg(L/2)\cos\theta$, tending to turn it clockwise; and the force $F$ from the stick provides a torque of $F(\ell/\tan\theta)$, tending to turn it counterclockwise. Equating these two torques gives

$$F = \frac{MgL}{2\ell}\sin\theta. \tag{1.73}$$

REMARKS: $F$ goes to zero as $\theta \to 0$, as it should.[12] And $F$ increases to $MgL/2\ell$, as $\theta \to \pi/2$, which isn't so obvious (the required torque from the stick is very small, but its lever arm is also very small). However, in the special case where the ladder is exactly vertical, no force is required. You can see that our calculations above are not valid in this case, because we divided by $\cos\theta$, which is zero when $\theta = \pi/2$.

The normal force at the pivot of the stick (which equals the vertical component of $F$, because the stick is massless) is equal to $MgL\sin\theta\cos\theta/2\ell$. This has a maximum value of $MgL/4\ell$ at $\theta = \pi/4$. ♣

16. **Stick on a circle**

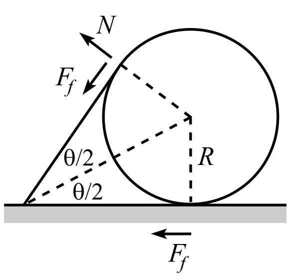

**Figure 1.53**

Let $N$ be the normal force between the stick and the circle, and let $F_f$ be the friction force between the ground and the circle (see Fig. 1.53). Then we immediately see that the friction force between the stick and the circle is also $F_f$, because the torques from the two friction forces on the circle must cancel.

Looking at torques on the stick around the point of contact with the ground, we have $Mg\cos\theta(L/2) = NL$, where $M$ is the mass of the stick and $L$ is its length. Therefore, $N = (Mg/2)\cos\theta$. Balancing the horizontal forces on the circle then gives $N\sin\theta = F_f + F_f\cos\theta$. So we have

$$F_f = \frac{N\sin\theta}{1+\cos\theta} = \frac{Mg\sin\theta\cos\theta}{2(1+\cos\theta)}\,. \tag{1.74}$$

But $M = \rho L$, and from Fig. 1.53 we have $L = R/\tan(\theta/2)$. Using the identity $\tan(\theta/2) = \sin\theta/(1+\cos\theta)$, we finally obtain

$$F_f = \frac{1}{2}\rho g R\cos\theta. \tag{1.75}$$

[12] For $\theta \to 0$, we would need to lengthen the ladder with a massless extension, because the stick would have to be very far to the right to remain perpendicular to the ladder.

In the limit $\theta \to \pi/2$, $F_f$ approaches zero, which makes sense. In the limit $\theta \to 0$ (which corresponds to a very long stick), the friction force approaches $\rho g R/2$, which isn't so obvious.

17. **Leaning sticks and circles**

Let $S_i$ be the $i$th stick, and let $C_i$ be the $i$th circle. The normal forces $C_i$ feels from $S_i$ and $S_{i+1}$ are equal in magnitude, because these two forces provide the only horizontal forces on the frictionless circle, so they must cancel. Let $N_i$ be this normal force.

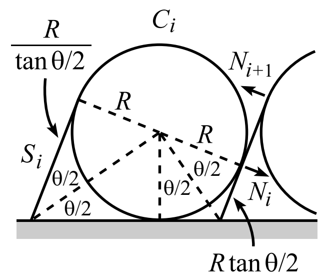


Figure 1.54

Look at the torques on $S_{i+1}$, relative to the hinge on the ground. The torques come from $N_i$, $N_{i+1}$, and the weight of $S_{i+1}$. From Fig. 1.54, we see that $N_i$ acts at a point which is a distance $R\tan(\theta/2)$ away from the hinge. Since the stick has a length $R/\tan(\theta/2)$, this point is a fraction $\tan^2(\theta/2)$ up along the stick. Therefore, balancing the torques on $S_{i+1}$ gives

$$\frac{1}{2}Mg\cos\theta + N_i\tan^2\frac{\theta}{2} = N_{i+1}. \tag{1.76}$$

$N_0$ is by definition 0, so we have $N_1 = (Mg/2)\cos\theta$ (as in the previous problem). If we successively use eq. (1.76), we see that $N_2$ equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)\big)$, and $N_3$ equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)+\tan^4(\theta/2)\big)$, and so on. In general,

$$N_i = \frac{Mg\cos\theta}{2}\left(1+\tan^2\frac{\theta}{2}+\tan^4\frac{\theta}{2}+\cdots+\tan^{2(i-1)}\frac{\theta}{2}\right). \tag{1.77}$$

In the limit $i\to\infty$, we may write this infinite geometric sum in closed form as

$$\lim_{i\to\infty} N_i \equiv N_\infty = \frac{Mg\cos\theta}{2}\left(\frac{1}{1-\tan^2(\theta/2)}\right). \tag{1.78}$$

Note that this is the solution to eq. (1.76), with $N_i = N_{i+1}$. So if a limit exists, it must equal this. Using $M = \rho L = \rho R/\tan(\theta/2)$, we can rewrite $N_\infty$ as

$$N_\infty = \frac{\rho R g\cos\theta}{2\tan(\theta/2)}\left(\frac{1}{1-\tan^2(\theta/2)}\right). \tag{1.79}$$

The identity $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ may then be used to write this as

$$N_\infty = \frac{\rho R g\cos^3(\theta/2)}{2\sin(\theta/2)}. \tag{1.80}$$

REMARKS: $N_\infty$ goes to infinity for $\theta\to 0$, which makes sense, because the sticks are very long. All of the $N_i$ are essentially equal to half the weight of a stick (in order to cancel the torque from the weight relative to the pivot). For $\theta\to\pi/2$, we see from eq. (1.80) that $N_\infty$ approaches $\rho R g/4$, which is not at all obvious; the $N_i$ start off at $N_1 = (Mg/2)\cos\theta \approx 0$, but gradually increase to $\rho R g/4$, which is a quarter of the weight of a stick.

Note that the horizontal force that must be applied to the last circle far to the right is $N_\infty\sin\theta = \rho R g\cos^4(\theta/2)$. This ranges from $\rho R g$ for $\theta\to 0$, to $\rho R g/4$ for $\theta\to\pi/2$. ♣

18. **Balancing the stick**

Let the stick go off to infinity in the positive $x$ direction, and let it be cut at $x = x_0$. Then the pivot point is located at $x = x_0+\ell$ (see Fig. 1.55). Let the density be $\rho(x)$. The condition that the total gravitational torque relative to $x_0+\ell$ equal zero is

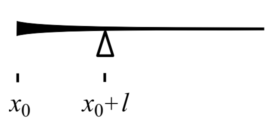


Figure 1.55

$$\tau = \int_{x_0}^{\infty}\rho(x)\big(x-(x_0+\ell)\big)g\,dx = 0. \tag{1.81}$$

We want this to equal zero for all $x_0$, so the derivative of $\tau$ with respect to $x_0$ must be zero. $\tau$ depends on $x_0$ through both the limits of integration and the integrand. In taking the derivative, the former dependence requires finding the value of the integrand at the limits, while the latter dependence requires taking the derivative of the integrand with respect to $x_0$, and then integrating. We obtain, using the fact that $\rho(\infty) = 0$,

$$0 = \frac{d\tau}{dx_0} = \ell\rho(x_0) - \int_{x_0}^{\infty} \rho(x)\, dx. \tag{1.82}$$

Taking the derivative of this equation with respect to $x_0$ gives

$$\ell\rho'(x_0) = -\rho(x_0). \tag{1.83}$$

The solution to this is (rewriting the arbitrary $x_0$ as $x$)

$$\rho(x) = Ae^{-x/\ell}. \tag{1.84}$$

We therefore see that the density decreases exponentially with $x$. The smaller $\ell$ is, the quicker it falls off. Note that the density at the pivot is $1/e$ times the density at the left end. And you can show that $1 - 1/e \approx 63\,\%$ of the mass is contained between the left end and the pivot.

19. **The spool**

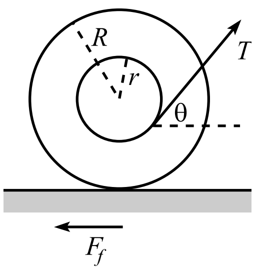


**Figure 1.56**

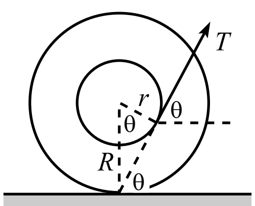


**Figure 1.57**

(a) Let $F_f$ be the friction force the ground provides. Balancing the horizontal forces on the spool gives (see Fig. 1.56)

$$T\cos\theta = F_f. \tag{1.85}$$

Balancing torques around the center of the spool gives

$$Tr = F_f R. \tag{1.86}$$

These two equations imply

$$\cos\theta = \frac{r}{R}\,. \tag{1.87}$$

The niceness of this result suggests that there is a quicker way to obtain it. And indeed, we see from Fig. 1.57 that $\cos\theta = r/R$ is the angle that causes the line of the tension to pass through the contact point on the ground. Since gravity and friction provide no torque around this point, the total torque around it is therefore zero, and the spool remains at rest.

(b) The normal force from the ground is

$$N = Mg - T\sin\theta. \tag{1.88}$$

Using eq. (1.85), the statement $F_f \le \mu N$ becomes $T\cos\theta \le \mu(Mg - T\sin\theta)$. Hence,

$$T \le \frac{\mu Mg}{\cos\theta + \mu\sin\theta}\,, \tag{1.89}$$

where $\theta$ is given in eq. (1.87).

(c) The maximum value of $T$ is given in (1.89). This depends on $\theta$, which in turn depends on $r$. We want to find the $r$ which minimizes this maximum $T$.

Taking the derivative with respect to $\theta$, we find that the $\theta$ that maximizes the denominator in eq. (1.89) is given by $\tan\theta_0 = \mu$. You can then show that the value of $T$ for this $\theta_0$ is

$$T_0 = \frac{\mu M g}{\sqrt{1+\mu^2}} = Mg\sin\theta_0. \tag{1.90}$$

To find the corresponding $r$, we can use eq. (1.87) to write $\tan\theta = \sqrt{R^2 - r^2}/r$. The relation $\tan\theta_0 = \mu$ then yields

$$r_0 = \frac{R}{\sqrt{1+\mu^2}}\,. \tag{1.91}$$

This is the $r$ that yields the smallest upper bound on $T$. In the limit $\mu = 0$, we have $\theta_0 = 0$, $T_0 = 0$, and $r_0 = R$. And in the limit $\mu = \infty$, we have $\theta_0 = \pi/2$, $T_0 = Mg$, and $r_0 = 0$.

# Chapter 2

# Using $F = ma$



The general goal of classical mechanics is to determine what happens to a given set of objects in a given physical situation. In order to figure this out, we need to know what makes the objects move the way they do. There are two main ways of going about this task. The first way, which you are undoubtedly familiar with, involves Newton's laws. This will be the subject of the present chapter. The second way, which is the more advanced one, is the *Lagrangian* method. This will be the subject of Chapter 5.

It should be noted that each of these methods is perfectly sufficient for solving any problem. They both produce the same information in the end, but they are based on vastly different principles. We'll talk more about this is Chapter 5.

## 2.1 Newton's Laws

Newton published his three laws in 1687 in his *Principia Mathematica*. The laws are fairly intuitive, although it seems a bit strange to attach the adjective "intuitive" to a set of statements that took millennia for humans to write down. The laws may be stated as follows.

- **First Law:** A body moves with constant velocity (which may be zero) unless acted on by a force.

- **Second Law:** The time rate of change of the momentum of a body equals the force acting on the body.

- **Third Law:** The forces two bodies apply to each other are equal in magnitude and opposite in direction.

We could discuss for days on end the degree to which these statements are physical laws, and the degree to which they are definitions. Sir Arthur Eddington once made the unflattering comment that the first law essentially says that "every particle continues in its state of rest or uniform motion in a straight line except insofar that it doesn't." Although Newton's laws might seem somewhat vacuous at first glance, there is actually a bit more content to them than Eddington's comment implies. Let's look at each in turn. The discussion will be brief, because we have to save time for other things in this book that we really *do* want to discuss for days on end.

### First Law

One thing this law does is give a definition of zero force.

Another thing it does is give a definition of an *inertial frame*, which is defined simply as a reference frame in which the first law holds. Since the term "velocity" is used, we have to state what frame of reference we are measuring the velocity with respect to. The first law does *not* hold in an arbitrary frame. For example, it fails in the frame of a spinning turntable.[1] Intuitively, an inertial frame is one that moves at constant speed. But this is ambiguous, because we have to say what the frame is moving at constant speed *with respect to.* At any rate, an inertial frame is simply defined as the special type of frame where the first law holds.

So, what we now have are two intertwined definitions of "force" and "inertial frame." Not much physical content here. But, however sparse in content the law is, it still holds for *all* particles. So if we have a frame in which one free particle moves with constant velocity, than *all* free particles move with constant velocity. This is a statement with content.

### Second Law

One thing this law does is give a definition of nonzero force. Momentum is defined[2] to be $m\mathbf{v}$. If $m$ is constant,[3] then the second law says that

$$\mathbf{F} = m\mathbf{a}, \tag{2.1}$$

where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law holds only in an inertial frame, which was defined by the first law.

> For things moving free or at rest,
> Observe what the first law does best.
> It defines a key frame,
> "Inertial" by name,
> Where the second law then is expressed.

So far, the second law merely gives a definition of $\mathbf{F}$. But the meaningful statement arises when we invoke the fact that the law holds for *all* particles. If the same force (for example, the same spring stretched by the same amount) acts on two

---

[1] It is, however, possible to modify things so that Newton's laws hold in such a frame, but we'll save this discussion for Chapter 9.

[2] We're doing everything nonrelativistically here, of course. Chapter 11 gives the relativistic modification of the $m\mathbf{v}$ expression.

[3] We'll assume in this chapter that $m$ is constant. But don't worry, we'll get plenty of practice with changing mass (in rockets and such) in Chapter 4.

particles, with masses $m_1$ and $m_2$, then eq. (2.1) says that their accelerations must be related by

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}\,. \tag{2.2}$$

This relation holds regardless of what the common force is. Therefore, once you've used one force to find the relative masses of two objects, then you know what the ratio of their $a$'s will be when they are subjected to any other force.

Of course, we haven't really defined *mass* yet. But eq. (2.2) gives an experimental method for determining an object's mass in terms of a standard (say, 1 kg) mass. All you have to do is compare its acceleration with that of the standard mass, when acted on by the same force.

There is also another piece of substance in this law, in that it says $\mathbf{F} = m\mathbf{a}$, instead of, say, $\mathbf{F} = m\mathbf{v}$, or $\mathbf{F} = m\,d^3\mathbf{x}/dt^3$. This issue is related to the first law. $\mathbf{F} = m\mathbf{v}$ is not viable, because the first law says that it is possible to have a velocity without a force. And $\mathbf{F} = md^3\mathbf{x}/dt^3$ would make the first law incorrect, because it would then be true that a particle moves with constant acceleration (instead of constant velocity) unless acted on by a force.

Note that $\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really three equations in one. In Cartesian coordinates, it says that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

### Third Law

This law essentially postulates that momentum is conserved (that is, not dependent on time). To see this, note that

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \frac{d(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)}{dt} \\ &= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 \\ &= \mathbf{F}_1 + \mathbf{F}_2, \end{aligned} \tag{2.3}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces acting on $m_1$ and $m_2$, respectively. This demonstrates that momentum conservation (that is, $d\mathbf{p}/dt = 0$) is equivalent to Newton's third law (that is, $\mathbf{F}_1 = -\mathbf{F}_2$.)

There isn't much left to be defined via this law, so the third law is one of pure content. It says that if you have two isolated particles interacting through some force, then their accelerations are opposite in direction and inversely proportional to their masses.

This third law cannot be a definition, because it's actually not always valid. It only holds for forces of the "pushing" and "pulling" type. It fails for the magnetic force, for example. In that case, momentum is carried off in the electromagnetic field (so the total momentum of the particles *and* the field is conserved). But we won't deal with fields here. Just particles. So the third law will always hold in any situation we're concerned with.

## 2.2 Free-body diagrams

The law that allows us to be quantitative is the second law. Given a force, we can apply $\mathbf{F} = m\mathbf{a}$ to find the acceleration. And knowing the acceleration, we can determine the behavior of a given object (that is, where it is and what its velocity is), provided that we are given the initial position and velocity. This process sometimes takes a bit of work, but there are two basic types of situations that commonly arise.

- In many problems, all you are given is a physical situation (for example, a block resting on a plane, strings connecting masses, etc.), and it is up to you to find all the forces acting on all the objects. These forces generally point in various directions, so it is easy to lose track of them. It therefore proves useful to isolate the objects and draw all the forces acting on each of them. This is the subject of the present section.
- In other problems, you are *given* the force explicitly as a function of time, position, or velocity, and the task immediately becomes the mathematical one of solving the $F = ma \equiv m\ddot{x}$ equation (we'll just deal with one dimension here). These *differential equations* can be difficult (or impossible) to solve exactly. They are the subject of Section 2.3.

Let's now consider the first of these two types of scenarios, where we are presented with a physical situation, and where we must determine all the forces involved. The term *free-body diagram* is used to denote a diagram with all the forces drawn on a given object. After drawing such a diagram for each object in the setup, we simply write down all the $F = ma$ equations they imply. The result will be a system of linear equations in various unknown forces and accelerations, for which we must then solve. This procedure is best understood through an example.

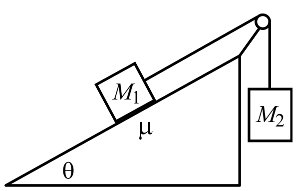

**Figure 2.1**

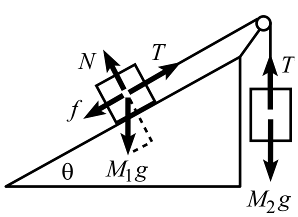

**Figure 2.2**

---

**Example (A plane and masses):** Mass $M_1$ is held on a plane with inclination angle $\theta$, and mass $M_2$ hangs over the side. The two masses are connected by a massless string which runs over a massless pulley (see Fig. 2.1). The coefficient of kinetic friction between $M_1$ and the plane is $\mu$. $M_1$ is released. Assuming that $M_2$ is sufficiently large so that $M_1$ gets pulled up the plane, what is the acceleration of the masses? What is the tension in the string?

**Solution:** The first thing to do is draw all the forces on the two masses. These are shown in Fig. 2.2. The forces on $M_2$ are gravity and the tension. The forces on $M_1$ are gravity, friction, the tension, and the normal force. Note that the friction force points down the plane, because we are assuming that $M_1$ moves up the plane.

Having drawn all the forces, we now simply have to write down all the $F = ma$ equations. When dealing with $M_1$, we could break things up into horizontal and vertical components, but it is much cleaner to use the components along and perpendicular to the plane.[4] These two components of $\mathbf{F} = m\mathbf{a}$, along with the vertical $F = ma$

[4]When dealing with inclined planes, one of these two coordinate systems will generally work much better than the other. Sometimes it's not clear which one, but if things get messy with one system, you can always try the other one.

equation for $M_2$, give

$$\begin{aligned} T - f - M_1 g \sin\theta &= M_1 a, \\ N - M_1 g \cos\theta &= 0, \\ M_2 g - T &= M_2 a, \end{aligned} \tag{2.4}$$

where we have used the fact that the two masses accelerate at the same rate (and we have defined the positive direction for $M_2$ to be downward). We have also used the fact that tension is the same at both ends of the string, because otherwise there would be a net force on some part of the string which would then have to undergo infinite acceleration, because it is massless.

There are four unknowns in eqs. (2.4) (namely $T$, $a$, $N$, and $f$), but only three equations. Fortunately, we have a fourth equation: $f = \mu N$. Using this in the second equation above gives $f = \mu M_1 g \cos\theta$. The first equation then becomes $T - \mu M_1 g \cos\theta - M_1 g \sin\theta = M_1 a$. Adding this to the third equation leave us with only $a$, so we find

$$a = \frac{g(M_2 - \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2}, \qquad \Longrightarrow \qquad T = \frac{M_1 M_2 g(1 + \mu\cos\theta + \sin\theta)}{M_1 + M_2}. \tag{2.5}$$

Note that in order for $M_1$ to move upward (that is, $a > 0$), we must have $M_2 > M_1(\mu\cos\theta + \sin\theta)$ . This is clear from looking at the forces along the plane.

REMARK: If we had instead assumed that $M_1$ was sufficiently large so that it slides down the plane, then the friction force would point up the plane, and we would have found, as you can check,

$$a = \frac{g(M_2 + \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 - \mu\cos\theta + \sin\theta). \tag{2.6}$$

In order for $M_1$ to move downward (that is, $a < 0$), we must have $M_2 < M_1(\sin\theta - \mu\cos\theta)$. Therefore, the range of $M_2$ for which the system doesn't move is $M_1(\sin\theta - \mu\cos\theta) < M_2 < M_1(\sin\theta + \mu\cos\theta)$. ♣

---

In problems like the one above, it is clear what things you should pick as the objects on which you're going to draw forces. But in other problems, where there are various different subsystems you can choose, you must be careful to include all the relevant forces on a given subsystem. Which subsystems you want to pick depends on what quantities you're trying to find. Consider the following example.

---

**Example (Platform and pulley):** A person stands on a platform-and-pulley system, as shown in Fig. 2.3. The masses of the platform, person, and pulley[5] are $M$, $m$, and $\mu$, respectively.[6] The rope is massless. Let the person pull up on the rope so that she has acceleration $a$ upwards.[7]

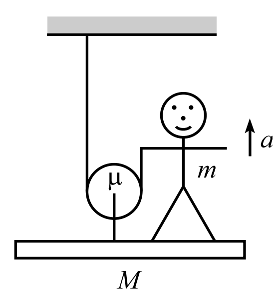


**Figure 2.3**

[5]Assume that the pulley's mass is concentrated at its center, so that we don't have to worry about any rotational dynamics (the subject of Chapter 7).

[6]My apologies for using $\mu$ as a mass here, since it usually denotes a coefficient of friction. Alas, there are only so many symbols for "$m$".

[7]Assume that the platform is somehow constrained to stay level, perhaps by having it run along some rails.

(a) What is the tension in the rope?

(b) What is the normal force between the person and the platform? What is the tension in the rod connecting the pulley to the platform?

**Solution:**

(a) To find the tension in the rope, we simply want to let our subsystem be the whole system (except the ceiling). If we imagine putting the system in a black box (to emphasize the fact that we don't care about any internal forces within the system), then the forces we see "protruding" from the box are the three weights ($Mg$, $mg$, and $\mu g$) downward, and the tension $T$ upward. Applying $F = ma$ to the whole system gives

$$T-(M+m+\mu)g = (M+m+\mu)a \qquad \Longrightarrow \qquad T = (M+m+\mu)(g+a). \quad (2.7)$$

(b) To find the normal force, $N$, between the person and the platform, and also the tension, $f$, in the rod connecting the pulley to the platform, it is not sufficient to consider the system as a whole. We must consider subsystems.

- Let's apply $F = ma$ to the person. The forces acting on the person are gravity, the normal force from the platform, and the tension from the rope (pulling downward on her hand). Therefore, we have

$$N - T - mg = ma. \quad (2.8)$$

- Now apply $F = ma$ to the platform. The forces acting on the platform are gravity, the normal force from the person, and the force upward from the rod. Therefore, we have

$$f - N - Mg = Ma. \quad (2.9)$$

- Now apply $F = ma$ to the pulley. The forces acting on the pulley are gravity, the force downward from the rod, and *twice* the tension in the rope (because it pulls up on both sides). Therefore, we have

$$2T - f - \mu g = \mu a. \quad (2.10)$$

Note that if we add up the three previous equations, we obtain the $F = ma$ equation in eq. (2.7), as should be the case, because the whole system is the sum of the three above subsystems. Eqs. (2.8) – (2.10) are three equations in the three unknowns ($T$, $N$, and $f$). Their sum yields the $T$ in (2.7), and then eqs. (2.8) and (2.10) give, respectively (as you can show),

$$N = (M + 2m + \mu)(g + a), \qquad \text{and} \qquad f = (2M + 2m + \mu)(g + a). \quad (2.11)$$

REMARK: You can also obtain these results by considering subsystems different from the ones we chose above. For example, you can choose the pulley-plus-platform subsystem, etc. But no matter how you choose to break up the system, you will need to produce three independent $F = ma$ statements in order to solve for the three unknowns, $T$, $N$, and $f$.

In problems like this one, it is easy to forget to include one of the forces, such as the second $T$ in eq. (2.10). The safest thing to do is to isolate each subsystem, draw a box around it, and then draw all the forces that "protrude" from the box. Fig. 2.4 shows the free-body diagram for the subsystem of the pulley. ♣

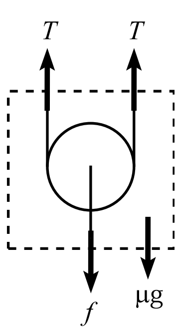


**Figure 2.4**

Another class of problems, similar to the previous example, goes by the name of *Atwood's machines*. An Atwood's machine is simply the name for any system that consists of a combination of masses, strings, and pulleys. In general, the pulleys and strings can have mass, but we'll just deal with massless ones in this chapter.

We'll do one example here, but additional (and stranger) setups are given in the exercises and problems for this chapter. As we'll see below, there are two basic steps in solving an Atwood's problem: (1) Write down all the $F = ma$ equations, and (2) Relate the accelerations of the various masses by noting that the length of the string doesn't change (a fact that we'll call "conservation of string").

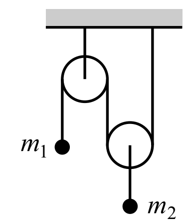

**Figure 2.5**

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**Example (An Atwood's machine):** Consider the pulley system in Fig. 2.5, with masses $m_1$ and $m_2$. The strings and pulleys are massless. What are the accelerations of the masses? What is the tension in the string?

**Solution:** The first thing to note is that the tension, $T$, is the same everywhere throughout the massless string, because otherwise there would be infinite acceleration. It then follows that the tension in the short string connected to $m_2$ is $2T$. This is true because there must be zero net force on the massless right pulley, because otherwise it would have infinite acceleration. The $F = ma$ equations on the two masses are therefore

$$\begin{aligned} T - m_1 g &= m_1 a_1, \\ 2T - m_2 g &= m_2 a_2. \end{aligned} \tag{2.12}$$

We now have two equations in the three unknowns, $a_1$, $a_2$, and $T$. So we need one more equation. This is the "conservation of string" fact, which relates $a_1$ and $a_2$. If we imagine moving $m_2$ and the right pulley up a distance $d$, then a length $2d$ of string has disappeared from the two parts of the string touching the right pulley. This string has to go somewhere, so it ends up in the part of the string touching $m_1$. Therefore, $m_1$ goes down by a distance $2d$. In other words, $y_1 = -2y_2$ (where $y_1$ and $y_2$ are measured relative to the initial locations of the masses). Taking two time derivatives of this statement gives our desired relation between $a_1$ and $a_2$,

$$a_1 = -2a_2. \tag{2.13}$$

Combining this with eqs. (2.12), we can now solve for $a_1$, $a_2$, and $T$. The result is

$$a_1 = g\frac{2m_2 - 4m_1}{4m_1 + m_2}, \qquad a_2 = g\frac{2m_1 - m_2}{4m_1 + m_2}, \qquad T = \frac{3m_1 m_2 g}{4m_1 + m_2}. \tag{2.14}$$

Remark: There are all sorts of limits and special cases that we can check here. A few are: (1) If $m_2 = 2m_1$, then eq. (2.14) gives $a_1 = a_2 = 0$, and $T = m_1 g$. Everything is at rest. (2) If $m_2 \gg m_1$, then eq. (2.14) gives $a_1 = 2g$, $a_2 = -g$, and $T = 3m_1 g$. In this case, $m_2$ is essentially in free fall, while $m_1$ gets yanked up with acceleration $2g$. The value of $T$ is exactly what is needed to make the net force on $m_1$ equal to $m_1(2g)$, because $T - m_1 g = 3m_1 g - m_1 g = m_1(2g)$. We'll let you check the case where $m_1 \gg m_2$. ♣

---

In the problems for this chapter, you'll encounter some strange Atwood's setups. But no matter how complicated they get, there are only two things you need to do to solve them, as mentioned above: (1) Write down the $F = ma$ equations for all the masses (which may involve relating the tensions in various strings), and (2) relate the accelerations of the masses, using "conservation of string".

> It may seem, with the angst it can bring,
> That an Atwood's machine's a harsh thing.
> But you just need to say
> That $F$ is $ma$,
> And use conservation of string!

## 2.3 Solving differential equations

Let's now consider the type of problem where we are *given* the force as a function of time, position, or velocity, and where our task is to solve the $F = ma \equiv m\ddot{x}$ differential equation to find the position, $x(t)$, as a function of time. In what follows, we will develop a few techniques for solving differential equations. The ability to apply these techniques dramatically increases the number of problems we can solve.

In general, the force $F$ can also be a function of higher derivatives of $x$, in addition to the quantities $t$, $x$, and $v \equiv \dot{x}$. But these cases don't arise much, so we won't worry about them. The $F = ma$ differential equation we want to solve is therefore (we'll just work in one dimension here)

$$m\ddot{x} = F(x, v, t). \tag{2.15}$$

In general, this equation cannot be solved exactly for $x(t)$.[8] But for most of the problems we will deal with, it can be solved. The problems we will encounter will often fall into one of three special cases, namely, where $F$ is a function of $t$ only, or $x$ only, or $v$ only. In all of these cases, we must invoke the given initial conditions, $x_0 \equiv x(t_0)$ and $v_0 \equiv v(t_0)$, to obtain our final solutions. These initial conditions will appear in the limits of the integrals in the following discussion.[9]

Note: You may want to just skim the following page and a half, and then refer back to it as needed. Don't try to memorize all the different steps. We present them only for completeness. The whole point here can basically be summarized by saying that sometimes you want to write $\ddot{x}$ as $dv/dt$, and sometimes you want to write it as $v\,dv/dx$ (see eq. (2.19)). Then you "simply" have to separate variables and integrate. We'll go through the three special cases, and then we'll do some examples.

[8] It can always be solved for $x(t)$ *numerically*, to any desired accuracy. This is discussed in Appendix D.

[9] It is no coincidence that we need *two* initial conditions to completely specify the solution to our *second*-order $F = m\ddot{x}$ differential equation. It is a general result (which we'll just accept here) that the solution to an $n$th-order differential equation has $n$ free parameters, which must then be determined from the initial conditions.

- *$F$ is a function of $t$ only: $F = F(t)$.*

  Since $a = d^2x/dt^2$, we just need to integrate $F = ma$ twice to obtain $x(t)$. Let's do this in a very systematic way, to get used to the general procedure. First, write $F = ma$ as

$$m\frac{dv}{dt} = F(t). \tag{2.16}$$

  Then separate variables and integrate both sides to obtain[10]

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')\,dt'. \tag{2.17}$$

  We have put primes on the integration variables so that we don't confuse them with the limits of integration. Eq. (2.17) yields $v$ as a function of $t$, $v(t)$. We then separate variables in $dx/dt = v(t)$ and integrate to obtain

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\,dt'. \tag{2.18}$$

  This yields $x$ as a function of $t$, $x(t)$. This procedure might seem like a cumbersome way to simply integrate something twice. That's because it is. But the technique proves more useful in the following case.

- *$F$ is a function of $x$ only: $F = F(x)$.*

  We will use

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{2.19}$$

  to write $F = ma$ as

$$mv\frac{dv}{dx} = F(x). \tag{2.20}$$

  Now separate variables and integrate both sides to obtain

$$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')\,dx'. \tag{2.21}$$

  The left side will contain the square of $v(x)$. Taking a square root, this gives $v$ as a function of $x$, $v(x)$. Separate variables in $dx/dt = v(x)$ to obtain

$$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'. \tag{2.22}$$

  This gives $t$ as a function of $x$, and hence (in principle) $x$ as a function of $t$, $x(t)$. The unfortunate thing about this case is that the integral in eq. (2.22) might not be doable. And even if it is, it might not be possible to invert $t(x)$ to produce $x(t)$.

[10] If you haven't seen such a thing before, the act of multiplying both sides by the infinitesimal quantity $dt'$ might make you feel a bit uneasy. But it is in fact quite legal. If you wish, you can imagine working with the small (but not infinitesimal) quantities $\Delta v$ and $\Delta t$, for which it is certainly legal to multiply both sides by $\Delta t$. Then you can take a discrete sum over many $\Delta t$ intervals, and then finally take the limit $\Delta t \to 0$, which results in eq. (2.17)

- *$F$ is a function of $v$ only: $F = F(v)$.*

  Write $F = ma$ as
  $$m\frac{dv}{dt} = F(v). \tag{2.23}$$
  Separate variables and integrate both sides to obtain
  $$m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt'. \tag{2.24}$$
  This yields $t$ as a function of $v$, and hence (in principle) $v$ as a function of $t$, $v(t)$. Integrate $dx/dt = v(t)$ to obtain $x(t)$ from
  $$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\, dt'. \tag{2.25}$$
  *Note:* In this $F = F(v)$ case, if you want to find $v$ as a function of $x$, $v(x)$, then you should write $a$ as $v(dv/dx)$ and integrate
  $$m\int_{v_0}^{v(x)} \frac{v'\, dv'}{F(v')} = \int_{x_0}^{x} dx'. \tag{2.26}$$
  You can then obtain $x(t)$ from eq. (2.22), if desired.

When dealing with the initial conditions, we have chosen to put them in the limits of integration above. If you wish, you can perform the integrals without any limits, and just tack on a constant of integration to your result. The constant is then determined from the initial conditions.

Again, as mentioned above, you do *not* have to memorize the above three procedures, because there are variations, depending on what you're given and what you want to solve for. All you have to remember is that $\ddot{x}$ can be written as either $dv/dt$ or $v\, dv/dx$. One of these will get the job done (namely, the one that makes only two out of the three variables, $t$, $x$, and $v$, appear in your differential equation). And then be prepared to separate variables and integrate as many times as needed.

$a$ is $dv$ by $dt$.
Is this useful? There's no guarantee.
If it leads to "Oh, heck!"'s,
Take $dv$ by $dx$,
And then write down its product with $v$.

---

**Example 1 (Gravitational force):** A particle of mass $m$ is subject to a constant force $F = -mg$. The particle starts at rest at height $h$. Because this constant force falls into all of the above three categories, we should be able to solve for $y(t)$ in two ways:

(a) Find $y(t)$ by writing $a$ as $dv/dt$.

(b) Find $y(t)$ by writing $a$ as $v\, dv/dy$.

**Solution:**

(a) $F = ma$ gives $dv/dt = -g$. Integrating this yields $v = -gt + C$, where $C$ is a constant of integration.[11] The initial condition $v(0) = 0$ gives $C = 0$. Therefore, $dy/dt = -gt$. Integrating this and using $y(0) = h$ gives

$$y = h - \frac{1}{2}gt^2. \tag{2.27}$$

(b) $F = ma$ gives $v\,dv/dy = -g$. Separating variables and integrating yields $v^2/2 = -gy + C$. The initial condition $v(0) = 0$ gives $v^2/2 = -gy + gh$. Therefore, $v \equiv dy/dt = -\sqrt{2g(h-y)}$. We have chosen the negative square root, because the particle is falling. Separating variables gives

$$\int \frac{dy}{\sqrt{h-y}} = -\sqrt{2g}\int dt. \tag{2.28}$$

This yields $2\sqrt{h-y} = \sqrt{2g}\,t$, where we have used the initial condition $y(0) = h$. Hence, $y = h - gt^2/2$, in agreement with part (a). The solution in part (a) was clearly the simpler one.

---

**Example 2 (Dropped ball):** A beach-ball is dropped from rest at height $h$. Assume[12] that the drag force from the air takes the form, $F_d = -\beta v$. Find the velocity and height as a function of time.

**Solution:** For simplicity in future formulas, let's write the drag force as $F_d = -\beta v \equiv -m\alpha v$ (so we won't have a bunch of $1/m$'s floating around). Taking upward to be the positive $y$ direction, the force on the ball is

$$F = -mg - m\alpha v. \tag{2.29}$$

Note that $v$ is negative here, because the ball is falling, so the drag force points upward, as it should. Writing $F = m\,dv/dt$, and separating variables, gives

$$\int_0^{v(t)} \frac{dv'}{g + \alpha v'} = -\int_0^t dt'. \tag{2.30}$$

Integration yields $\ln(1 + \alpha v/g) = -\alpha t$. Exponentiation then gives

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{2.31}$$

Writing $dy/dt \equiv v(t)$, and then separating variables and integrating to obtain $y(t)$, yields

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'. \tag{2.32}$$

[11] We'll do this example by adding on constants of integration which are then determined from the initial conditions. We'll do the following example by putting the initial conditions in the limits of integration.

[12] The drag force is roughly proportional to $v$ as long as the speed is fairly slow. For large speeds, the drag force is roughly proportional to $v^2$.

Therefore,

$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{2.33}$$

REMARKS:

(a) Let's look at some limiting cases. If $t$ is very small (more precisely, if $\alpha t \ll 1$), then we can use $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in $t$. You can show that eq. (2.31) gives $v(t) \approx -gt$. This makes sense, because the drag force is negligible at the start, so the ball is essentially in free fall. And eq. (2.33) gives $y(t) \approx h - gt^2/2$, as expected.

We can also look at large $t$. In this case, $e^{-\alpha t}$ is essentially equal to zero, so eq. (2.31) gives $v(t) \approx -g/\alpha$. (This is the "terminal velocity." Its value makes sense, because it is the velocity for which the total force, $-mg - m\alpha v$, vanishes.) And eq. (2.33) gives $y(t) \approx h - (g/\alpha)t + g/\alpha^2$. Interestingly, we see that for large $t$, $g/\alpha^2$ is the distance our ball lags behind another ball which started out already at the terminal velocity, $g/\alpha$.

(b) The velocity of the ball obtained in eq. (2.31) depends on $\alpha$, which was defined via $F_d = -m\alpha v$. We explicitly wrote the $m$ here just to make all of our formulas look a little nicer, but it should *not* be inferred that the velocity of the ball is independent of $m$. The coefficient $\beta \equiv m\alpha$ depends (in some complicated way) on the cross-sectional area, $A$, of the ball. Therefore, $\alpha \propto A/m$. Two balls of the same size, one made of lead and one made of styrofoam, will have the same $A$ but different $m$'s. Hence, their $\alpha$'s will be different, and they will fall at different rates.

For heavy objects in a thin medium such as air, $\alpha$ is small, so the drag effects are not very noticeable over short distances. Heavy objects fall at roughly the same rate. If the air were a bit thicker, different objects would fall at noticeably different rates, and maybe it would have taken Galileo a bit longer to come to his conclusions.

What would you have thought, Galileo,
If instead you dropped cows and did say, "Oh!
To lessen the sound
Of the moos from the ground,
They should fall not through air, but through mayo!" ♣

## 2.4 Projectile motion

Consider a ball thrown through the air, not necessarily vertically. We will neglect air resistance in the following discussion.

Let $x$ and $y$ be the horizontal and vertical positions, respectively. The force in the $x$-direction is $F_x = 0$, and the force in the $y$-direction is $F_y = -mg$. So $\mathbf{F} = m\mathbf{a}$ gives

$$\ddot{x} = 0, \qquad \text{and} \qquad \ddot{y} = -g. \tag{2.34}$$

Note that these two equations are "decoupled." That is, there is no mention of $y$ in the equation for $\ddot{x}$, and vice-versa. The motions in the $x$- and $y$-directions are therefore completely independent.

REMARK: The classic demonstration of the independence of the $x$- and $y$-motions is the following. Fire a bullet horizontally (or, preferably, just imagine firing a bullet horizontally), and at the same time drop a bullet from the height of the gun. Which bullet will hit the ground first? (Neglect air resistance, the curvature of the earth, etc.) The answer is that they will hit the ground at the same time, because the effect of gravity on the two $y$-motions is exactly the same, independent of what is going on in the $x$-direction. ♣

If the initial position and velocity are $(X, Y)$ and $(V_x, V_y)$, then we can easily integrate eqs. (2.34) to obtain

$$\begin{aligned} \dot{x}(t) &= V_x, \\ \dot{y}(t) &= V_y - gt. \end{aligned} \tag{2.35}$$

Integrating again gives

$$\begin{aligned} x(t) &= X + V_x t, \\ y(t) &= Y + V_y t - \frac{1}{2}gt^2. \end{aligned} \tag{2.36}$$

These equations for the speeds and positions are all you need to solve a projectile problem.

---

**Example (Throwing a ball):**

(a) For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum horizontal distance by the time it returns to the ground? Assume that the ground is horizontal, and that the ball is released from ground level.

(b) What is the optimal angle if the ground is sloped upward at an angle $\beta$ (or downward, if $\beta$ is negative)?

**Solution:**

(a) Let the inclination angle be $\theta$, and let the initial speed be $V$. Then the horizontal speed is always $V_x = V\cos\theta$, and the initial vertical speed is $V_y = V\sin\theta$.

The first thing we need to do is find the time $t$ in the air. We know that the vertical speed is zero at time $t/2$, because the ball is moving horizontally at its highest point. So the second of eqs. (2.35) gives $V_y = g(t/2)$. Therefore, $t = 2V_y/g$.[13]

The first of eqs. (2.36) tells us that the horizontal distance traveled is $d = V_x t$. Using $t = 2V_y/g$ in this gives

$$d = \frac{2V_xV_y}{g} = \frac{V^2(2\sin\theta\cos\theta)}{g} = \frac{V^2\sin 2\theta}{g}\,. \tag{2.37}$$

The $\sin 2\theta$ factor has a maximum at

$$\theta = \frac{\pi}{4}\,. \tag{2.38}$$

[13] Alternatively, the time of flight can be found from the second of eqs. (2.36), which says that the ball returns to the ground when $V_y t = gt^2/2$. We will have to use this type of strategy in part (b), where the trajectory is not symmetric around the maximum.

The maximum horizontal distance traveled is then $d_{\max} = V^2/g$.

REMARKS: For $\theta = \pi/4$, you can show that the maximum height achieved is $V^2/4g$. This may be compared to the maximum height of $V^2/2g$ (as you can show) if the ball is thrown straight up. Note that any possible distance you might want to find in this problem must be proportional to $V^2/g$, by dimensional analysis. The only question is what the numerical factor is. ♣

(b) As in part (a), the first thing we need to do is find the time $t$ in the air. If the ground is sloped at an angle $\beta$, then the equation for the line of the ground is

$$y = (\tan\beta)x. \tag{2.39}$$

The path of the ball is given in terms of $t$ by

$$x = (V\cos\theta)t, \qquad \text{and} \qquad y = (V\sin\theta)t - \frac{1}{2}gt^2. \tag{2.40}$$

We must solve for the $t$ that makes $y = (\tan\beta)x$, because this gives the place where the path of the ball intersects the line of the ground. Using eqs. (2.40), we find that $y = (\tan\beta)x$ when

$$t = \frac{2V}{g}(\sin\theta - \tan\beta\cos\theta). \tag{2.41}$$

(There is, of course, also the solution $t = 0$.) Plugging this into the expression for $x$ in eq. (2.40) gives

$$x = \frac{2V^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta). \tag{2.42}$$

We must now maximize this value for $x$, which is equivalent to maximizing the distance along the slope. Setting the derivative with respect to $\theta$ equal to zero, and using the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we find $\tan\beta = -\cot 2\theta$. This can be rewritten as $\tan\beta = -\tan(\pi/2 - 2\theta)$. Therefore, $\beta = -(\pi/2 - 2\theta)$, so we have

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right). \tag{2.43}$$

In other words, the throwing angle should bisect the angle between the ground and the vertical.

REMARKS: For $\beta \approx \pi/2$, we have $\theta \approx \pi/2$, as should be the case. For $\beta = 0$, we have $\theta = \pi/4$, as we found in part (a). And for $\beta \approx -\pi/2$, we have $\theta \approx 0$, which makes sense.

Substituting the value of $\theta$ from eq. (2.43) into eq. (2.42), you can show (after a bit of algebra) that the maximum distance traveled along the tilted ground is

$$d = \frac{x}{\cos\beta} = \frac{V^2/g}{1 + \sin\beta}. \tag{2.44}$$

This checks in the various limits for $\beta$. ♣

Along with the bullet example mentioned above, another classic example of the independence of the $x$- and $y$-motions is the "hunter and monkey" problem. In it, a hunter aims an arrow (made of styrofoam, of course) at a monkey hanging from a branch in a tree. The monkey, thinking he's being clever, tries to avoid the arrow by letting go of the branch right when he sees the arrow released. The unfortunate consequence of this action is that he *will* get hit, because gravity acts on both him and the arrow in the same way; they both fall the same distance relative to where they would have been if there were no gravity. And the monkey *would* get hit in such a case, because the arrow is initially aimed at him. You can work this out in Exercise 16, in a more peaceful setting involving fruit.

If a monkey lets go of a tree,
The arrow will hit him, you see,
Because both heights are pared
By a half $gt^2$
From what they would be with no $g$.

## 2.5 Motion in a plane, polar coordinates

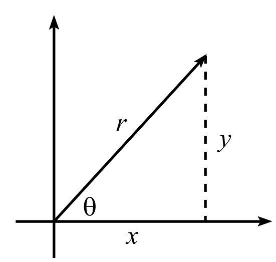

**Figure 2.6**

When dealing with problems where the motion lies in a plane, it is often convenient to work with polar coordinates, $r$ and $\theta$. These are related to the Cartesian coordinates by (see Fig. 2.6)

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta. \tag{2.45}$$

Depending on the problem, either Cartesian or polar coordinates will be easier to use. It is usually clear from the setup which is better. For example, if the problem involves circular motion, then polar coordinates are a good bet. But to use polar coordinates, we need to know what form Newton's second law takes in terms of them. Therefore, the goal of the present section is to determine what $\mathbf{F} = m\mathbf{a} \equiv m\ddot{\mathbf{r}}$ looks like when written in terms of polar coordinates.

At a given position $\mathbf{r}$ in the plane, the basis vectors in polar coordinates are $\hat{\mathbf{r}}$, which is a unit vector pointing in the radial direction; and $\hat{\boldsymbol{\theta}}$, which is a unit vector pointing in the counterclockwise tangential direction. In polar coords, a general vector may therefore be written as

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{2.46}$$

Note that the directions of the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors depend, of course, on $\mathbf{r}$.

Since the goal of this section is to find $\ddot{\mathbf{r}}$, we must, in view of eq. (2.46), get a handle on the time derivative of $\hat{\mathbf{r}}$. And we'll eventually need the derivative of $\hat{\boldsymbol{\theta}}$, too. In contrast with the fixed Cartesian basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), the polar basis vectors ($\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$) change as a point moves around in the plane.

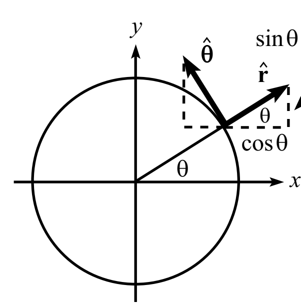

**Figure 2.7**

We can find $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ in the following way. In terms of the Cartesian basis, Fig. 2.7 shows that

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}},\\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}.\end{aligned} \tag{2.47}$$

Taking the time derivative of these equations gives

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= -\sin\theta\,\dot\theta\hat{\mathbf{x}} + \cos\theta\,\dot\theta\hat{\mathbf{y}},\\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\,\dot\theta\hat{\mathbf{x}} - \sin\theta\,\dot\theta\hat{\mathbf{y}}.\end{aligned} \tag{2.48}$$

Using eqs. (2.47), we arrive at the nice clean expressions,

$$\dot{\hat{\mathbf{r}}} = \dot\theta\hat{\boldsymbol{\theta}}, \qquad \text{and} \qquad \dot{\hat{\boldsymbol{\theta}}} = -\dot\theta\hat{\mathbf{r}}. \tag{2.49}$$

These relations are fairly evident if we look at what happens to the basis vectors as $\mathbf{r}$ moves a tiny distance in the tangential direction. Note that the basis vectors do not change as $\mathbf{r}$ moves in the radial direction.

We can now start differentiating eq. (2.46). One derivative gives (yes, the product rule works fine here)

$$\begin{aligned}\dot{\mathbf{r}} &= \dot r\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\\ &= \dot r\hat{\mathbf{r}} + r\dot\theta\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.50}$$

This makes sense, because $\dot r$ is the speed in the radial direction, and $r\dot\theta$ is the speed in the tangential direction, which is often written as $\omega r$ (where $\omega \equiv \dot\theta$ is the angular speed, or "angular frequency").[14]

Differentiating eq. (2.50) then gives

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot r\hat{\mathbf{r}} + \dot r\dot{\hat{\mathbf{r}}} + \dot r\dot\theta\hat{\boldsymbol{\theta}} + r\ddot\theta\hat{\boldsymbol{\theta}} + r\dot\theta\dot{\hat{\boldsymbol{\theta}}}\\ &= \ddot r\hat{\mathbf{r}} + \dot r(\dot\theta\hat{\boldsymbol{\theta}}) + \dot r\dot\theta\hat{\boldsymbol{\theta}} + r\ddot\theta\hat{\boldsymbol{\theta}} + r\dot\theta(-\dot\theta\hat{\mathbf{r}})\\ &= (\ddot r - r\dot\theta^2)\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.51}$$

Finally, equating $m\ddot{\mathbf{r}}$ with $\mathbf{F} \equiv F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ gives the radial and tangential forces as

$$\begin{aligned}F_r &= m(\ddot r - r\dot\theta^2),\\ F_\theta &= m(r\ddot\theta + 2\dot r\dot\theta).\end{aligned} \tag{2.52}$$

(See Exercise 32 for a slightly different derivation of these equations.) Let's look at each of the four terms on the right-hand sides of eqs. (2.52).

- The $m\ddot r$ term is quite intuitive. For radial motion, it simply states that $F = ma$ along the radial direction.
- The $mr\ddot\theta$ term is also quite intuitive. For circular motion, it states that $F = ma$ along the tangential direction, because $r\ddot\theta$ is the second derivative of the distance $r\theta$ along the circumference.

[14]For $r\dot\theta$ to be the tangential speed, we must measure $\theta$ in radians and not degrees. Then $r\theta$ is by definition the distance along the circumference, so $r\dot\theta$ is the speed along the circumference.

- The $-mr\dot{\theta}^2$ term is also fairly clear. For circular motion, it says that the radial force is $-m(r\dot{\theta})^2/r = -mv^2/r$, which is the familiar force that causes the centripetal acceleration, $v^2/r$. See Problem 19 for an alternate (and quicker) derivation of this $v^2/r$ result.

- The $2m\dot{r}\dot{\theta}$ term isn't so obvious. It is called the *Coriolis* force. There are various ways to look at this term. One is that it exists in order to keep angular momentum conserved. We'll have a great deal to say about the Coriolis force in Chapter 9.

---

**Example (Circular pendulum):** A mass hangs from a massless string of length $\ell$. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle $\beta$ with the vertical (see Fig. 2.8). What is the angular frequency, $\omega$, of this motion?

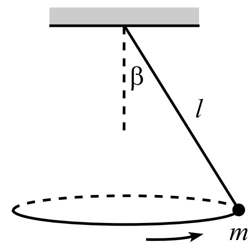


**Figure 2.8**

**Solution:** The mass travels in a circle, so the horizontal radial force must be $F_r = mr\dot{\theta}^2 \equiv mr\omega^2$ (with $r = \ell\sin\beta$), directed radially inward. The forces on the mass are the tension in the string, $T$, and gravity, $mg$ (see Fig. 2.9). There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions gives, respectively,

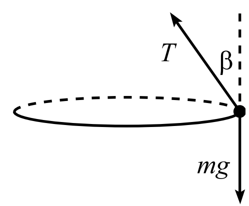


**Figure 2.9**

$$\begin{aligned} T\cos\beta &= mg, \\ T\sin\beta &= m(\ell\sin\beta)\omega^2. \end{aligned} \tag{2.53}$$

Solving for $\omega$ gives

$$\omega = \sqrt{\frac{g}{\ell\cos\beta}}\,. \tag{2.54}$$

Note that if $\beta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which equals the frequency of a plane pendulum of length $\ell$. And if $\beta \approx 90^\circ$, then $\omega \to \infty$, which makes sense.

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## 2.6 Exercises

*Section 2.2: Free-body diagrams*

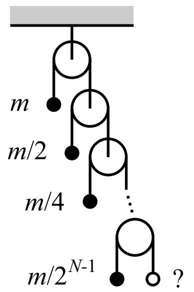


**Figure 2.10**

1. **A peculiar Atwood's machine**

   The Atwood's machine in Fig. 2.10 consists of $N$ masses, $m$, $m/2$, $m/4$, ..., $m/2^{N-1}$. All the pulleys and strings are massless, as usual.

   (a) Put a mass $m/2^{N-1}$ at the free end of the bottom string. What are the accelerations of all the masses?

   (b) Remove the mass $m/2^{N-1}$ (which was arbitrarily small, for very large $N$) that was attached in part (a). What are the accelerations of all the masses, now that you've removed this infinitesimal piece?

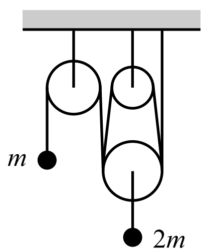


**Figure 2.11**

2. **Double-loop Atwood's** $*$

   Consider the Atwood's machine shown in Fig. 2.11. It consists of three pulleys, a short piece of string connecting one mass to the bottom pulley, and a continuous long piece of string that wraps twice around the bottom side of the bottom pulley, and once around the top side of the top two pulleys. The two masses are $m$ and $2m$. Assume that the parts of the string connecting the pulleys are essentially vertical. Find the accelerations of the masses.

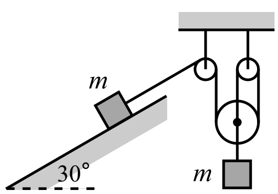


**Figure 2.12**

3. **Atwood's and a plane** $*$

   Consider the Atwoods machine shown in Fig. 2.12, with two masses $m$. The plane is frictionless, and it is inclined at a 30° angle. Find the accelerations of the masses.

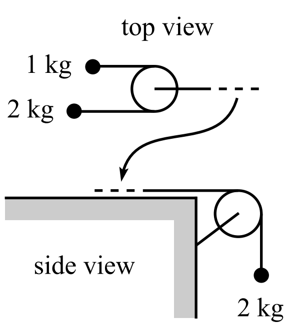


**Figure 2.13**

4. **Atwood's on a table** $*$

   Consider the Atwood's machine shown in Fig. 2.13, Masses of 1 kg and 2 kg lie on a frictionless table, connected by a string which passes around a pulley. The pulley is connected to another mass of 2 kg, which hangs down over another pulley, as shown. Find the accelerations of all three masses.

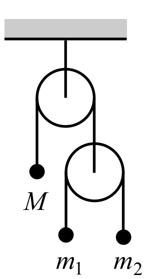


**Figure 2.14**

5. **Keeping the mass still** $*$

   In the Atwood's machine in Fig. 2.14, what should $M$ be (in terms of $m_1$ and $m_2$) so that it doesn't move?

6. **Three-mass Atwood's** $**$

Consider the Atwood's machine in Fig. 2.15, with masses $m$, $2m$, and $3m$. Find the accelerations of all three masses.

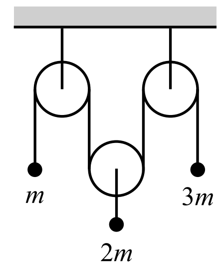



**Figure 2.15**

7. **Accelerating plane** $**$

A block of mass $m$ rests on a plane inclined at angle $\theta$. The coefficient of static friction between the block and the plane is $\mu$. The plane is accelerated to the right with acceleration $a$ (which may be negative); see Fig. 2.16. For what range of $a$ does the block remain at rest with respect to the plane?

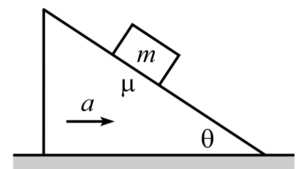



**Figure 2.16**

8. **Accelerating cylinders** $**$

Three identical cylinders are arranged in a triangle as shown in Fig. 2.17, with the bottom two lying on the ground. The ground and the cylinders are frictionless. You apply a constant horizontal force (directed to the right) on the left cylinder. Let $a$ be the acceleration you give to the system. For what range of $a$ will all three cylinders remain in contact with each other?

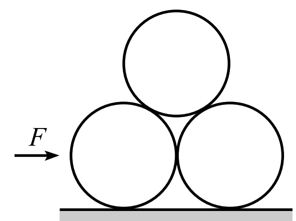



**Figure 2.17**

*Section 2.3: Solving differential equations*

9. $-bv^2$ **force** $*$

A particle of mass $m$ is subject to a force $F(v) = -bv^2$. The initial position is zero, and the initial speed is $v_0$. Find $x(t)$.

10. $-kx$ **force** $**$

A particle of mass $m$ is subject to a force $F(x) = -kx$. The initial position is zero, and the initial speed is $v_0$. Find $x(t)$.

11. $kx$ **force** $**$

A particle of mass $m$ is subject to a force $F(x) = kx$. The initial position is zero, and the initial speed is $v_0$. Find $x(t)$.

12. **Motorcycle circle** $***$

A motorcyclist wishes to travel in a circle of radius $R$ on level ground. The coefficient of friction between the tires and the ground is $\mu$. The motorcycle starts at rest. What is the minimum distance the motorcycle must travel in order to achieve its maximum allowable speed (that is, the speed above which it will skid out of the circular path)?

*Section 2.4: Projectile motion*

13. **Dropped balls**

A ball is dropped from height $4h$. After it has fallen a distance $d$, a second ball is dropped from height $h$. What should $d$ be (in terms of $h$) so that the balls hit the ground at the same time?

14. **Equal distances**

    At what angle should a ball be thrown so that its maximum height equals the horizontal distance traveled?

15. **Redirected horizontal motion** $*$

    A ball is dropped from rest at height $h$, and it bounces off a surface at height $y$, with no loss in speed. The surface is inclined at 45°, so that the ball bounces off horizontally. What should $y$ be so that the ball travels the maximum horizontal distance?

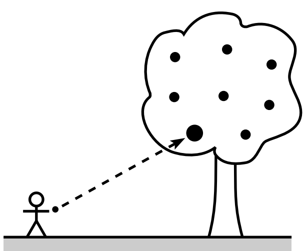


**Figure 2.18**

16. **Newton's apple** $*$

    Newton is tired of apples falling on his head, so he decides to throw a rock at one of the larger and more formidable looking apples positioned directly above his favorite sitting spot. Forgetting all about his work on gravitation, he aims the rock directly at the apple (see Fig. 2.18). To his surprise, the apple falls from the tree just as he releases the rock. Show, by calculating the rock's height when it reaches the horizontal position of the apple, that the rock will hit the apple.[15]

17. **Throwing at a wall** $*$

    You throw a ball with speed $V_0$ at a vertical wall, a distance $\ell$ away. At what angle should you throw the ball, so that it hits the wall at a maximum height? Assume $\ell < V_0^2/g$ (why?).

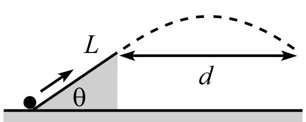



**Figure 2.19**

18. **Firing a cannon** $**$

    A cannon, when aimed vertically, is observed to fire a ball to a maximum height of $L$. Another ball is then fired with this same speed, but with the cannon now aimed up along a plane of length $L$, inclined at an angle $\theta$, as shown in Fig. 2.19. What should $\theta$ be, so that the ball travels the largest horizontal distance, $d$, by the time it returns to the height of the top of the plane?

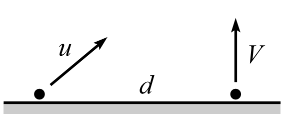



**Figure 2.20**

19. **Colliding projectiles** $*$

    Two balls are fired from ground level, a distance $d$ apart. The right one is fired vertically with speed $V$; see Fig. 2.20. You wish to simultaneously fire the left one at the appropriate velocity $\vec{u}$ so that it collides with the right ball when they reach their highest point. What should $\vec{u}$ be (give the horizontal and vertical components)? Given $d$, what should $V$ be so that the speed $u$ is minimum?

[15]This problem suggests a way in which William Tell and his son might survive their ordeal if they were plopped down on a planet with an unknown gravitational constant (provided that the son weren't too short or $g$ weren't too big).

20. **Throwing in the wind** ∗

A ball is thrown horizontally to the right, from the top of a vertical cliff of height $h$. A wind blows horizontally to the left, and assume (simplistically) that the effect of the wind is to provide a constant force to the left, equal in magnitude to the weight of the ball. How fast should the ball be thrown, so that it lands at the foot of the cliff?

21. **Throwing in the wind again** ∗

A ball is thrown eastward across level ground. A wind blows horizontally to the east, and assume (simplistically) that the effect of the wind is to provide a constant force to the east, equal in magnitude to the weight of the ball. At what angle $\theta$ should the ball be thrown, so that it travels the maximum horizontal distance?

22. **Increasing gravity** ∗

At $t = 0$ on the planet Gravitus Increasicus, a projectile is fired with speed $V_0$ at an angle $\theta$ above the horizontal. This planet is a strange one, in that the acceleration due to gravity increases linearly with time, starting with a value of zero when the projectile is fired. In other words, $g(t) = \beta t$, where $\beta$ is a given constant. What horizontal distance does the projectile travel? What should $\theta$ be so that this horizonal distance is maximum?

23. **Cart, ball, and plane** ∗∗

A cart rolls down an inclined plane. A ball is fired from the cart, perpendicularly to the plane. Will the ball eventually land in the cart? *Hint*: Choose your coordinate system wisely.

*Section 2.5: Motion in a plane, polar coordinates*

24. **Low-orbit satellite**

What is the speed of a satellite whose orbit is just above the earth's surface? Give the numerical value.

25. **Weight at the equator** ∗

A person stands on a scale at the equator. If the earth somehow stopped spinning but kept its same shape, would the reading on the scale increase or decrease? By what fraction?

26. **Banking an airplane** ∗

An airplane flies at speed $v$ in a horizontal circle of radius $R$. At what angle should the plane be banked so that you don't feel like you are getting flung to the side in your seat?

27. **Car on a banked track** ∗∗

A car travels around a circular banked track with radius $R$. The coefficient of friction between the tires and the track is $\mu$. What is the maximum allowable speed, above which the car slips?

28. **Driving on tilted ground** $**$

A driver encounters a large tilted parking lot, where the angle of the ground with respect to the horizontal is $\theta$. The driver wishes to drive in a circle of radius $R$, at constant speed. The coefficient of friction between the tires and the ground is $\mu$.

(a) What is the largest speed the driver can have if he wants to avoid slipping?

(b) What is the largest speed the driver can have, assuming he is concerned only with whether or not he slips at one of the "side" points on the circle (that is, halfway between the top and bottom points; see Fig. 2.21)?

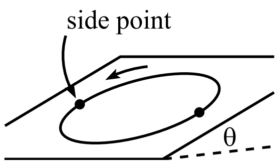



**Figure 2.21**

29. **Rolling wheel** $*$

If you paint a dot on the rim of a rolling wheel, the coordinates of the dot may be written as[16]

$$(x, y) = (R\theta + R\sin\theta, R + R\cos\theta). \quad (2.55)$$

The path of the dot is called a *cycloid*. Assume that the wheel is rolling at constant speed, which implies $\theta = \omega t$.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$ of the dot.

(b) At the instant the dot is at the top of the wheel, it may be considered to be moving along the arc of a circle. What is the radius of this circle in terms of $R$? *Hint*: You know $v$ and $a$.

30. **Bead on a hoop** $**$

A bead rests on top of a frictionless hoop of radius $R$ which lies in a vertical plane. The bead is given a tiny push so that it slides down and around the hoop. At what points on the hoop (specify them by giving the angular position relative to the top) is the bead's acceleration vertical?[17] What is this vertical acceleration? *Note*: We haven't studied conservation of energy yet, but use the fact that the bead's speed after it has fallen a height $h$ is given by $v = \sqrt{2gh}$.

31. **Another bead on a hoop** $**$

A bead rests on top of a frictionless hoop of radius $R$ which lies in a vertical plane. The bead is given a tiny push so that it slides down and around the hoop. At what points on the hoop (specify them by giving the angular position relative to the horizontal) is the bead's acceleration horizontal? As in the previous exercise, use $v = \sqrt{2gh}$.

---

[16] This can be shown by writing $(x, y)$ as $(R\theta, R) + (R\sin\theta, R\cos\theta)$. The first term here is the position of the center of the wheel, and the second term is the position of the dot relative to the center, where $\theta$ is measured clockwise from the top.

[17] One such point is the bottom of the hoop. Another point is technically the top, where $a = 0$. Find the other two more interesting points (one on each side).

32. **Derivation of $F_r$ and $F_\theta$** $**$

In Cartesian coordinates, a general vector takes the form,

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ &= r\cos\theta\,\hat{\mathbf{x}} + r\sin\theta\,\hat{\mathbf{y}}. \end{aligned} \tag{2.56}$$

Derive eqs. (2.52) by taking two derivatives of this expression for $\mathbf{r}$, and then using eqs. (2.47) to show that the result can be written in the form of eq. (2.51). Note that unlike $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change with time.

33. **A force $F_\theta = 2\dot{r}\dot{\theta}$** $**$

Consider a particle that feels an angular force only, of the form $F_\theta = 2m\dot{r}\dot{\theta}$. (As in Problem 21, there's nothing all that physical about this force; it simply makes the $F = ma$ equations solvable.) Show that the trajectory takes the form of an exponential spiral, that is, $r = Ae^{\theta}$.

34. **A force $F_\theta = 3\dot{r}\dot{\theta}$** $**$

Consider a particle that feels an angular force only, of the form $F_\theta = 3m\dot{r}\dot{\theta}$. (As in the previous exercise, we're solving this problem simply because we can.) Show that $\dot{r} = \sqrt{Ar^4 + B}$. Also, show that the particle reaches $r = \infty$ in a finite time.

## 2.7 Problems

*Section 2.2: Free-body diagrams*

1. **Sliding down a plane** $**$

   (a) A block starts at rest and slides down a frictionless plane inclined at angle $\theta$. What should $\theta$ be so that the block travels a given horizontal distance in the minimum amount of time?

   (b) Same question, but now let there be a coefficient of kinetic friction, $\mu$, between the block and the plane.

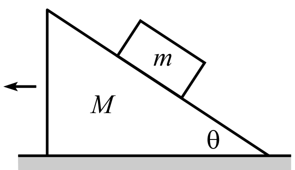



**Figure 2.22**

2. **Moving plane** $***$

   A block of mass $m$ is held motionless on a frictionless plane of mass $M$ and angle of inclination $\theta$ (see Fig. 2.22). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?

3. **Sliding sideways on plane** $***$

   A block is placed on a plane inclined at angle $\theta$. The coefficient of friction between the block and the plane is $\mu = \tan\theta$. The block is given a kick so that it initially moves with speed $V$ horizontally along the plane (that is, in the direction perpendicular to the direction pointing straight down the plane). What is the speed of the block after a very long time?

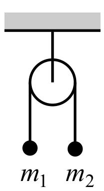



**Figure 2.23**

4. **Atwood's machine**

   A massless pulley hangs from a fixed support. A massless string connecting two masses, $m_1$ and $m_2$, hangs over the pulley (see Fig. 2.23). Find the acceleration of the masses and the tension in the string.

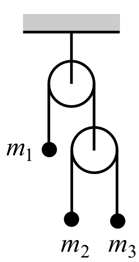



**Figure 2.24**

5. **Double Atwood's machine** $**$

   A double Atwood's machine is shown in Fig. 2.24, with masses $m_1$, $m_2$, and $m_3$. What are the accelerations of the masses?

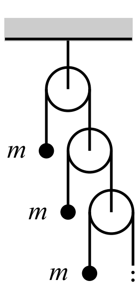



**Figure 2.25**

6. **Infinite Atwood's machine** $***$

   Consider the infinite Atwood's machine shown in Fig. 2.25. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to $m$, and all the pulleys and strings are massless. The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?[18]

[18] You may define this infinite system as follows. Consider it to be made of $N$ pulleys, with a non-zero mass replacing what would have been the $(N+1)$st pulley. Then take the limit as $N \to \infty$.

7. **Line of pulleys** *

$N + 2$ equal masses hang from a system of pulleys, as shown in Fig. 2.26. What are the accelerations of all the masses?

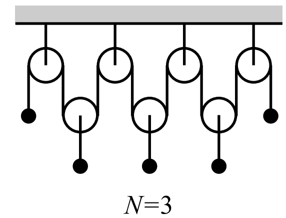



**Figure 2.26**

8. **Ring of pulleys** **

Consider the system of pulleys shown in Fig. 2.27. The string (which is a loop with no ends) hangs over $N$ fixed pulleys. $N$ masses, $m_1$, $m_2$, …, $m_N$, are attached to $N$ pulleys that hang on the string. What are the accelerations of all the masses?

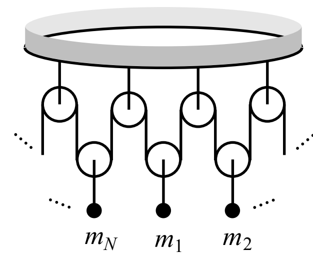



**Figure 2.27**

*Section 2.3: Solving differential equations*

9. **Exponential force**

A particle of mass $m$ is subject to a force $F(t) = me^{-bt}$. The initial position and speed are zero. Find $x(t)$.

10. **Falling chain** **

A chain of length $\ell$ is held stretched out on a frictionless horizontal table, with a length $y_0$ hanging down through a hole in the table. The chain is released. As a function of time, find the length that hangs down through the hole (don't bother with $t$ after the chain loses contact with the table). Also, find the speed of the chain right when it loses contact with the table.

11. **Circling around a pole** **

A mass, which is free to move on a horizontal frictionless plane, is attached to one end of a massless string which wraps partially around a frictionless vertical pole of radius $r$ (see the top view in Fig. 2.28). You hold onto the other end of the string. At $t = 0$, the mass has speed $v_0$ in the tangential direction along the dotted circle of radius $R$ shown.

Your task is to pull on the string so that the mass keeps moving along the dotted circle. You are required to do this in such a way that the string remains in contact with the pole at all times. (You will have to move your hand around the pole, of course.) What is the the speed of the mass as a function of time?

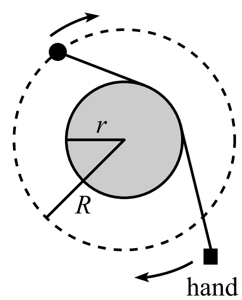



**Figure 2.28**

12. **Throwing a beach ball** ***

A beach ball is thrown upward with initial speed $v_0$. Assume that the drag force from the air is $F = -m\alpha v$. What is the speed of the ball, $v_f$, when it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

13. **Balancing a pencil** ***

Consider a pencil that stands upright on its tip and then falls over. Let's idealize the pencil as a mass $m$ sitting at the end of a massless rod of length $\ell$.[19]

[19] It actually involves only a trivial modification to do the problem correctly using the moment of inertia and the torque. But the point-mass version will be quite sufficient for the present purposes.

(a) Assume that the pencil makes an initial (small) angle $\theta_0$ with the vertical, and that its initial angular speed is $\omega_0$. The angle will eventually become large, but while it is small (so that $\sin\theta \approx \theta$), what is $\theta$ as a function of time?

(b) You might think that it would be possible (theoretically, at least) to make the pencil balance for an arbitrarily long time, by making the initial $\theta_0$ and $\omega_0$ sufficiently small.

However, it turns out that due to Heisenberg's uncertainty principle (which puts a constraint on how well we can know the position and momentum of a particle), it is impossible to balance the pencil for more than a certain amount of time. The point is that you can't be sure that the pencil is initially both at the top *and* at rest. The goal of this problem is to be quantitative about this. The time limit is sure to surprise you.

Without getting into quantum mechanics, let's just say that the uncertainty principle says (up to factors of order 1) that $\Delta x \Delta p \geq \hbar$ (where $\hbar = 1.06 \cdot 10^{-34}$ Js is Planck's constant). The implications of this are somewhat vague, but we'll just take it to mean that the initial conditions satisfy $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$.

With this condition, find the maximum time it can take your solution in part (a) to become of order 1. In other words, determine (roughly) the maximum time the pencil can balance. Assume $m = 0.01\,\text{kg}$, and $\ell = 0.1\,\text{m}$.

*Section 2.4: Projectile motion*

14. **Throwing a ball from a cliff** $**$

A ball is thrown with speed $v$ from the edge of a cliff of height $h$. At what inclination angle should it be thrown so that it travels the maximum horizontal distance? What is this maximum distance? Assume that the ground below the cliff is horizontal.

15. **Redirected motion** $**$

A ball is dropped from rest at height $h$, and it bounces off a surface at height $y$ (with no loss in speed). The surface is inclined so that the ball bounces off at an angle of $\theta$ with respect to the horizontal. What should $y$ and $\theta$ be so that the ball travels the maximum horizontal distance?

16. **Maximum trajectory length** $***$

A ball is thrown at speed $v$ from zero height on level ground. Let $\theta_0$ be the angle at which the ball should be thrown so that the distance traveled *through the air* is maximum. Show that $\theta_0$ satisfies

$$\sin\theta_0 \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right) = 1. \tag{2.57}$$

You can show numerically that $\theta_0 \approx 56.5°$.

17. **Maximum trajectory area** $*$

    A ball is thrown at speed $v$ from zero height on level ground. At what angle should it be thrown so that the area under the trajectory is maximum?

18. **Bouncing ball** $*$

    A ball is thrown straight upward so that it reaches a height $h$. It falls down and bounces repeatedly. After each bounce, it returns to a certain fraction $f$ of its previous height. Find the total distance traveled, and also the total time, before it comes to rest. What is its average speed?

*Section 2.5: Motion in a plane, polar coordinates*

19. **Centripetal acceleration** $*$

    Show that the acceleration of a particle moving in a circle is $v^2/r$. To do this, draw the position and velocity vectors at two nearby times, and then make use of some similar triangles.

20. **Free particle** $**$

    Consider a free particle in a plane. Using Cartesian coordinates, it is trivial to show that the particle moves in a straight line. The task of this problem is to demonstrate this result in a much more cumbersome way, using eqs. (2.52). More precisely, show that $\cos\theta = r_0/r$ for a free particle, where $r_0$ is the radius at closest approach to the origin, and $\theta$ is measured with respect to this radius.

21. **A force $F_\theta = \dot{r}\dot{\theta}$** $**$

    Consider a particle that feels an angular force only, of the form $F_\theta = m\dot{r}\dot{\theta}$. (There's nothing all that physical about this force. It simply makes the $F = ma$ equations solvable.) Show that $\dot{r} = \sqrt{A\ln r + B}$, where $A$ and $B$ are constants of integration, determined by the initial conditions.

## 2.8 Solutions

1. **Sliding down a plane**

(a) The component of gravity along the plane in $g \sin\theta$. The acceleration in the horizontal direction is therefore $a_x = (g\sin\theta)\cos\theta$. Our goal is to maximize $a_x$. By taking the derivative, or by noting that $\sin\theta\cos\theta = (\sin 2\theta)/2$, we obtain $\theta = \pi/4$.

(b) The normal force from the plane is $mg\cos\theta$, so the kinetic friction force is $\mu mg\cos\theta$. The acceleration along the plane is therefore $g(\sin\theta - \mu\cos\theta)$, and so the acceleration in the horizontal direction is $a_x = g(\sin\theta - \mu\cos\theta)\cos\theta$. We want to maximize this. Setting the derivative equal to zero gives

$$(\cos^2\theta - \sin^2\theta) + 2\mu\sin\theta\cos\theta = 0 \quad\Longrightarrow\quad \cos 2\theta + \mu\sin 2\theta = 0$$
$$\Longrightarrow\quad \tan 2\theta = -\frac{1}{\mu}\,. \qquad (2.58)$$

For $\mu \to 0$, this gives the $\pi/4$ result from part (a). For $\mu \to \infty$, we obtain $\theta \approx \pi/2$, which makes sense.

REMARK: The time to travel a horizontal distance $d$ is obtained from $a_x t^2/2 = d$. In part (a), this gives a minimum time of $2\sqrt{d/g}$. In part (b), you can show that the maximum $a_x$ is $(g/2)(\sqrt{1+\mu^2} - \mu)$, and that this leads to a minimum time of $2\sqrt{d/g}\sqrt{\sqrt{1+\mu^2}+\mu}$. This has the correct $\mu \to 0$ limit, and it behaves like $2\sqrt{2\mu d/g}$ for $\mu \to \infty$. ♣

2. **Moving plane**

Let $N$ be the normal force between the block and the plane. Note that we *cannot* assume that $N = mg\cos\theta$, because the plane recoils. We can see that $N = mg\cos\theta$ is in fact incorrect, because in the limiting case where $M = 0$, we have no normal force at all.

The various $F = ma$ equations (vertical and horizontal for the block, and horizontal for the plane) are

$$\begin{aligned} mg - N\cos\theta &= ma_y, \\ N\sin\theta &= ma_x, \\ N\sin\theta &= MA_x, \end{aligned} \qquad (2.59)$$

where we have chosen the positive directions for $a_y$, $a_x$, and $A_x$ to be downward, rightward, and leftward, respectively. There are four unknowns here: $a_x$, $a_y$, $A_x$, and $N$. So we need one more equation. This fourth equation is the constraint that the block remains in contact with the plane. The horizontal distance between the block and its starting point on the plane is $(a_x + A_x)t^2/2$, and the vertical distance is $a_y t^2/2$. The ratio of these distances must equal $\tan\theta$ if the block is to remain on the plane. Therefore, we must have

$$\frac{a_y}{a_x + A_x} = \tan\theta. \qquad (2.60)$$

Using eqs. (2.59), this becomes

$$\frac{g - \frac{N}{m}\cos\theta}{\frac{N}{m}\sin\theta + \frac{N}{M}\sin\theta} = \tan\theta$$
$$\Longrightarrow \quad N = g\left(\sin\theta\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right) + \frac{\cos\theta}{m}\right)^{-1}. \qquad (2.61)$$

(In the limit $M \to \infty$, this reduces to $N = mg\cos\theta$, as it should.) Having found $N$, the third of eqs. (2.59) gives $A_x$, which may be written as

$$A_x = \frac{N\sin\theta}{M} = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}\,. \tag{2.62}$$

Remarks: For given $M$ and $m$, you can show that the angle $\theta_0$ that maximizes $A_x$ is

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}\,. \tag{2.63}$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, eq. (2.62) gives $A_x \approx g/\tan\theta$. This makes sense, because $m$ falls essentially straight down, and the plane gets squeezed out to the left.

In the limit $M \gg m$, we have $A_x \approx g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $a_x = (M/m)A_x \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of $a_x$ implies that the acceleration of $m$ *along* the plane is equal to $a_x/\cos\theta \approx g\sin\theta$, as expected. ♣

3. **Sliding sideways on plane**

The normal force from the plane is $N = mg\cos\theta$. Therefore, the friction force on the block is $\mu N = (\tan\theta)N = mg\sin\theta$. This force acts in the direction opposite to the motion. The block also feels the gravitational force of $mg\sin\theta$ pointing down the plane.

Because the magnitudes of the friction force and the gravitational force along the plane are equal, the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting $v$ be the speed of the block, and letting $v_y$ be the component of the velocity in the direction down the plane, we therefore have

$$v + v_y = C, \tag{2.64}$$

where $C$ is a constant. $C$ is given by its initial value, which is $V + 0 = V$. The final value of $C$ is $V_f + V_f = 2V_f$ (where $V_f$ is the final speed of the block), because the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = \frac{V}{2}\,. \tag{2.65}$$

4. **Atwood's machine**

Let $T$ be the tension in the string, and let $a$ be the acceleration of $m_1$ (with upward taken to be positive). Then $-a$ is the acceleration of $m_2$. So we have

$$\begin{aligned} T - m_1 g &= m_1 a, \\ T - m_2 g &= m_2(-a). \end{aligned} \tag{2.66}$$

Solving these two equations for $a$ and $T$ gives

$$a = \frac{(m_2 - m_1)g}{m_2 + m_1}\,, \qquad \text{and} \qquad T = \frac{2m_1 m_2 g}{m_2 + m_1}\,. \tag{2.67}$$

Remarks: As a double-check, $a$ has the correct limits when $m_2 \gg m_1$, $m_1 \gg m_2$, and $m_2 = m_1$ (namely $a \approx g$, $a \approx -g$, and $a = 0$, respectively).

As far as $T$ goes, if $m_1 = m_2 \equiv m$, then $T = mg$, as it should. And if $m_1 \ll m_2$, then $T \approx 2m_1 g$. This is correct, because it makes the net upward force on $m_1$ equal to $m_1 g$, which means that its acceleration is $g$ upward, which is consistent with the fact that $m_2$ is essentially in free fall. ♣

5. **Double Atwood's machine**

Let the tension in the lower string be $T$. Then the tension in the upper string is $2T$ (by balancing the forces on the bottom pulley). The three $F = ma$ equations are therefore (with all the $a$'s taken to be positive upward)

$$\begin{aligned} 2T - m_1 g &= m_1 a_1, \\ T - m_2 g &= m_2 a_2, \\ T - m_3 g &= m_3 a_3. \end{aligned} \tag{2.68}$$

And conservation of string says that the acceleration of $m_1$ is

$$a_1 = -\left(\frac{a_2 + a_3}{2}\right). \tag{2.69}$$

This follows from the fact that the average position of $m_2$ and $m_3$ moves the same distance as the bottom pulley, which in turn moves the same distance (but in the opposite direction) as $m_1$.

We now have four equations in the four unknowns, $a_1$, $a_2$, $a_3$, and $T$. With a little work, we can solve for the accelerations,

$$\begin{aligned} a_1 &= g\frac{4m_2m_3 - m_1(m_2+m_3)}{4m_2m_3 + m_1(m_2+m_3)}, \\ a_2 &= -g\frac{4m_2m_3 + m_1(m_2-3m_3)}{4m_2m_3 + m_1(m_2+m_3)}, \\ a_3 &= -g\frac{4m_2m_3 + m_1(m_3-3m_2)}{4m_2m_3 + m_1(m_2+m_3)}. \end{aligned} \tag{2.70}$$

REMARKS: There are many limits we can check here. A couple are: (1) If $m_2 = m_3 = m_1/2$, then all the $a$'s are zero, which is correct. (2) If $m_3$ is much less than both $m_1$ and $m_2$, then $a_1 = -g$, $a_2 = -g$, and $a_3 = 3g$. To understand this $3g$, convince yourself that if $m_1$ and $m_2$ go down by $d$, then $m_3$ goes up by $3d$.

Note that $a_1$ can be written as

$$a_1 = g\frac{\frac{4m_2m_3}{(m_2+m_3)} - m_1}{\frac{4m_2m_3}{(m_2+m_3)} + m_1}. \tag{2.71}$$

In view of the result of Problem 4 in eq. (2.67), we see that as far as $m_1$ is concerned, the $m_2, m_3$ pulley system acts just like a mass of $4m_2m_3/(m_2+m_3)$. This has the expected properties of equaling zero when either $m_2$ or $m_3$ is zero, and equaling $2m$ if $m_2 = m_3 \equiv m$. ♣

6. **Infinite Atwood's machine**

**First Solution:** If the strength of gravity on the earth were multiplied by a factor $\eta$, then the tension in all of the strings in the Atwood's machine would likewise be multiplied by $\eta$. This is true because the only way to produce a quantity with the units of tension (that is, force) is to multiply a mass by $g$. Conversely, if we put the Atwood's machine on another planet and discover that all of the tensions are multiplied by $\eta$, then we know that the gravity there must be $\eta g$.

Let the tension in the string above the first pulley be $T$. Then the tension in the string above the second pulley is $T/2$ (because the pulley is massless). Let the downward acceleration of the second pulley be $a_2$. Then the second pulley effectively lives in a world where gravity has strength $g - a_2$.

Consider the subsystem of all the pulleys except the top one. This infinite subsystem is identical to the original infinite system of all the pulleys. Therefore, by the arguments in the first paragraph above, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_2}\,, \tag{2.72}$$

which gives $a_2 = g/2$. But $a_2$ is also the acceleration of the top mass, so our answer is $g/2$.

Remarks: You can show that the relative acceleration of the second and third pulleys is $g/4$, and that of the third and fourth is $g/8$, etc. The acceleration of a mass far down in the system therefore equals $g(1/2 + 1/4 + 1/8 + \cdots) = g$, which makes intuitive sense.

Note that $T = 0$ also makes eq. (2.72) true. But this corresponds to putting a mass of zero at the end of a finite pulley system (see the following solution). ♣

**Second Solution:** Consider the following auxiliary problem.

**Problem:** Two setups are shown below in Fig. 2.29. The first contains a hanging mass $m$. The second contains a pulley, over which two masses, $m_1$ and $m_2$, hang. Let both supports have acceleration $a_s$ downward. What should $m$ be, in terms of $m_1$ and $m_2$, so that the tension in the top string is the same in both cases?

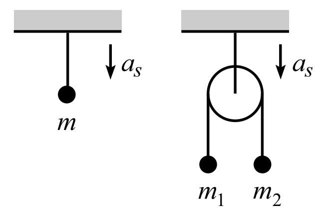


**Figure 2.29**

**Answer:** In the first case, we have

$$mg - T = ma_s. \tag{2.73}$$

In the second case, let $a$ be the acceleration of $m_2$ relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} m_1 g - \frac{T}{2} &= m_1(a_s - a), \\ m_2 g - \frac{T}{2} &= m_2(a_s + a). \end{aligned} \tag{2.74}$$

Note that if we define $g' \equiv g - a_s$, then we may write the above three equations as

$$\begin{aligned} mg' &= T, \\ m_1 g' &= \frac{T}{2} - m_1 a, \\ m_2 g' &= \frac{T}{2} + m_2 a. \end{aligned} \tag{2.75}$$

Eliminating $a$ from the last two of these equations gives $T = 4m_1 m_2 g'/(m_1 + m_2)$. Using this value of $T$ in the first equation then gives

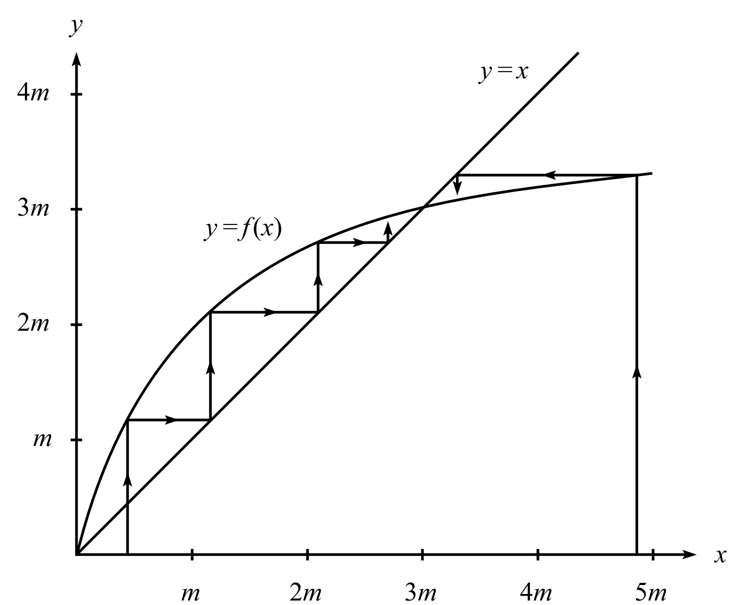
$$m = \frac{4m_1 m_2}{m_1 + m_2}\,. \tag{2.76}$$

Note that the value of $a_s$ is irrelevant. We effectively have a fixed support in a world where the acceleration due to gravity is $g'$ (see eqs. (2.75)), and the answer can't depend on $g'$, by dimensional analysis. This auxiliary problem shows that the two-mass system in the second case may be equivalently treated as a mass $m$, given by eq. (2.76), as far as the upper string is concerned. ■

Now let's look at our infinite Atwood's machine. Assume that the system has $N$ pulleys, where $N \to \infty$. Let the bottom mass be $x$. Then the auxiliary problem shows that the bottom two masses, $m$ and $x$, may be treated as an effective mass $f(x)$, where

$$\begin{aligned} f(x) &= \frac{4mx}{m+x} \\ &= \frac{4x}{1+(x/m)} . \end{aligned} \tag{2.77}$$

We may then treat the combination of the mass $f(x)$ and the next $m$ as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass $m$ and a mass $f^{(N-1)}(x)$ hanging over the top pulley. So we must determine the behavior of $f^N(x)$, as $N \to \infty$. This behavior is clear if we look at the following plot of $f(x)$.




Note that $x = 3m$ is a fixed point of $f(x)$. That is, $f(3m) = 3m$. This plot shows that no matter what $x$ we start with, the iterations approach $3m$ (unless we start at $x = 0$, in which case we remain there). These iterations are shown graphically by the directed lines in the plot. After reaching the value $f(x)$ on the curve, the line moves horizontally to the $x$ value of $f(x)$, and then vertically to the value $f(f(x))$ on the curve, and so on.

Therefore, since $f^N(x) \to 3m$ as $N \to \infty$, our infinite Atwood's machine is equivalent to (as far as the top mass is concerned) just two masses, $m$ and $3m$. You can then quickly show that that the acceleration of the top mass is $g/2$.

Note that as far as the support is concerned, the whole apparatus is equivalent to a mass $3m$. So $3mg$ is the upward force exerted by the support.

7. **Line of pulleys**

Let $m$ be the common mass, and let $T$ be the tension in the string. Let $a$ be the acceleration of the end masses, and let $a'$ be the acceleration of the other $N$ masses, with upward taken to be positive. Note that these $N$ accelerations are indeed all equal, because the same net force acts on all of the internal $N$ masses, namely $2T$ upwards and $mg$ downwards. The $F = ma$ equations for the end and internal masses are, respectively,

$$\begin{aligned} T - mg &= ma, \\ 2T - mg &= ma'. \end{aligned} \tag{2.78}$$

But the string has fixed length. Therefore,

$$N(2a') + a + a = 0. \tag{2.79}$$

Eliminating $T$ from eqs. (2.78) gives $a' = 2a + g$. Combining this with eq. (2.79) then gives

$$a = -\frac{Ng}{2N+1}, \qquad \text{and} \qquad a' = \frac{g}{2N+1}. \tag{2.80}$$

REMARKS: For $N = 1$, we have $a = -g/3$ and $a' = g/3$. For larger $N$, $a$ increases in magnitude and approaches $-g/2$ for $N \to \infty$, and $a'$ decreases in magnitude and approaches zero for $N \to \infty$.

The signs of $a$ and $a'$ in eq. (2.80) may be surprising. You might think that if, say, $N = 100$, then these 100 masses will "'win" out over the two end masses, so that the $N$ masses will fall. But this is not correct, because there are many ($2N$, in fact) tensions acting up on the $N$ masses. They do *not* act like a mass $Nm$ hanging below one pulley. In fact, two masses of $m/2$ on the ends will balance any number $N$ of masses in the interior (with the help of the upward forces from the top row of pulleys). ♣

8. **Ring of pulleys**

Let $T$ be the tension in the string. Then $F = ma$ for $m_i$ gives

$$2T - m_i g = m_i a_i, \tag{2.81}$$

with upward taken to be positive. The $a_i$'s are related by the fact that the string has fixed length, which implies that the sum of the displacements of all the masses is zero. In other words,

$$a_1 + a_2 + \cdots + a_N = 0. \tag{2.82}$$

If we divide eq. (2.81) by $m_i$, and then add the $N$ such equations together, we obtain, using eq. (2.82),

$$2T\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right) - Ng = 0. \tag{2.83}$$

Substituting this value for $T$ into (2.81) gives

$$a_i = g\left(\frac{N}{m_i\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right)} - 1\right). \tag{2.84}$$

A few special cases are: If all the masses are equal, then all $a_i = 0$. If $m_k = 0$ (and all the others are not zero), then $a_k = (N-1)g$, and all the other $a_i = -g$.

9. **Exponential force**

We are given $\ddot{x} = e^{-bt}$. Integrating this with respect to time gives $v(t) = -e^{-bt}/b + A$. Integrating again gives $x(t) = e^{-bt}/b^2 + At + B$. The initial condition, $v(0) = 0$, gives $-1/b + A = 0 \Longrightarrow A = 1/b$. And the initial condition, $x(0) = 0$, gives $1/b^2 + B = 0 \Longrightarrow B = -1/b^2$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}\,. \tag{2.85}$$

Limits: For $t \to \infty$, $v$ approaches $1/b$, and $x$ approaches $t/b - 1/b^2$. We see that the particle eventually lags a distance $1/b^2$ behind another particle that started at the same position but with speed $v = 1/b$.

10. **Falling chain**

Let the density of the chain be $\rho$, and let $y(t)$ be the length hanging down through the hole at time $t$. Then the total mass is $\rho\ell$, and the mass hanging below the hole is $\rho y$. The net downward force on the chain is $(\rho y)g$, so $F = ma$ gives

$$\rho g y = (\rho\ell)\ddot{y} \qquad \Longrightarrow \qquad \ddot{y} = \frac{g}{\ell}y. \tag{2.86}$$

At this point, there are two ways we can proceed:

**First method:** Since we have a function whose second derivative is proportional to itself, a good bet for the solution is an exponential function. And indeed, a quick check shows that the solution is

$$y(t) = Ae^{\alpha t} + Be^{-\alpha t}, \qquad \text{where} \quad \alpha \equiv \sqrt{\frac{g}{\ell}}\,. \tag{2.87}$$

Taking the derivative of this to obtain $\dot{y}(t)$, and using the given information that $\dot{y}(0) = 0$, we find $A = B$. Using $y(0) = y_0$, we then find $A = B = y_0/2$. So the length that hangs below the hole is

$$y(t) = \frac{y_0}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) \equiv y_0 \cosh(\alpha t). \tag{2.88}$$

And the speed is

$$\dot{y}(t) = \frac{\alpha y_0}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) \equiv \alpha y_0 \sinh(\alpha t). \tag{2.89}$$

The time $T$ that satisfies $y(T) = \ell$ is given by $\ell = y_0 \cosh(\alpha T)$. Using $\sinh x = \sqrt{\cosh^2 x - 1}$, we find that the speed of the chain right when it loses contact with the table is

$$\dot{y}(T) = \alpha y_0 \sinh(\alpha T) = \alpha\sqrt{\ell^2 - y_0^2} \equiv \sqrt{g\ell}\sqrt{1 - \eta_0^2}, \tag{2.90}$$

where $\eta_0 \equiv y_0/\ell$ is the initial fraction hanging below the hole.

If $\eta_0 \approx 0$, then the speed at time $T$ is $\sqrt{g\ell}$ (this quickly follows from conservation of energy, which is the subject of Chapter 4). Also, you can show that eq. (2.88) implies that $T$ goes to infinity logarithmically as $\eta_0 \to 0$.

**Second method:** Write $\ddot{y}$ as $v\,dv/dy$ in eq. (2.86), and then separate variables and integrate to obtain

$$\int_0^v v\,dv = \alpha^2 \int_{y_0}^y y\,dy \qquad \Longrightarrow \qquad v^2 = \alpha^2(y^2 - y_0^2), \tag{2.91}$$

where $\alpha \equiv \sqrt{g/\ell}$. Now write $v$ as $dy/dt$ and separate variables again to obtain

$$\int_{y_0}^{y} \frac{dy}{\sqrt{y^2 - y_0^2}} = \alpha \int_0^t dt. \tag{2.92}$$

The integral on the left-hand side is $\cosh^{-1}(y/y_0)$, so we arrive at

$$y(t) = y_0 \cosh(\alpha t), \tag{2.93}$$

in agreement with eq. (2.88). The solution proceeds as above. However, an easier way to obtain the final speed with this method is to simply use the result for $v$ in eq. (2.91). This tells us that the speed of the chain when it leaves the table (that is, when $y = \ell$) is $v = \alpha\sqrt{\ell^2 - y_0^2}$, in agreement with eq. (2.90).

11. **Circling around a pole**

Let $F$ be the tension in the string. At the mass, the angle between the string and the radius of the dotted circle is $\theta = \sin^{-1}(r/R)$. In terms of $\theta$, the radial and tangential $F = ma$ equations are

$$\begin{aligned} F\cos\theta &= \frac{mv^2}{R}, \qquad \text{and} \\ F\sin\theta &= m\dot{v}. \end{aligned} \tag{2.94}$$

Dividing these two equations gives $\tan\theta = (R\dot{v})/v^2$. Separating variables and integrating gives

$$\begin{aligned} \int_{v_0}^{v} \frac{dv}{v^2} &= \frac{\tan\theta}{R}\int_0^t dt \\ \Longrightarrow \quad \frac{1}{v_0} - \frac{1}{v} &= \frac{(\tan\theta)t}{R} \\ \Longrightarrow \quad v(t) &= \left(\frac{1}{v_0} - \frac{(\tan\theta)t}{R}\right)^{-1}. \end{aligned} \tag{2.95}$$

REMARK: Note that $v$ becomes infinite when

$$t = T \equiv \frac{R}{v_0 \tan\theta}. \tag{2.96}$$

In other words, you can keep the mass moving in the desired circle only up to time $T$. After that, it is impossible. (Of course, it will become impossible, for all practical purposes, long before $v$ becomes infinite.) The total distance, $d = \int v\, dt$, is infinite, because this integral diverges (barely, like a log) as $t$ approaches $T$. ♣

12. **Throwing a beach ball**

On both the way up and the way down, the total force on the ball is

$$F = -mg - m\alpha v. \tag{2.97}$$

On the way up, $v$ is positive, so the drag force points downward, as it should. And on the way down, $v$ is negative, so the drag force points upward.

Our strategy for finding $v_f$ will be to produce two different expressions for the maximum height, $h$, and then equate them. We'll find these two expressions by considering the upward and then the downward motion of the ball. In doing so, we will need to write the acceleration of the ball as $a = v\, dv/dy$.

For the upward motion, $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= mv\frac{dv}{dy} \\ \Longrightarrow \qquad \int_0^h dy &= -\int_{v_0}^0 \frac{v\,dv}{g+\alpha v}\,. \end{aligned} \tag{2.98}$$

where we have taken advantage of the fact that we know that the speed of the ball at the top is zero. Writing $v/(g + \alpha v)$ as $[1 - g/(g + \alpha v)]/\alpha$, we may evaluate the integral to obtain

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{2.99}$$

Now let us consider the downward motion. Let $v_f$ be the final speed, which is a positive quantity. The final velocity is then the negative quantity, $-v_f$. Using $F = ma$, we similarly obtain

$$\int_h^0 dy = -\int_0^{-v_f} \frac{v\,dv}{g+\alpha v}\,. \tag{2.100}$$

Performing the integration (or just replacing the $v_0$ in eq. (2.99) with $-v_f$) gives

$$h = -\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{2.101}$$

Equating the expressions for $h$ in eqs. (2.99) and (2.101) gives an implicit equation for $v_f$ in terms of $v_0$,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_f}\right). \tag{2.102}$$

Remarks: In the limit of small $\alpha$ (more precisely, in the limit $\alpha v_0/g \ll 1$), we can use $\ln(1+x) = x - x^2/2 + \cdots$ to obtain approximate values for $h$ in eqs. (2.99) and (2.101). The results are, as expected,

$$h \approx \frac{v_0^2}{2g}, \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g}\,. \tag{2.103}$$

We can also make approximations for large $\alpha$ (or large $\alpha v_0/g$). In this limit, the log term in eq. (2.99) is negligible, so we obtain $h \approx v_0/\alpha$. And eq. (2.101) gives $v_f \approx g/\alpha$, because the argument of the log must be very small in order to give a very large negative number, which is needed to produce a positive $h$ on the left-hand side. There is no way to relate $v_f$ and $h$ is this limit, because the ball quickly reaches the terminal velocity of $-g/\alpha$ (which is the velocity that makes the net force equal to zero), independent of $h$. ♣

Let's now find the times it takes for the ball to go up and to go down. We'll present two methods for doing this.

**First method:** Let $T_1$ be the time for the upward path. If we write the acceleration of the ball as $a = dv/dt$, then $F = ma$ gives

$$\begin{aligned} -mg - m\alpha v &= m\frac{dv}{dt} \\ \Longrightarrow \qquad \int_0^{T_1} dt &= -\int_{v_0}^0 \frac{dv}{g+\alpha v}\,. \end{aligned} \tag{2.104}$$

$$T_1 = \frac{1}{\alpha}\ln\left(1+\frac{\alpha v_0}{g}\right). \tag{2.105}$$

In a similar manner, we find that the time $T_2$ for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1-\frac{\alpha v_f}{g}\right). \tag{2.106}$$

Therefore,

$$T_1+T_2 = \frac{1}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_f}\right). \tag{2.107}$$

Using eq. (2.102), we have

$$T_1+T_2 = \frac{v_0+v_f}{g}. \tag{2.108}$$

This is shorter than the time in vacuum (namely $2v_0/g$) because $v_f < v_0$.

**Second method:** The very simple form of eq. (2.108) suggests that there is a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (2.108). This procedure only works, of course, because the drag force is proportional to $v$.

REMARKS: The fact that the time here is shorter than the time in vacuum isn't obvious. On one hand, the ball doesn't travel as high in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any $\alpha$, you can use eq. (2.105) to show that $T_1 < v_0/g$. But $T_2$ is harder to get a handle on, because it is given in terms of $v_f$. But in the limit of large $\alpha$, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum. ♣

13. **Balancing a pencil**

(a) The component of gravity in the tangential direction is $mg\sin\theta \approx mg\theta$. Therefore, the tangential $F = ma$ equation is $mg\theta = m\ell\ddot{\theta}$, which may be written as $\ddot{\theta} = (g/\ell)\theta$. The general solution to this equation is[20]

$$\theta(t) = Ae^{t/\tau} + Be^{-t/\tau}, \qquad \text{where } \tau \equiv \sqrt{\ell/g}. \tag{2.109}$$

The constants $A$ and $B$ are found from the initial conditions,

$$\begin{aligned} \theta(0) = \theta_0 \qquad &\Longrightarrow \qquad A + B = \theta_0, \\ \dot{\theta}(0) = \omega_0 \qquad &\Longrightarrow \qquad (A-B)/\tau = \omega_0. \end{aligned} \tag{2.110}$$

Solving for $A$ and $B$, and then plugging them into eq. (2.109) gives

$$\theta(t) = \frac{1}{2}(\theta_0 + \omega_0\tau)\,e^{t/\tau} + \frac{1}{2}(\theta_0 - \omega_0\tau)\,e^{-t/\tau}. \tag{2.111}$$

[20] If you want, you can derive this by separating variables and integrating. The solution is essentially the same as in the second method presented in the solution to Problem 10.

(b) The constants $A$ and $B$ will turn out to be small (they will each be of order $\sqrt{\hbar}$). Therefore, by the time the positive exponential has increased enough to make $\theta$ of order 1, the negative exponential will have become negligible. We will therefore ignore the latter term from here on. In other words,

$$\theta(t) \approx \frac{1}{2}\left(\theta_0 + \omega_0\tau\right)e^{t/\tau}. \tag{2.112}$$

The goal is to keep $\theta$ small for as long as possible. Hence, we want to minimize the coefficient of the exponential, subject to the uncertainty-principle constraint, $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$. This constraint gives $\omega_0 \geq \hbar/(m\ell^2\theta_0)$. Therefore,

$$\theta(t) \geq \frac{1}{2}\left(\theta_0 + \frac{\hbar\tau}{m\ell^2\theta_0}\right)e^{t/\tau}. \tag{2.113}$$

Taking the derivative with respect to $\theta_0$ to minimize the coefficient, we find that the minimum value occurs at

$$\theta_0 = \sqrt{\frac{\hbar\tau}{m\ell^2}}\,. \tag{2.114}$$

Substituting this back into eq. (2.113) gives

$$\theta(t) \geq \sqrt{\frac{\hbar\tau}{m\ell^2}}\; e^{t/\tau}. \tag{2.115}$$

Setting $\theta \approx 1$, and then solving for $t$ gives (using $\tau \equiv \sqrt{\ell/g}$)

$$t \leq \frac{1}{4}\sqrt{\frac{\ell}{g}}\ln\left(\frac{m^2\ell^3 g}{\hbar^2}\right). \tag{2.116}$$

With the given values, $m = 0.01\,\text{kg}$ and $\ell = 0.1\,\text{m}$, along with $g = 10\,\text{m/s}^2$ and $\hbar = 1.06 \cdot 10^{-34}\,\text{Js}$, we obtain

$$t \leq \frac{1}{4}(0.1\,\text{s})\ln(9 \cdot 10^{61}) \approx 3.5\,\text{s}. \tag{2.117}$$

No matter how clever you are, and no matter how much money you spend on the newest, cutting-edge pencil-balancing equipment, you can never get a pencil to balance for more than about four seconds.

REMARKS: This smallness of this answer is quite amazing. It is remarkable that a quantum effect on a macroscopic object can produce an everyday value for a time scale. Basically, the point here is that the fast exponential growth of $\theta$ (which gives rise to the log in the final result for $t$) wins out over the smallness of $\hbar$, and produces a result for $t$ of order 1. When push comes to shove, exponential effects always win.

The above value for $t$ depends strongly on $\ell$ and $g$, through the $\sqrt{\ell/g}$ term. But the dependence on $m$, $\ell$, and $g$ in the log term is very weak. If $m$ were increased by a factor of 1000, for example, the result for $t$ would increase by only about 10%. Note that this implies that any factors of order 1 that we neglected throughout this problem are completely irrelevant. They will appear in the argument of the log term, and will thus have negligible effect.

Note that dimensional analysis, which is generally a very powerful tool, won't get you too far in this problem. The quantity $\sqrt{\ell/g}$ has dimensions of time, and the quantity

$\eta \equiv m^2\ell^3 g/\hbar^2$ is dimensionless (it is the only such quantity), so the balancing time must take the form,

$$t \approx \sqrt{\frac{\ell}{g}}\, f(\eta), \tag{2.118}$$

where $f$ is some function. If the leading term in $f$ were a power (even, for example, a square root), then $t$ would essentially be infinite ($t \approx 10^{30}$ s for the square root). But $f$ in fact turns out to be a log (which you can't determine without solving the problem), which completely cancels out the smallness of $\hbar$, reducing an essentially infinite time down to a few seconds. ♣

14. **Throwing a ball from a cliff**

Let the inclination angle be $\theta$. Then the horizontal speed is $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$. The time it takes for the ball to hit the ground is given by $h + (v\sin\theta)t - gt^2/2 = 0$. Therefore,

$$t = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right), \qquad \text{where} \;\; \beta \equiv \frac{2gh}{v^2}\,. \tag{2.119}$$

(The "−" solution for $t$ from the quadratic formula corresponds to the ball being thrown backwards down through the cliff.) The horizontal distance traveled is $d = (v\cos\theta)t$, which gives

$$d = \frac{v^2}{g}\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right). \tag{2.120}$$

We want to maximize this function of $\theta$. Taking the derivative, multiplying through by $\sqrt{\sin^2\theta + \beta}$, and setting the result equal to zero, gives

$$(\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\theta + \beta} = \sin\theta\big(\beta - (\cos^2\theta - \sin^2\theta)\big). \tag{2.121}$$

Using $\cos^2\theta = 1 - \sin^2\theta$, and then squaring and simplifying this equation, gives an optimal angle of

$$\sin\theta_{\max} = \frac{1}{\sqrt{2+\beta}} \equiv \frac{1}{\sqrt{2 + 2gh/v^2}}\,. \tag{2.122}$$

Plugging this into eq. (2.120), and simplifying, gives a maximum distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1+\beta} \equiv \frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}}\,. \tag{2.123}$$

Remarks: If $h = 0$, then we obtain $\theta_{\max} = \pi/4$ and $d_{\max} = v^2/g$, in agreement with the example in Section 2.4. If $h \to \infty$ or $v \to 0$, then $\theta \approx 0$, which makes sense.

If we make use of conservation of energy (discussed in Chapter 4), it turns out that the final speed of the ball when it hits the ground is $v_f = \sqrt{v^2 + 2gh}$. The maximum distance in eq. (2.123) may therefore be written as (with $v_i \equiv v$ being the initial speed)

$$d_{\max} = \frac{v_i v_f}{g}\,. \tag{2.124}$$

Note that this is symmetric in $v_i$ and $v_f$, as it must be, because we could imagine the trajectory running backwards. Also, it equals zero if $v_i$ is zero, as it should. We can also write the angle $\theta$ in eq. (2.122) in terms of $v_f$ (instead of $h$). You can show that the result is $\tan\theta = v_i/v_f$. You can further show that this implies that the initial and final velocities are perpendicular to each other. The simplicity of all these results suggests that there is an easier way to derive them, but I have no clue what it is. ♣

15. **Redirected motion**

**First Solution:** We will use the results of Problem 14, namely eqs. (2.123) and (2.122), which say that an object projected from height $y$ at speed $v$ travels a maximum horizontal distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1 + \frac{2gy}{v^2}}\,, \tag{2.125}$$

and the optimal angle yielding this distance is

$$\sin\theta = \frac{1}{\sqrt{2 + 2gy/v^2}}\,. \tag{2.126}$$

In the problem at hand, the object is dropped from a height $h$, so conservation of energy (or integration of $mv\,dv/dy = -mg$) says that the speed at height $y$ is

$$v = \sqrt{2g(h-y)}. \tag{2.127}$$

Plugging this into eq. (2.125) shows that the maximum horizontal distance, as a function of $y$, is

$$d_{\max}(y) = 2\sqrt{h(h-y)}. \tag{2.128}$$

This is maximum when $y = 0$, in which case the distance is $d_{\max} = 2h$. Eq. (2.126) then gives the associated optimal angle as $\theta = 45^\circ$.

**Second Solution:** Assume that the greatest distance, $d_0$, is obtained when $y = y_0$ and $\theta = \theta_0$. And let the speed at $y_0$ be $v_0$. We will show that $y_0$ must be 0. We will do this by assuming that $y_0 \neq 0$ and explicitly constructing a situation that yields a greater distance.

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height $y_0$, it is traveling at an angle $\theta_0$ with respect to the horizontal. When it reaches the height $y_0$, the ball will have speed $v_0$ (by conservation of energy), so it will travel a horizontal distance $d_0$ from this point. But the ball already traveled a nonzero horizontal distance on its way up to the height $y_0$. We have therefore constructed a situation that yields a distance greater than $d_0$. Hence, the optimal setup must have $y_0 = 0$. Therefore, the maximum distance is obtained when $y = 0$, in which case the example in Section 2.4 says that the optimal angle is $\theta = 45^\circ$.

If we want the ball to go even further, we can simply dig a (wide enough) hole in the ground and have the ball bounce from the bottom of the hole.

16. **Maximum trajectory length**

Let $\theta$ be the angle at which the ball is thrown. Then the coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The ball reaches its maximum height at $t = v\sin\theta/g$, so the length of the trajectory is

$$\begin{aligned} L &= 2\int_0^{v\sin\theta/g}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt \\ &= 2\int_0^{v\sin\theta/g}\sqrt{(v\cos\theta)^2 + (v\sin\theta - gt)^2}\,dt \\ &= 2v\cos\theta\int_0^{v\sin\theta/g}\sqrt{1 + \left(\tan\theta - \frac{gt}{v\cos\theta}\right)^2}\,dt. \end{aligned} \tag{2.129}$$

Letting $z \equiv \tan\theta - gt/v\cos\theta$, we obtain

$$L = -\frac{2v^2\cos^2\theta}{g}\int_{\tan\theta}^{0}\sqrt{1+z^2}\,dz. \tag{2.130}$$

We can either look up this integral, or we can derive it by making a $z \equiv \sinh\alpha$ substitution. The result is

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\cdot\frac{1}{2}\left(z\sqrt{1+z^2}+\ln\left(z+\sqrt{1+z^2}\right)\right)\Big|_0^{\tan\theta} \\ &= \frac{v^2}{g}\left(\sin\theta+\cos^2\theta\,\ln\left(\frac{\sin\theta+1}{\cos\theta}\right)\right). \end{aligned} \tag{2.131}$$

As a double-check, you can verify that $L = 0$ when $\theta = 0$, and $L = v^2/g$ when $\theta = 90°$. Taking the derivative of eq. (2.131) to find the maximum, we obtain

$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\left(\frac{\cos\theta}{1+\sin\theta}\right)\frac{\cos^2\theta+(1+\sin\theta)\sin\theta}{\cos^2\theta}. \tag{2.132}$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right). \tag{2.133}$$

Finally, you can show numerically that the solution for $\theta$ is $\theta_0 \approx 56.5°$.

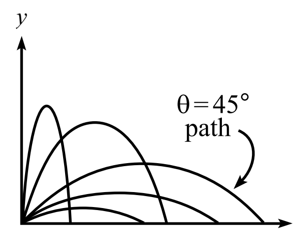



**Figure 2.30**

REMARK: A few possible trajectories are shown Fig. 2.30. Since it is well known that $\theta = 45°$ provides the maximum *horizontal* distance, it follows from the figure that the $\theta_0$ yielding the arc of maximum *length* must satisfy $\theta_0 \geq 45°$. The exact angle, however, requires the above detailed calculation. ♣

17. **Maximum trajectory area**

Let $\theta$ be the angle at which the ball is thrown. Then the coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The total time in the air is $2(v\sin\theta)/g$, so the area under the trajectory is

$$\begin{aligned} A &= \int_0^{x_{\max}} y\,dx \\ &= \int_0^{2v\sin\theta/g}\left((v\sin\theta)t - \frac{gt^2}{2}\right)v\cos\theta\,dt \\ &= \frac{2v^4}{3g^2}\sin^3\theta\cos\theta. \end{aligned} \tag{2.134}$$

Taking the derivative, we find that the maximum occurs when $\tan\theta = \sqrt{3}$, that is, when

$$\theta = 60°. \tag{2.135}$$

The maximum area is then $A_{\max} = \sqrt{3}v^4/8g^2$. Note that by dimensional analysis, we know that the area, which has dimensions of distance squared, must be proportional to $v^4/g^2$.

18. **Bouncing ball**

The ball travels $2h$ during the first up-and-down journey. It travels $2hf$ during the second, then $2hf^2$ during the third, and so on. Therefore, the total distance traveled is

$$\begin{aligned} D &= 2h(1 + f + f^2 + f^3 + \cdots) \\ &= \frac{2h}{1-f} \,. \end{aligned} \tag{2.136}$$

The time it takes to fall down during the first up-and-down is obtained from $h = gt^2/2$. Therefore, the time for the first up-and-down equals $2t = 2\sqrt{2h/g}$. Likewise, the time for the second up-and-down equals $2\sqrt{2(hf)/g}$. Each successive up-and-down time decreases by a factor of $\sqrt{f}$, so the total time is

$$\begin{aligned} T &= 2\sqrt{\frac{2h}{g}}\left(1 + f^{1/2} + f^1 + f^{3/2} + \cdots\right) \\ &= 2\sqrt{\frac{2h}{g}} \cdot \frac{1}{1-\sqrt{f}} \,. \end{aligned} \tag{2.137}$$

The average speed equals

$$\frac{D}{T} = \frac{\sqrt{gh/2}}{1+\sqrt{f}} \,. \tag{2.138}$$

REMARK: The average speed for $f \approx 1$ is roughly half of the average speed for $f \approx 0$. This may seem somewhat counterintuitive, because in the $f \approx 0$ case the ball slows down far more quickly than in the $f \approx 1$ case. But the $f \approx 0$ case consists of essentially only one bounce, and the average speed for that one bounce is the largest of any bounce. Both $D$ and $T$ are smaller for $f \approx 0$ than for $f \approx 1$, but $T$ is smaller by a larger factor. ♣

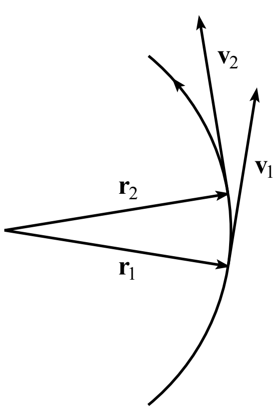



**Figure 2.31**

19. **Centripetal acceleration**

The position and velocity vectors at two nearby times are shown in Fig. 2.31. Their differences, $\Delta\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ and $\Delta\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1$, are shown in Fig. 2.32. The angle between the $\mathbf{v}$'s is the same as the angle between the $\mathbf{r}$'s, because each $\mathbf{v}$ makes a right angle with the corresponding $\mathbf{r}$. The triangles in Fig. 2.32 are therefore similar, so we have

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{r} \,, \tag{2.139}$$

where $r \equiv |\mathbf{r}|$ and $v \equiv |\mathbf{v}|$. Dividing eq. (2.139) through by $\Delta t$ gives

$$\frac{1}{v}\left|\frac{\Delta\mathbf{v}}{\Delta t}\right| = \frac{1}{r}\left|\frac{\Delta\mathbf{r}}{\Delta t}\right| \quad \Longrightarrow \quad \frac{|\mathbf{a}|}{v} = \frac{|\mathbf{v}|}{r} \quad \Longrightarrow \quad a = \frac{v^2}{r} \,. \tag{2.140}$$

We have assumed that $\Delta t$ is infinitesimal here, which allows us to get rid of the $\Delta$'s in favor of instantaneous quantities.

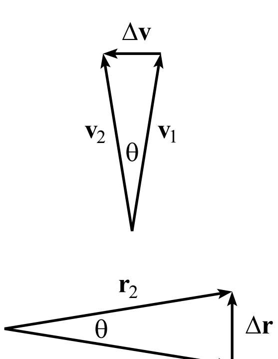



**Figure 2.32**

20. **Free particle**

For zero force, eqs. (2.52) give

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2, \\ r\ddot{\theta} &= -2\dot{r}\dot{\theta}. \end{aligned} \tag{2.141}$$

Separating variables in the second equation and integrating yields

$$\int \frac{\ddot{\theta}}{\dot{\theta}} = -\int \frac{2\dot{r}}{r} \qquad \Longrightarrow \qquad \ln\dot{\theta} = -2\ln r + C \qquad \Longrightarrow \qquad \dot{\theta} = \frac{D}{r^2}\,, \qquad (2.142)$$

where $D = e^C$ is a constant of integration, determined by the initial conditions.[21] Substituting this value of $\dot{\theta}$ into the first of eqs. (2.141), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r^2}\right)^2 \qquad \Longrightarrow \qquad \int \ddot{r}\dot{r} = D^2\int \frac{\dot{r}}{r^3} \qquad \Longrightarrow \qquad \frac{\dot{r}^2}{2} = -\frac{D^2}{2r^2} + E. \quad (2.143)$$

We want $\dot{r} = 0$ when $r = r_0$, which implies that $E = D^2/2r_0^2$. Therefore,

$$\dot{r} = V\sqrt{1 - \frac{r_0^2}{r^2}}\,, \qquad (2.144)$$

where $V \equiv D/r_0$. Separating variables and integrating gives

$$\int \frac{r\dot{r}}{\sqrt{r^2 - r_0^2}} = V \qquad \Longrightarrow \qquad \sqrt{r^2 - r_0^2} = Vt \qquad \Longrightarrow \qquad r = \sqrt{r_0^2 + (Vt)^2}\,, \qquad (2.145)$$

where the constant of integration is zero, because we have chosen $t = 0$ to correspond with $r = r_0$. Plugging this value for $r$ into the $\dot{\theta} = D/r^2 \equiv Vr_0/r^2$ result in eq. (2.142) gives

$$\int d\theta = \int \frac{Vr_0\,dt}{r_0^2 + (Vt)^2} \qquad \Longrightarrow \qquad \theta = \tan^{-1}\left(\frac{Vt}{r_0}\right) \qquad \Longrightarrow \qquad \cos\theta = \frac{r_0}{\sqrt{r_0^2 + (Vt)^2}}\,. \qquad (2.146)$$

Finally, combining this with the result for $r$ in eq. (2.145) gives $\cos\theta = r_0/r$, as desired.

21. **A force** $F_\theta = \dot{r}\dot{\theta}$

With the given force, eqs. (2.52) become

$$\begin{aligned} 0 &= m(\ddot{r} - r\dot{\theta}^2), \\ m\dot{r}\dot{\theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \qquad (2.147)$$

The second of these equations gives $-\dot{r}\dot{\theta} = r\ddot{\theta}$. Therefore,

$$\int \frac{\ddot{\theta}}{\dot{\theta}} = -\int \frac{\dot{r}}{r} \qquad \Longrightarrow \qquad \ln\dot{\theta} = -\ln r + C \qquad \Longrightarrow \qquad \dot{\theta} = \frac{D}{r}\,, \qquad (2.148)$$

where $D = e^C$ is a constant of integration, determined by the initial conditions. Substituting this value of $\dot{\theta}$ into the first of eqs. (2.147), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r}\right)^2 \qquad \Longrightarrow \qquad \int \ddot{r}\dot{r} = D^2\int \frac{\dot{r}}{r} \qquad \Longrightarrow \qquad \frac{\dot{r}^2}{2} = D^2\ln r + E. \quad (2.149)$$

Therefore,

$$\dot{r} = \sqrt{A\ln r + B}\,, \qquad (2.150)$$

where $A \equiv 2D^2$ and $B \equiv 2E$.

[21]The statement that $r^2\dot{\theta}$ is constant is simply the statement of conservation of angular momentum, because $r^2\dot{\theta} = r(r\dot{\theta}) = rv_\theta$. More on this in Chapters 6 and 7.

# Chapter 3

# Oscillations



In this chapter we will discuss oscillatory motion. The simplest examples of such motion are a swinging pendulum and a mass on a spring, but it is possible to make a system more complicated by introducing a damping force and/or an external driving force. We will study all of these cases.

We are interested in oscillatory motion for two reasons. First, we study it because we *can* study it. This is one of the few systems in physics where we can solve the motion exactly. There's nothing wrong with looking under the lamppost every now and then. Second, oscillatory motion is ubiquitous in nature, for reasons that will become clear in Section 4.2. If there was ever a type of physical system worthy of study, this is it.

We'll jump right into some math in Section 3.1. And then in Section 3.2 we'll show how the math is applied to the physics.

## 3.1 Linear differential equations

A *linear differential equation* is one in which $x$ and its time derivatives enter only through their first powers. An example is $3\ddot{x}+7\dot{x}+x=0$. An example of a nonlinear differential equation is $3\ddot{x}+7\dot{x}^2+x=0$.

If the right-hand side of the equation is zero, then we use the term *homogeneous* differential equation. If the right-hand side is some function of $t$, as in the case of $3\ddot{x}-4\dot{x}=9t^2-5$, then we use the term *inhomogeneous* differential equation. The goal of this chapter is to learn how to solve these two types of equations. Linear differential equations come up again and again in physics, so we had better find a systematic method of solving them.

The techniques that we will use are best learned through examples, so let's solve a few differential equations, starting with some simple ones. Throughout this chapter, $x$ will be understood to be a function of $t$. Hence, a dot will denote time differentiation.

---

**Example 1 ($\dot{x}=ax$):** This is a very simple differential equation. There are two ways (at least) to solve it.

**First method:** Separate variables to obtain $dx/x = a\,dt$, and then integrate to obtain $\ln x = at + c$. Exponentiate to obtain

$$x = Ae^{at}, \tag{3.1}$$

where $A \equiv e^c$ is a constant factor. $A$ is determined by the value of $x$ at, say, $t = 0$.

**Second method:** Guess an exponential solution, that is, one of the form $x = Ae^{\alpha t}$. Substitution into $\dot{x} = ax$ immediately gives $\alpha = a$. Therefore, the solution is $x = Ae^{at}$. Note that we can't solve for $A$, due to the fact that our differential equation is homogeneous and linear in $x$ (translation: $A$ cancels out). $A$ is determined from the initial condition.

This method may seem a bit silly, and somewhat cheap. But as we will see below, guessing these exponential functions (or sums of them) is actually the most general thing we can try, so the method is indeed quite general.

REMARK: Using this method, you may be concerned that although we have found one solution, we might have missed another one. But the general theory of differential equations says that a first-order linear equation has only one independent solution (we'll just accept this fact here). So if we find one solution, then we know that we've found the whole thing. ♣

---

**Example 2 ($\ddot{x} = ax$):** If $a$ is negative, then this equation describes the oscillatory motion of, say, a spring. If $a$ is positive, then it describes exponentially growing or decaying motion. There are two ways (at least) to solve this equation.

**First method:** We can use the separation-of-variables method of Section 2.3 here, because our system is one in which the force depends on only the position $x$. But this method is rather cumbersome, as you found if you did Exercise 2.10 or 2.11. It will certainly work, but in the case where our equation is a *linear* function of $x$, there is a much simpler method:

**Second method:** As in the first example above, we can guess a solution of the form $x(t) = Ae^{\alpha t}$ and then find out what $\alpha$ must be. Again, we can't solve for $A$, because it cancels out. Plugging $Ae^{\alpha t}$ into $\ddot{x} = ax$ gives $\alpha = \pm\sqrt{a}$. We have therefore found two solutions. The most general solution is an arbitrary linear combination of these,

$$x(t) = Ae^{\sqrt{a}\,t} + Be^{-\sqrt{a}\,t}, \tag{3.2}$$

as you can quickly check. $A$ and $B$ are determined from the initial conditions.

VERY IMPORTANT REMARK: The fact that the sum of two different solutions is again a solution to our equation is a monumentally important property of *linear* differential equations. This property does *not* hold for nonlinear differential equations, for example $\ddot{x}^2 = x$, because the act of squaring after adding the two solutions produces a cross term which destroys the equality, as you should check.

This property is called the *principle of superposition*. That is, superimposing two solutions yields another solution. This quality makes theories in physics that are governed by linear equations *much* easier to deal with than those that are governed by nonlinear ones. General Relativity, for example, is permeated with nonlinear equations, and solutions to most General Relativity systems are extremely difficult to come by.

For equations with one main condition
(Those linear), you have permission
To take your solutions,
With firm resolutions,
And add them in superposition. ♣

Let's say a little more about the solution in eq. (3.2). If $a$ is negative, then let's define $a \equiv -\omega^2$, where $\omega$ is a real number. The solution now becomes $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$, this can be written in terms of trig functions, if desired. Various ways of writing the solution are:

$$
\begin{aligned}
x(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\
x(t) &= C\cos\omega t + D\sin\omega t, \\
x(t) &= E\cos(\omega t + \phi_1), \\
x(t) &= F\sin(\omega t + \phi_2).
\end{aligned} \tag{3.3}
$$

The various constants here are related to each other. For example, $C = E\cos\phi_1$ and $D = -E\sin\phi_1$, which follow from the cosine sum formula. Note that there are two free parameters in each of the above expressions for $x(t)$. These parameters are determined from the initial conditions (say, the position and speed at $t = 0$). Depending on the specifics of a given problem, one of the above forms will work better than the others.

If $a$ is positive, then let's define $a \equiv \omega^2$, where $\omega$ is a real number. The solution in eq. (3.2) now becomes $x(t) = Ae^{\omega t} + Be^{-\omega t}$. Using $e^{\theta} = \cosh\theta + \sinh\theta$, this can be written in terms of hyperbolic trig functions, if desired. Various ways of writing the solution are:

$$
\begin{aligned}
x(t) &= Ae^{\omega t} + Be^{-\omega t} \\
x(t) &= C\cosh\omega t + D\sinh\omega t, \\
x(t) &= E\cosh(\omega t + \phi_1), \\
x(t) &= F\sinh(\omega t + \phi_2).
\end{aligned} \tag{3.4}
$$

Again, the various constants are related to each other. If you are unfamiliar with the hyperbolic trig functions, a few facts are listed in Appendix A.

REMARKS: Although the solution in eq. (3.2) is completely correct for both signs of $a$, it is generally more illuminating to write the negative-$a$ solutions in either the trig forms or the $e^{\pm i\omega t}$ exponential form where the $i$'s are explicit.

As in the first example above, you may be concerned that although we have found two solutions to the equation, we might have missed others. But the general theory of differential equations says that our second-order linear equation has only two independent solutions. Therefore, having found two independent solutions, we know that we've found them all. ♣

The usefulness of this method of guessing exponential solutions cannot be overemphasized. It may seem somewhat restrictive, but it works. The examples in the remainder of this chapter should convince you of this.

This is our method, essential,
For equations we solve, differential.
It gets the job done,

And it's even quite fun.
We just try a routine exponential.

---

**Example 3 ($\ddot{x} + 2\gamma\dot{x} + ax = 0$):** This will be our last mathematical example, and then we'll start doing some physics. As we will see later, this example pertains to a damped harmonic oscillator. We have put a factor of 2 in the coefficient of $\dot{x}$ here to make some later formulas look nicer.

Note that the force in this example (if we allow ourselves to think physically for a moment) is $-2\gamma\dot{x} - ax$ (times $m$), which depends on both $v$ and $x$. Our methods of Section 2.3 therefore don't apply. This leaves us with only our method of guessing an exponential solution, $Ae^{\alpha t}$. Plugging this into the given equation, and cancelling the nonzero factor of $Ae^{\alpha t}$, gives

$$\alpha^2 + 2\gamma\alpha + a = 0. \tag{3.5}$$

The solutions for $\alpha$ are

$$-\gamma \pm \sqrt{\gamma^2 - a}. \tag{3.6}$$

Call these $\alpha_1$ and $\alpha_2$. Then the general solution to our equation is

$$\begin{aligned} x(t) &= Ae^{\alpha_1 t} + Be^{\alpha_2 t} \\ &= e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right). \end{aligned} \tag{3.7}$$

Well, well, our method of trying $Ae^{\alpha t}$ doesn't look so trivial anymore...

If $\gamma^2 - a < 0$, then we can write our answer in terms of sines and cosines, so we have oscillatory motion that decreases in time due to the $e^{-\gamma t}$ factor (or it increases, if $\gamma < 0$, but this is rarely physical). If $\gamma^2 - a > 0$, then we have exponential motion. We'll talk more about these different possibilities in Section 3.3.

---

In general, if we have an $n$-th order homogeneous linear differential equation,

$$\frac{d^n x}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + c_1\frac{dx}{dt} + c_0 x = 0, \tag{3.8}$$

then our strategy is to guess an exponential solution, $x(t) = Ae^{\alpha t}$, and to then (in theory) solve the resulting $n$th order equation (namely $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0$) for $\alpha$, to obtain the solutions $\alpha_1, \ldots, \alpha_n$. The general solution for $x(t)$ is then

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \cdots + A_n e^{\alpha_n t}, \tag{3.9}$$

where the $A_i$ are determined from the initial conditions. In practice, however, we will rarely encounter differential equations of degree higher than 2. Note: if some of the $\alpha_i$ happen to be equal, then eq. (3.9) is not valid, so a modification is needed. We will encounter such a situation in Section 3.3.

## 3.2 Simple harmonic motion

Let's now do some real live physical problems. We'll start with simple harmonic motion. This is the motion undergone by a particle subject to a force $F(x) = -kx$.

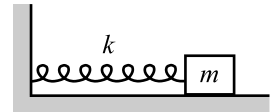



**Figure 3.1**

The classic system that undergoes simple harmonic motion is a mass attached to a spring (see Fig. 3.1). A typical spring has a force of the form $F(x) = -kx$, where $x$ is the displacement from equilibrium. This is "Hooke's law," and it holds as long as the spring isn't stretched too far; eventually this expression breaks down for any real spring. Assuming a $-kx$ force, $F = ma$ gives $-kx = m\ddot{x}$, or

$$\ddot{x} + \omega^2 x = 0, \qquad \text{where } \omega \equiv \sqrt{\frac{k}{m}}\,. \tag{3.10}$$

This is simply the equation we studied in Example 2 in the previous section. From eq. (3.3), the solution to may be written as

$$x(t) = A\cos(\omega t + \phi), \tag{3.11}$$

where $A$ and $\phi$ are determined from the initial conditions. This trig solution shows that the system will oscillate back and forth forever in time.

Remark: The constants $A$ and $\phi$ are determined from the initial conditions. If, for example, $x(0) = 0$ and $\dot{x}(0) = v$, then we must have $A\cos\phi = 0$ and $-A\omega\sin\phi = v$. Hence, $\phi = \pi/2$, and $A = -v/\omega$. Therefore, the solution is $x(t) = -(v/\omega)\cos(\omega t + \pi/2)$. This looks a little nicer if we write it as $x(t) = (v/\omega)\sin(\omega t)$. So, given these initial conditions, we could have arrived at this result a little quicker if we had chosen the "sin" solution in eq. (3.3). ♣

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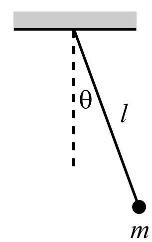


**Figure 3.2**

**Example (Simple pendulum):** Another classic system that undergoes (approximately) simple harmonic motion is the simple pendulum, that is, a mass that hangs on a massless string and swings in a vertical plane.

Let $\ell$ be the length of the string, and let $\theta$ be the angle the string makes with the vertical (see Fig. 3.2). Then the gravitational force on the mass in the tangential direction is $-mg\sin\theta$. So $F = ma$ in the tangential direction gives

$$-mg\sin\theta = m(\ell\ddot{\theta}) \tag{3.12}$$

The tension in the string exactly cancels the radial component of gravity, so the radial $F = ma$ serves only to tell us the tension, which we won't need here.

We will now enter the realm of approximations and assume that the amplitude of the oscillations is small. Without this approximation, the problem cannot be solved in closed form. Assuming $\theta$ is small, we can use $\sin\theta \approx \theta$ in eq. (3.12) to obtain

$$\ddot{\theta} + \omega^2\theta = 0, \qquad \text{where } \omega \equiv \sqrt{\frac{g}{\ell}}\,. \tag{3.13}$$

Therefore,

$$\theta(t) = A\cos(\omega t + \phi), \tag{3.14}$$

where $A$ and $\phi$ are determined from the initial conditions.

The true motion is arbitrarily close to this, for sufficiently small amplitudes. Exercise 8 deals with the higher-order corrections to the motion in the case where the amplitude is not small.

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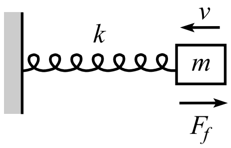

**Figure 3.3**

## 3.3 Damped harmonic motion

Consider a mass $m$ attached to the end of a spring with spring constant $k$. Let the mass be subject to a drag force proportional to its velocity, $F_f = -bv$; see Fig. 3.3.[1] What is the position as a function of time?[2]

The force on the mass is $F = -b\dot{x} - kx$. So $F = m\ddot{x}$ gives

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \tag{3.15}$$

where $2\gamma \equiv b/m$, and $\omega \equiv \sqrt{k/m}$. But this is exactly the equation we solved in Example 3 in Section 3.1 (with $a \to \omega^2$). Now, however, we have the physical restrictions that $\gamma > 0$ and $\omega^2 > 0$. Letting $\Omega^2 \equiv \gamma^2 - \omega^2$ for simplicity, we may write the solution in eq. (3.7) as

$$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right), \qquad \text{where } \ \Omega^2 \equiv \gamma^2 - \omega^2. \tag{3.16}$$

There are three cases to consider.

### Case 1: Underdamping ($\Omega^2 < 0$)

In this case, $\omega > \gamma$. Since $\Omega$ is imaginary, let us define the real number $\tilde{\omega} \equiv \sqrt{\omega^2 - \gamma^2}$, so that $\Omega = i\tilde{\omega}$. Eq. (3.16) then gives

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right) \\ &\equiv e^{-\gamma t}C\cos(\tilde{\omega}t + \phi). \end{aligned} \tag{3.17}$$

These two forms are equivalent. Depending on the given problem, one of these expressions will inevitably work better than the other. Or perhaps one of the other forms in eq. (3.3) (times $e^{-\gamma t}$) will be the most useful one.

Using $e^{i\theta} = \cos\theta + i\sin\theta$, the constants in eq. (3.17) are related by $A + B = C\cos\phi$ and $A - B = iC\sin\phi$. In a physical problem, $x(t)$ is real, so we must have $A^* = B$, where the star denotes complex conjugation. The two constants $A$ and $B$, or $C$ and $\phi$, are determined from the initial conditions.

The cosine form makes it apparent that the motion is harmonic motion whose amplitude decreases in time, due to the $e^{-\gamma t}$ factor. A plot of such motion is shown in Fig. 3.4. Note that the frequency of the motion, $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$, is less than the natural frequency, $\omega$, of the undamped oscillator.

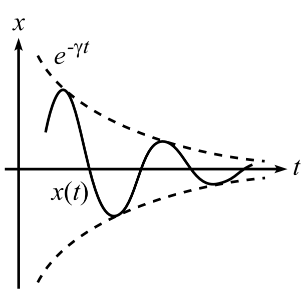

**Figure 3.4**

REMARKS: If $\gamma$ is very small, then $\tilde{\omega} \approx \omega$, which makes sense because we almost have an undamped oscillator. If $\gamma$ is very close to $\omega$, then $\tilde{\omega} \approx 0$. So the oscillations are very

[1]The subscript $f$ stands for "friction" here. We'll have to save the letter $d$ for "driving" in the next section.

[2]We choose to study this $F_f = -bv$ damping force because (1) it is linear in $x$, which will allow us to solve for the motion, and (2) it is a perfectly realistic force; an object moving at a slow speed through a fluid will generally experience a drag force proportional to its velocity. Note that this $F_f = -bv$ force is *not* the force that a mass would feel if it were placed on a table with friction. In that case the drag force would be (roughly) constant.

slow. Of course, for very small $\tilde{\omega}$ it is hard to even tell that the oscillations exist, because they will damp out on a time scale of order $1/\gamma$, which will be short compared to the long time scale of the oscillations, $1/\tilde{\omega}$. ♣

### Case 2: Overdamping ($\Omega^2 > 0$)

In this case, $\omega < \gamma$. $\Omega$ is real (and taken to be positive), so eq. (3.16) gives

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}. \tag{3.18}$$

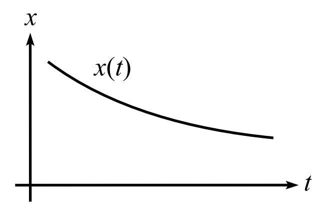

**Figure 3.5**

There is no oscillatory motion in this case; see Fig. 3.5. Note that $\gamma > \Omega \equiv \sqrt{\gamma^2 - \omega^2}$, so both of the exponents are negative. The motion therefore goes to zero for large $t$. This had better be the case, because a real spring is certainly not going to have the motion head off to infinity. If we had obtained a positive exponent somehow, we'd know we had made a mistake.

REMARKS: If $\gamma$ is just slightly larger than $\omega$, then $\Omega \approx 0$, so the two terms in (3.18) are roughly equal, and we essentially have exponential decay, according to $e^{-\gamma t}$. If $\gamma \gg \omega$ (that is, strong damping), then $\Omega \approx \gamma$, so the first term in (3.18) dominates, and we essentially have exponential decay according to $e^{-(\gamma-\Omega)t}$. We can be somewhat quantitative about this by approximating $\Omega$ as $\Omega \equiv \sqrt{\gamma^2 - \omega^2} = \gamma\sqrt{1 - \omega^2/\gamma^2} \approx \gamma(1 - \omega^2/2\gamma^2)$. Hence, the exponential behavior goes like $e^{-\omega^2 t/2\gamma}$. This is slow decay (that it, slow compared to $t \sim 1/\omega$), which makes sense if the damping is very strong. The mass slowly creeps back to the origin, as in the case of a weak spring immersed in molasses. ♣

### Case 3: Critical damping ($\Omega^2 = 0$)

In this case, $\gamma = \omega$. Eq. (3.15) therefore becomes $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$. In this special case, we have to be careful in solving our differential equation. The solution in eq. (3.16) is not valid, because in the procedure leading to eq. (3.7), the roots $\alpha_1$ and $\alpha_2$ are equal (to $-\gamma$), so we have really found only one solution, $e^{-\gamma t}$. We'll just invoke here the result from the theory of differential equations that says that in this special case, the other solution is of the form $te^{-\gamma t}$.

REMARK: You should check explicitly that $te^{-\gamma t}$ solves the equation $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$. Or if you want to, you can derive it in the spirit of Problem 1. In the more general case where there are $n$ identical roots in the procedure leading to eq. (3.9) (call them all $\alpha$), the $n$ independent solutions to the differential equation are $t^k e^{\alpha t}$, for $0 \le k \le (n-1)$. But more often than not, there are no repeated roots, so you don't have to worry about this. ♣

Our solution is therefore of the form

$$x(t) = e^{-\gamma t}(A + Bt). \tag{3.19}$$

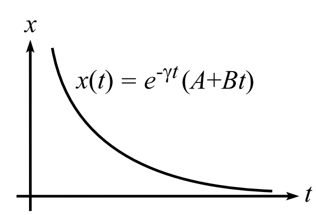

**Figure 3.6**

The exponential factor eventually wins out over the $Bt$ term, so the motion goes to zero for large $t$ (see Fig. 3.6).

If we are given a spring with a fixed $\omega$, and if we look at the system for different values of $\gamma$, then critical damping (when $\gamma = \omega$) is the case where the motion converges to zero in the quickest way (which is like $e^{-\omega t}$). This is true because in the underdamped case ($\gamma < \omega$), the envelope of the oscillatory motion goes like $e^{-\gamma t}$, which goes to zero slower than $e^{-\omega t}$, because $\gamma < \omega$. And in the overdamped case ($\gamma > \omega$), the dominant piece is the $e^{-(\gamma-\Omega)t}$ term. And as you can verify, if $\gamma > \omega$ then $\gamma - \Omega \equiv \gamma - \sqrt{\gamma^2 - \omega^2} < \omega$, so this motion also goes to zero slower than $e^{-\omega t}$.

Critical damping is very important in many real systems, such as screen doors and shock absorbers, where the goal is to have the system head to zero (without overshooting and bouncing around) as fast as possible.

## 3.4 Driven (and damped) harmonic motion

Before we examine driven harmonic motion, we must learn how to solve a new type of differential equation. How can we solve something of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = C_0 e^{i\omega_0 t}, \tag{3.20}$$

where $\gamma$, $a$, $\omega_0$, and $C_0$ are given quantities? This is an inhomogeneous differential equation, due to the term on the right-hand side. It's not very physical, because the right-hand side is complex, but let's not worry about this for now. Equations of this sort will come up again and again, and fortunately there is a nice and easy (although sometimes messy) method for solving them. As usual, the method involves making a reasonable guess, plugging it in, and seeing what condition comes out.

Since we have the $e^{i\omega_0 t}$ sitting on the right-hand side of eq. (3.20), let's guess a solution of the form $x(t) = Ae^{i\omega_0 t}$. $A$ will depend on $\omega_0$, among other things, as we will see. Plugging this guess into eq. (3.20) and cancelling the non-zero factor of $e^{i\omega_0 t}$, we obtain

$$(-\omega_0^2)A + 2\gamma(i\omega_0)A + aA = C_0. \tag{3.21}$$

Solving for $A$, we find that our solution for $x$ is

$$x(t) = \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}. \tag{3.22}$$

Note the differences between this technique and the one in Example 3 in Section 3.1. In that example, the goal was to determine the $\alpha$ in $x(t) = Ae^{\alpha t}$. And there was no way to solve for $A$; the initial conditions determined $A$. But in the present technique, the $\omega_0$ in $x(t) = Ae^{i\omega_0 t}$ is a given quantity, and the goal is to solve for $A$ in terms of the given constants. Therefore, in the solution in eq. (3.22), there are *no free constants* to be determined from the initial conditions. We've found one particular solution, and we're stuck with it. The term *particular solution* is what people use for eq. (3.22).

With no freedom to adjust the solution in eq. (3.22), how can we satisfy an arbitrary set of initial conditions? Fortunately, eq. (3.22) does not represent the most general solution to eq. (3.20). The most general solution is the sum of our particular solution in eq. (3.22), *plus* the "homogeneous" solution we found in eq. (3.7). This sum is certainly a solution, because the solution in eq. (3.7) was explicitly constructed to yield zero when plugged into the left-hand side of eq. (3.20).

Therefore, tacking it onto our particular solution doesn't change the equality in eq. (3.20), because the left side is linear. The principle of superposition has saved the day. The complete solution to eq. (3.20) is therefore

$$x(t) = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right) + \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right)e^{i\omega_0 t}, \tag{3.23}$$

where $A$ and $B$ are determined from the initial conditions.

With superposition in mind, it is clear what the strategy should be if we have a slightly more general equation to solve, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}. \tag{3.24}$$

Simply solve the equation with only the first term on the right. Then solve the equation with only the second term on the right. Then add the two solutions. And then add on the homogeneous solution from eq. (3.7). We are able to apply the principle of superposition because the left-hand side of eq. (3.24) is linear.

Finally, let's look at the case where we have many such terms on the right-hand side, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}. \tag{3.25}$$

We simply have to solve $N$ different equations, each with only one of the $N$ terms on the right-hand side. Then we add up all the solutions, and then we add on the homogeneous solution from eq. (3.7). If $N$ is infinite, that's fine; we'll just have to add up an infinite number of solutions. This is the principle of superposition at its best.

REMARK: The previous paragraph, combined with a basic result from Fourier analysis, allows us to solve (in principle) any equation of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = f(t). \tag{3.26}$$

Fourier analysis says that any (nice enough) function $f(t)$ may be decomposed into its Fourier components,

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t} d\omega. \tag{3.27}$$

In this continuous sum, the functions $g(\omega)$ take the place of the coefficients $C_n$ in eq. (3.25). So, if $S_\omega(t)$ is the solution for $x(t)$ when there is only the term $e^{i\omega t}$ on the right-hand side of eq. (3.26) (that is, $S_\omega(t)$ is the solution given in eq. (3.22), without the $C_0$ factor), then the principle of superposition tells us that the complete particular solution to (3.26) is

$$x(t) = \int_{-\infty}^{\infty} g(\omega)S_\omega(t)\, d\omega. \tag{3.28}$$

Finding the coefficients $g(\omega)$ is the hard part (or, rather, the messy part), but we won't get into that here. We won't do anything with Fourier analysis in this book, but we just wanted to let you know that it *is* possible to solve (3.26) for any function $f(t)$. Most of the functions we'll consider will be nice functions like $\cos\omega_0 t$, for which the Fourier decomposition is simply the finite sum, $\cos\omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$. ♣

Let's now do a physical example.

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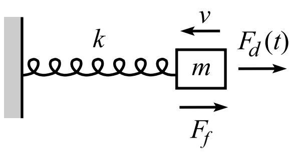


**Figure 3.7**

**Example (Damped and driven spring):** Consider a spring with spring constant $k$. A mass $m$ at the end of the spring is subject to a drag force proportional to its velocity, $F_f = -bv$. The mass is also subject to a driving force, $F_d(t) = F_d \cos\omega_d t$ (see Fig. 3.7). What is its position as a function of time?

**Solution:** The force on the mass is $F(x, \dot{x}, t) = -b\dot{x} - kx + F_d \cos\omega_d t$. So $F = ma$ gives

$$\begin{aligned} \ddot{x} + 2\gamma\dot{x} + \omega^2 x &= F\cos\omega_d t \\ &= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right). \end{aligned} \tag{3.29}$$

where $2\gamma \equiv b/m$, $\omega \equiv \sqrt{k/m}$, and $F \equiv F_d/m$. Note that there are two different frequencies here, $\omega$ and $\omega_d$, which need not have anything to do with each other. Eq. (3.22), along with the principle of superposition, tells us that our particular solution is

$$x_p(t) = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right)e^{i\omega_d t} + \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right)e^{-i\omega_d t}. \tag{3.30}$$

The complete solution is the sum of this particular solution and the homogeneous solution from eq. (3.16).

Let's now eliminate the $i$'s in eq. (3.30) (which we had better be able to do, because $x$ must be real), and write $x$ in terms of sines and cosines. Getting the $i$'s out of the denominators, and using $e^{i\theta} = \cos\theta + i\sin\theta$, we find (after a little work)

$$x_p(t) = \left(\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\sin\omega_d t. \tag{3.31}$$

Remarks: If you want, you can solve eq. (3.29) simply by taking the real part of the solution to eq. (3.20), that is, the $x(t)$ in eq. (3.22). This is true because if we take the real part of eq. (3.20), we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\big(\mathrm{Re}(x)\big) + 2\gamma\frac{d}{dt}\big(\mathrm{Re}(x)\big) + a\big(\mathrm{Re}(x)\big) &= \mathrm{Re}\left(C_0 e^{i\omega_0 t}\right) \\ &= C_0\cos(\omega_0 t). \end{aligned} \tag{3.32}$$

In other words, if $x$ satisfies eq. (3.20) with a $C_0 e^{i\omega_0 t}$ on the right-hand side, then $\mathrm{Re}(x)$ satisfies it with a $C_0\cos(\omega_0 t)$ on the right.

At any rate, it is clear that (with $C_0 = F$) the real part of the solution in eq. (3.22) does indeed give the result in eq. (3.31), because in eq. (3.30) we simply took half of a quantity plus its complex conjugate, which is the real part.

If you don't like using complex numbers, another way of solving eq. (3.29) is to keep it in the form with the $\cos\omega_d t$ on the right, and simply guess a solution of the form $A\cos\omega_d t + B\sin\omega_d t$, and then solve for $A$ and $B$ (this is the task of Problem 5). The result will be eq. (3.31). ♣

We can now write eq. (3.31) in a very simple form. If we define

$$R \equiv \sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}, \tag{3.33}$$

then we can rewrite eq. (3.31) as

$$\begin{aligned} x_p(t) &= \frac{F}{R}\left(\frac{(\omega^2-\omega_d^2)}{R}\cos\omega_d t + \frac{2\gamma\omega_d}{R}\sin\omega_d t\right) \\ &\equiv \frac{F}{R}\cos(\omega_d t-\phi), \end{aligned} \tag{3.34}$$

where $\phi$ is defined by

$$\cos\phi = \frac{\omega^2-\omega_d^2}{R}, \qquad \sin\phi = \frac{2\gamma\omega_d}{R} \qquad \Longrightarrow \qquad \tan\phi = \frac{2\gamma\omega_d}{\omega^2-\omega_d^2}. \tag{3.35}$$

The triangle describing the angle $\phi$ is shown in Fig. 3.8. Note that $0 \le \phi \le \pi$, because $\sin\phi$ is positive.

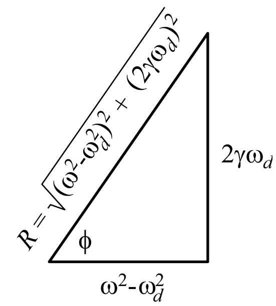


**Figure 3.8**

Recalling the homogeneous solution in eq. (3.16), we can write the complete solution to eq. (3.29) as

$$x(t) = \frac{F}{R}\cos(\omega_d t-\phi) + e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right). \tag{3.36}$$

The constants $A$ and $B$ are determined from the initial conditions. Note that if there is any damping at all in the system (that is, if $\gamma > 0$), then the homogeneous part of the solution goes to zero for large $t$, and we are left with only the particular solution. In other words, the system approaches a definite $x(t)$, namely $x_p(t)$, independent of the initial conditions.

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**Resonance**

The amplitude of the motion given in eq. (3.34) is proportional to

$$\frac{1}{R} = \frac{1}{\sqrt{(\omega^2-\omega_d^2)^2 + (2\gamma\omega_d)^2}}. \tag{3.37}$$

Given $\omega_d$ and $\gamma$, this is maximum when $\omega = \omega_d$. Given $\omega$ and $\gamma$, it is maximum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$, as you can show in Exercise 15. But for weak damping (that is, $\gamma \ll \omega$, which is usually the case we are concerned with), this reduces to $\omega_d \approx \omega$ also.

The term *resonance* is used to describe the situation where the amplitude of the oscillations is as large as possible. It is quite reasonable that this is achieved when the driving frequency equals the frequency of the spring. But what is the value of the phase $\phi$ at resonance? Using eq. (3.35), we see that $\phi$ satisfies $\tan\phi \approx \infty$ when $\omega_d \approx \omega$. Therefore, $\phi = \pi/2$ (it is indeed $\pi/2$, and not $-\pi/2$, because the $\sin\phi$ in eq. (3.35) is positive), and the motion of the particle lags the driving force by a quarter of a cycle at resonance. For example, when the particle moves rightward past the origin (which means it has a quarter of a phase to go before it hits the maximum value of $x$), the force is already at its maximum. And when the particle makes it out to the maximum value of $x$, the force is already back to zero.

The fact the force is maximum when the particle is moving fastest makes sense from an energy point of view. If you want the amplitude to become large, then you will need to give the system as much energy as you can. That is, you must do as much work as possible on the system. And in order to do as much work as possible, you should have your force act over as large a distance as possible, which means that you should apply your force when the particle is moving fastest, that is, as it speeds past the origin. And similarly, you don't want to waste your force when the particle is barely moving near the endpoints of its motion.

In short, $v$ is the derivative of $x$ and therefore a quarter cycle ahead of $x$ (which is a general property of a sinusoidal function, as you can show). Since we want the force to be in phase with $v$ at resonance (by the above energy argument), we see that the force is also a quarter cycle ahead of $x$.

**The phase $\phi$**

Eq. (3.35) gives the phase of the motion as

$$\tan\phi = \frac{2\gamma\omega_d}{\omega^2 - \omega_d^2}\,, \tag{3.38}$$

where $0 \le \phi \le \pi$. Let's look at a few cases for $\omega_d$ (not necessarily at resonance) and see what the resulting phase $\phi$ is. Using eq. (3.38), we have:

- $\omega_d \approx 0 \Longrightarrow \phi \approx 0$. This means that the motion is in phase with the force. Intuitively, the mass moves very slowly if $\omega_d \approx 0$, so the motion basically just follows the force. A little more mathematically: Since there is essentially no acceleration, the net force is always essentially zero. This means that the driving force always essentially balances the spring force (that is, the two forces are 180° out of phase), because the damping force is negligible (since $v \approx 0$). But the spring force is 180° out of phase with the motion (because of the minus sign in $F = -kx$). Therefore, the driving force is in phase with the motion.

- $\omega_d \approx \omega_0 \Longrightarrow \phi \approx \pi/2$. This is the case of resonance, discussed above.

- $\omega_d \approx \infty \Longrightarrow \phi \approx \pi$. This means that the motion is out of phase with the force. The reason for this is the following. If $\omega_d \approx \infty$, then the mass moves back and forth very quickly. From eq. (3.37), we see that the amplitude is proportional to $1/\omega_d^2$. It then follows that the velocity goes like $1/\omega_d$. Therefore, both $x$ and $v$ are always small; the mass hardly moves. But if $x$ and $v$ are always small, then the spring and damping forces can be ignored. So we basically have a mass that feels only one force, the driving force. But we already understand very well a situation where a mass is subject to only one oscillating force: a mass on a spring. Now, the mass can't tell if it's being driven by an oscillating driving force, or being pushed and pulled by an oscillating spring force. They both feel the same. Therefore, both phases must be the same. But in the spring case, the minus sign in $F = -kx$ tells us that the force is 180° out of phase with the motion. Hence, the same result holds in the $\omega_d \approx \infty$ case.

## 3.5 Coupled oscillators

In the previous sections, we have dealt with only one function of time, $x(t)$. What if we have two functions of time, say $x(t)$ and $y(t)$, which are related by a pair of "coupled" differential equations? For example, we might have

$$\begin{aligned} 2\ddot{x} + \omega^2(5x - 3y) &= 0, \\ 2\ddot{y} + \omega^2(5y - 3x) &= 0. \end{aligned} \tag{3.39}$$

We'll assume $\omega^2 > 0$ here, but this isn't necessary. We call these equations "coupled" because there are $x$'s and $y$'s in both of them, and it is not immediately obvious how to separate them to solve for $x$ and $y$. There are two methods (at least) of solving these equations.

**First method:** Sometimes it is easy, as in this case, to find certain linear combinations of the given equations for which nice things happen. If we take the sum, we find

$$(\ddot{x} + \ddot{y}) + \omega^2(x + y) = 0. \tag{3.40}$$

This equation involves $x$ and $y$ only in the combination of their sum, $x + y$. With $z \equiv x + y$, eq. (3.40) is just our old friend, $\ddot{z} + \omega^2 z = 0$. The solution is

$$x + y = A_1 \cos(\omega t + \phi_1), \tag{3.41}$$

where $A_1$ and $\phi_1$ are determined from initial conditions. We may also take the difference of eqs. (3.39). The result is

$$(\ddot{x} - \ddot{y}) + 4\omega^2(x - y) = 0. \tag{3.42}$$

This equation involves $x$ and $y$ only in the combination of their difference, $x - y$. The solution is

$$x - y = A_2 \cos(2\omega t + \phi_2), \tag{3.43}$$

Taking the sum and difference of eqs. (3.41) and (3.43), we find that $x(t)$ and $y(t)$ are given by

$$\begin{aligned} x(t) &= B_1 \cos(\omega t + \phi_1) + B_2 \cos(2\omega t + \phi_2), \\ y(t) &= B_1 \cos(\omega t + \phi_1) - B_2 \cos(2\omega t + \phi_2), \end{aligned} \tag{3.44}$$

where the $B_i$'s are half of the $A_i$'s.

The strategy of this solution was simply to fiddle around and try to form differential equations that involved only one combination of the variables, namely eqs. (3.40) and (3.42). This allowed us to write down the familiar solution for these combinations, as in eqs. (3.41) and (3.43).

We've managed to solve our equations for $x$ and $y$. However, the more interesting thing that we've done is produce the equations (3.41) and (3.43). The combinations $(x + y)$ and $(x - y)$ are called the *normal coordinates* of the system. These are the combinations that oscillate with one pure frequency. The motion of $x$ and $y$ will, in general, look rather complicated, and it may be difficult to tell that the motion is really made up of just the two frequencies in eq. (3.44). But if you plot the values of $(x + y)$ and $(x - y)$ as time goes by, for *any* motion of the system, then you will find nice sinusoidal graphs, even if $x$ and $y$ are each behaving in a rather unpleasant manner.

**Second method:** In the above method, it was fairly easy to guess which combinations of eqs. (3.39) produced equations involving just one combination of $x$ and $y$, eqs. (3.40) and (3.42). But surely there are problems in physics where the guessing isn't so easy. What do we do then? Fortunately, there is a fail-proof method for solving for $x$ and $y$. It proceeds as follows.

In the spirit of Section 3.1, let us try a solution of the form $x = Ae^{i\alpha t}$ and $y = Be^{i\alpha t}$, which we will write, for convenience, as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}. \tag{3.45}$$

It is not obvious that there should exist solutions for $x$ and $y$ that have the same $t$ dependence, but let's try it and see what happens. Note that we've explicitly put the $i$ in the exponent, but there's no loss of generality here. If $\alpha$ happens to be imaginary, then the exponent is real. It's personal preference whether or not you put the $i$ in.

Plugging our guess into eqs. (3.39), and dividing through by $e^{i\omega t}$, we find

$$\begin{aligned} 2A(-\alpha^2) + 5A\omega^2 - 3B\omega^2 &= 0, \\ 2B(-\alpha^2) + 5B\omega^2 - 3A\omega^2 &= 0, \end{aligned} \tag{3.46}$$

or equivalently, in matrix form,

$$\begin{pmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.47}$$

This homogeneous equation for $A$ and $B$ has a nontrivial solution (that is, one where $A$ and $B$ aren't both 0) only if the matrix is *not* invertible. This is true because if it were invertible, then we could simply multiply through by the inverse to obtain $(A, B) = (0, 0)$.

When is a matrix invertible? There is a straightforward (although tedious) method for finding the inverse of a matrix. It involves taking cofactors, taking a transpose, and dividing by the determinant. The step that concerns us here is the division by the determinant. The inverse will exist if and only if the determinant is not zero. So we see that eq. (3.47) has a nontrivial solution only if the determinant equals zero. Because we seek a nontrivial solution, we must therefore have

$$\begin{aligned} 0 &= \begin{vmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{vmatrix} \\ &= 4\alpha^4 - 20\alpha^2\omega^2 + 16\omega^4. \end{aligned} \tag{3.48}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm 2\omega$. We have therefore found four types of solutions. If $\alpha = \pm\omega$, then we can plug this back into eq. (3.47) to obtain $A = B$. (Both equations give this same result. This was essentially the point of setting the determinant equal to zero.) And if $\alpha = \pm 2\omega$, then eq. (3.47) gives $A = -B$. (Again, the equations are redundant.) Note that we cannot solve specifically for $A$ and $B$, but only for their ratio. Adding up our four solutions according to the principle of superposition, we see that $x$ and $y$ take the general form (written in vector form for the sake of simplicity and bookkeeping),

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &\quad + A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2i\omega t}. \end{aligned} \tag{3.49}$$

The four $A_i$ are determined from the initial conditions.

We can rewrite eq. (3.49) in a somewhat cleaner form. If the coordinates $x$ and $y$ describe the positions of particles, they must be real. Therefore, $A_1$ and $A_2$ must be complex conjugates, and likewise for $A_3$ and $A_4$. If we then define some $\phi$'s and $B$'s via $A_2^* = A_1 \equiv (B_1/2)e^{i\phi_1}$ and $A_4^* = A_3 \equiv (B_2/2)e^{i\phi_2}$, we may rewrite our solution in the form, as you can verify,

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\omega t + \phi_2), \tag{3.50}$$

where the $B_i$ and $\phi_i$ are real (and are determined from the initial conditions). We have therefore reproduced the result in eq. (3.44).

It is clear from eq. (3.50) that the combinations $x + y$ and $x - y$ (the normal coordinates) oscillate with the pure frequencies, $\omega$ and $2\omega$, respectively. The combination $x + y$ makes the $B_2$ terms disappear, and the combination $x - y$ makes the $B_1$ terms disappear.

It is also clear that if $B_2 = 0$, then $x = y$ at all times, and they both oscillate with frequency $\omega$. And if $B_1 = 0$, then $x = -y$ at all times, and they both oscillate with frequency $2\omega$. These two pure-frequency motions are called the *normal modes*. They are labeled by the vectors $(1, 1)$ and $(1, -1)$, respectively. In describing a normal mode, both the vector and the frequency should be stated. The significance of normal modes will become clear in the following example.

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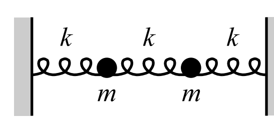

**Figure 3.9**

**Example (Two masses, three springs):** Consider two masses, $m$, connected to each other and to two walls by three springs, as shown in Fig. 3.9. The three springs have the same spring constant $k$. Find the positions of the masses as functions of time. What are the normal coordinates? What are the normal modes?

**Solution:** Let $x_1(t)$ and $x_2(t)$ be the positions of the left and right masses, respectively, relative to their equilibrium positions. Then the middle spring is stretched a distance $x_2 - x_1$. Therefore, the force on the left mass is $-kx_1 + k(x_2 - x_1)$, and the force on the right mass is $-kx_2 - k(x_2 - x_1)$. It's easy to make a mistake on the sign of the second term in these expressions. You can double check the sign by, say, looking at the force when $x_2$ is very big. At any rate, the second terms must have opposite signs in the two expressions, by Newton's third law.

With these forces, $F = ma$ on each mass gives, with $\omega^2 = k/m$,

$$\begin{aligned} \ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ \ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0. \end{aligned} \tag{3.51}$$

These are rather friendly-looking coupled equations, and we can see that the sum and difference are the useful combinations to take. The sum gives

$$(\ddot{x}_1 + \ddot{x}_2) + \omega^2(x_1 + x_2) = 0, \tag{3.52}$$

and the difference gives

$$(\ddot{x}_1 - \ddot{x}_2) + 3\omega^2(x_1 - x_2) = 0. \tag{3.53}$$

The solutions to these equations are the normal coordinates,

$$\begin{aligned} x_1 + x_2 &= A_+ \cos(\omega t + \phi_+), \\ x_1 - x_2 &= A_- \cos(\sqrt{3}\omega t + \phi_-). \end{aligned} \tag{3.54}$$

Taking the sum and difference of these normal coordinates, we have

$$\begin{aligned} x_1(t) &= B_+ \cos(\omega t + \phi_+) + B_- \cos(\sqrt{3}\omega t + \phi_-), \\ x_2(t) &= B_+ \cos(\omega t + \phi_+) - B_- \cos(\sqrt{3}\omega t + \phi_-), \end{aligned} \tag{3.55}$$

where the $B$'s are half of the $A$'s. They are determined from the initial conditions, along with the $\phi$'s.

Remark: We can also derive eqs. (3.55) by using the determinant method. Letting $x_1 = Ae^{i\alpha t}$ and $x_2 = Be^{i\alpha t}$ in eqs. (3.51), we see that for there to be a nontrivial solution for $A$ and $B$, we must have

$$\begin{aligned} 0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= \alpha^4 - 4\alpha^2\omega^2 + 3\omega^4. \end{aligned} \tag{3.56}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm\sqrt{3}\omega$. If $\alpha = \pm\omega$, then eq. (3.51) gives $A = B$. If $\alpha = \pm\sqrt{3}\omega$, then eq. (3.51) gives $A = -B$. The solutions for $x_1$ and $x_2$ therefore take the general form

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &\quad + A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\sqrt{3}i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\sqrt{3}i\omega t} \\ \Longrightarrow &\quad B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_+) + B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_-). \end{aligned} \tag{3.57}$$

This is equivalent to eq. (3.55). ♣

The normal modes are obtained by setting either $B_-$ or $B_+$ equal to zero in eq. (3.55) or eq. (3.57). Therefore, the normal modes are $(1,1)$ and $(1,-1)$. How do we visualize these?

The mode $(1,1)$ oscillates with frequency $\omega$. In this case (where $B_- = 0$), we have $x_1(t) = x_2(t) = B_+ \cos(\omega t + \phi_+)$ at all times. So the masses simply oscillate back and forth in the same manner, as shown in Fig. 3.10. It is clear that such motion has frequency $\omega$, because as far as the masses are concerned, the middle spring is effectively not there, so each mass moves under the influence of only one spring, and therefore has frequency $\omega$.

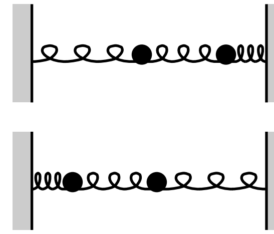
**Figure 3.10**

The mode $(1,-1)$ oscillates with frequency $\sqrt{3}\omega$. In this case (where $B_+ = 0$), we have $x_1(t) = -x_2(t) = B_- \cos(\sqrt{3}\omega t + \phi_-)$ at all times. So the masses oscillate back and forth with opposite displacements, as shown in Fig. 3.11. It is clear that this mode should have a frequency larger than that for the other mode, because the middle spring is stretched (or compressed), so the masses feel a larger force. But it takes a little thought to show that the frequency is $\sqrt{3}\omega$.[3]

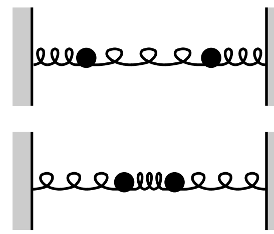
**Figure 3.11**

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Remark: The normal mode $(1,1)$ above is associated with the normal coordinate $x_1 + x_2$. They both involve the frequency $\omega$. However, this association is *not* due to the fact that the coefficients of both $x_1$ and $x_2$ in this normal coordinate are equal to 1. Rather, it is due to the fact that the *other* normal mode, namely $(x_1, x_2) \propto (1,-1)$, gives no contribution to the sum $x_1 + x_2$.

There are a few too many 1's floating around in the above example, so it's hard to see which results are meaningful and which results are coincidence. But the following example should clear things up. Let's say we solved a problem using the determinant method, and we found the solution to be

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \cos(\omega_2 t + \phi_2). \tag{3.58}$$

Then $5x + y$ is the normal coordinate associated with the normal mode $(3,2)$, which has frequency $\omega_1$. (This is true because there is no $\cos(\omega_2 t + \phi_2)$ dependence in the combination $5x + y$.) And similarly, $2x - 3y$ is the normal coordinate associated with the normal mode $(1,-5)$, which has frequency $\omega_2$ (because there is no $\cos(\omega_1 t + \phi_1)$ dependence in the combination $2x - 3y$). ♣

Another Remark: Note the difference between the types of differential equations we solved in the previous chapter in Section 2.3, and the types we solved throughout this chapter. The former dealt with forces that did not have to be linear in $x$ or $\dot{x}$, but which had to depend on only $x$, or only $\dot{x}$, or only $t$. The latter dealt with forces that could depend on all three of these quantities, but which had to be linear in $x$ and $\dot{x}$. ♣

[3]If you want to obtain this $\sqrt{3}\omega$ result without going through all of the above work, just note that the center of the middle spring doesn't move. Therefore, it acts like two "half springs," each with spring constant $2k$ (verify this). Hence, each mass is effectively attached to a "$k$" spring and a "$2k$" spring, yielding a total effective spring constant of $3k$. Thus the $\sqrt{3}$.

## 3.6 Exercises

*Section 3.1: Linear differential equations*

1. $kx$ **force** $*$

   A particle of mass $m$ is subject to a force $F(x) = kx$. What is the most general form of $x(t)$? If the particle starts out at $x_0$, what is the one special value of the initial velocity for which the particle doesn't eventually get far away from the origin?

2. **Rope on a pulley** $**$

   A rope of length $L$ and mass density $\rho\,\text{kg/m}$ hangs over a massless pulley. Initially, the ends of the rope are a distance $x_0$ above and below their average position. The rope is given an initial speed. If you want the rope to not eventually fall off the pulley, what should this initial speed be?

*Section 3.2: Simple harmonic motion*

3. **Amplitude**

   Find the amplitude of the motion given by $x(t) = C\cos\omega t + D\sin\omega t$.

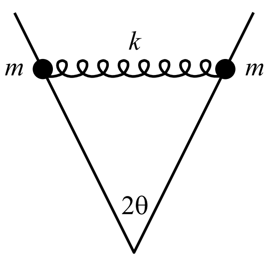


**Figure 3.12**

4. **Angled rails** $*$

   Two particles of mass $m$ are constrained to move along two horizontal rails that make an angle of $2\theta$ with respect to each other, as shown in Fig. 3.12. They are connected by a spring with spring constant $k$. What is the frequency of oscillations for the motion where the spring remains parallel to the position shown?


**Figure 3.13**

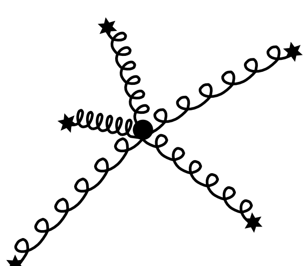

**Figure 3.14**

5. **Springs all over** $**$

   (a) A mass $m$ is attached to two springs that have equilibrium lengths equal to zero. The other ends of the springs are fixed at two points (see Fig. 3.13). The two spring constants are equal. The mass sits at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Ignore gravity, although you actually don't need to.)

   (b) A mass $m$ is attached to a number of springs that have equilibrium lengths equal to zero. The other ends of the springs are fixed at various points in space (see Fig. 3.14). The spring constants are all the same. The mass sits at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Again, ignore gravity, although you actually don't need to.)

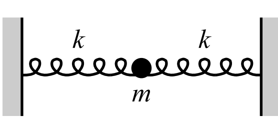


**Figure 3.15**

6. **Removing a spring** $*$

   The springs in Fig. 3.15 are at their natural equilibrium length. The mass oscillates along the line of the springs with amplitude $d$. At the moment (let this be $t = 0$) when the mass is at position $x = d/2$ (and moving to the right), the right spring is removed. What is the resulting $x(t)$? What is the amplitude of the new oscillation?

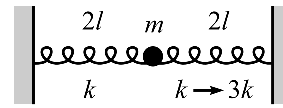


**Figure 3.16**

7. **Changing $k$** $**$

   Two springs each have spring constant $k$ and relaxed length $\ell$. They are both stretched a distance $\ell$ and attached to a mass $m$ and two walls, as shown in Fig. 3.16. At a given instant, the right spring constant is somehow magically changed to $3k$ (the relaxed length remains $\ell$). At a time $t = \frac{\pi}{4}\sqrt{\frac{m}{k}}$ later, what is the mass's position? What is its speed?

8. **Corrections to the pendulum** $***$

   (a) For small oscillations, the period of a pendulum is approximately $T \approx 2\pi\sqrt{\ell/g}$, independent of the amplitude, $\theta_0$. For finite oscillations, show that the exact expression for $T$ is

   $$T = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \,. \tag{3.59}$$

   (b) Find an approximation to this $T$, up to second order in $\theta_0^2$, in the following way. Make use of the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$ to write $T$ in terms of sines (because it's more convenient to work with quantities that go to zero as $\theta \to 0$). Then make the change of variables, $\sin x \equiv \sin(\theta/2)/\sin(\theta_0/2)$. Finally, expand your integrand in powers of $\theta_0$, and perform the integrals to show that[4]

   $$T \approx 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right). \tag{3.60}$$

*Section 3.3: Damped harmonic motion*

9. **Crossing the origin**

   Show that an overdamped or critically damped oscillator can cross the origin at most once.

10. **Strong damping** $*$

    In the strong damping ($\gamma \gg \omega$) case discussed in the remark in the overdamping subsection, we saw that $x(t) \propto e^{-\omega^2 t/2\gamma}$. Show that this can be written as $x(t) \propto e^{-kt/b}$, where $b$ is the coefficient of the damping force. And then explain, by looking at the forces on the mass, why this makes sense.

11. **Minimum speed** $*$

    A critically damped oscillator with natural frequency $\omega$ starts out at position $x_0$. What is the minimum initial speed it must have if it is to cross the origin?

[4]If you like this sort of thing, you can show that the next term in the parentheses is $(11/3072)\theta_0^4$. But be careful, this fourth-order correction comes from two terms.

12. **Another minimum speed** **

    An overdamped oscillator with natural frequency $\omega$ and damping coefficient $\gamma$ starts out at position $x_0$. What is the minimum initial speed it must have if it is to cross the origin?

13. **Maximum speed** **

    A mass on the end of a spring is released from rest at position $x_0$. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be critically damped. Show that the maximum speed of the mass in the first case is $e$ times the maximum speed in the second case.[5]

14. **Work** *

    A damped oscillator has initial position $x_0$ and speed $v_0$. After a long time, it will essentially be at rest at the origin. Therefore, by the work-energy theorem, the work done by the damping force must equal $-kx_0^2/2 - mv_0^2/2$. Verify that this is true. *Hint*: It's a bit messy to find $\dot{x}$ in terms of the initial conditions and then calculate the desired integral. An easier way is to use the $F = ma$ equation to rewrite one of the $\dot{x}$'s in your integral.

*Section 3.4: Driven (and damped) harmonic motion*

15. **Resonance**

    Given $\omega$ and $\gamma$, show that the $R$ in eq. (3.33) is minimum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$ (unless this is imaginary, in which case the minimum occurs at $\omega_d = 0$).

16. **No damping force** *

    A particle of mass $m$ is subject to a spring force, $-kx$, and also a driving force, $F_d \cos \omega_d t$. But there is no damping force. Find a solution for $x(t)$ by guessing $x(t) = A \cos \omega_d t + B \sin \omega_d t$. If you write your solution for $x(t)$ in the form $C \cos(\omega_d t - \phi)$, what are $C$ and $\phi$? Be careful about the phase.

17. **No spring force** *

    A particle of mass $m$ is subject to a damping force, $-bv$, and also a driving force, $F_d \cos \omega_d t$. But there is no spring force. Find a solution for $x(t)$ by guessing $x(t) = A \cos \omega_d t + B \sin \omega_d t$. If you write your solution for $x(t)$ in the form $C \cos(\omega_d t - \phi)$, what are $C$ and $\phi$?

*Section 3.5: Coupled oscillators*

[5] The fact that the maximum speeds differ by a fixed numerical factor follows from dimensional analysis, which tells us that the maximum speed in the first case must be proportional to $\omega x_0$. And since $\gamma = \omega$ in the critical-damping case, the damping doesn't introduce a new parameter, so the maximum speed has no choice but to again be proportional to $\omega x_0$. But showing that the maximum speeds differ by the nice factor of $e$ requires a calculation.

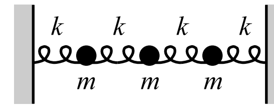


**Figure 3.17**

18. **Springs between walls** ∗∗

    Four identical springs and three identical masses lie between two walls (see Fig. 3.17). Find the normal modes.

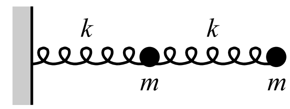


**Figure 3.18**

19. **Springs and one wall** ∗∗

    Two identical springs and two identical masses are attached to a wall as shown in Fig. 3.18. Find the normal modes.

20. **Coupled and damped** ∗∗

    The system in the example in Section 3.5 is modified by immersing it in a fluid so that both masses feel a damping force, $F_f = -bv$. Solve for $x_1(t)$ and $x_2(t)$.

21. **Coupled and driven** ∗∗

    The system in the example in Section 3.5 is modified by subjecting the left mass to a driving force $F_d \cos(2\omega t)$, and the right mass to a driving force $2F_d \cos(2\omega t)$, where $\omega = \sqrt{k/m}$. Find the particular solution for $x_1(t)$ and $x_2(t)$.

## 3.7 Problems

*Section 3.1: Linear differential equations*

1. **A limiting case** $*$

   Consider the equation $\ddot{x} = ax$. If $a = 0$, then the solution to $\ddot{x} = 0$ is of course $x(t) = C + Dt$. Show that in the limit $a \to 0$, eq. (3.2) reduces to this form. *Note:* $a \to 0$ is a very sloppy way of saying what we mean. What is the proper way to write this limit?

*Section 3.2: Simple harmonic motion*

2. **Average tension** $**$

   Is the average (over time) tension in the string of a pendulum larger or smaller than $mg$? By how much? As usual, assume that the angular amplitude $A$ is small.

*Section 3.3: Damped harmonic motion*

3. **Maximum speed** $**$

   A mass on the end of a spring (with natural frequency $\omega$) is released from rest at position $x_0$. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be overdamped (with damping coefficient $\gamma$). Find the ratio of the maximum speed in the former case to that in the latter. What is the ratio in the limit of strong damping ($\gamma \gg \omega$)? In the limit of critical damping?

*Section 3.4: Driven (and damped) harmonic motion*

4. **Exponential force** $*$

   A particle of mass $m$ is subject to a force $F(t) = me^{-bt}$. The initial position and speed are both zero. Find $x(t)$.[6]

5. **Driven oscillator** $*$

   Derive eq. (3.31) by guessing a solution of the form $x(t) = A\cos\omega_d t + B\sin\omega_d t$ in eq. (3.29).

*Section 3.4: Coupled oscillators*

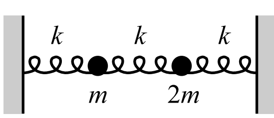

**Figure 3.19**

6. **Unequal masses** $**$

   Three identical springs and two masses, $m$ and $2m$, lie between two walls as shown in Fig. 3.19. Find the normal modes.

[6]This problem was already given as Problem 2.9, but solve it here by guessing an exponential function, in the spirit of Section 3.4.

7. **Driven mass on a circle** $**$

   Two identical masses $m$ are constrained to move on a horizontal hoop. Two identical springs with spring constant $k$ connect the masses and wrap around the hoop (see Fig. 3.20). One mass is subject to a driving force, $F_d \cos \omega_d t$. Find the particular solution for the motion of the masses.

8. **Springs on a circle** $****$

   (a) Two identical masses $m$ are constrained to move on a horizontal hoop. Two identical springs with spring constant $k$ connect the masses and wrap around the hoop (see Fig. 3.21). Find the normal modes.

   (b) Three identical masses are constrained to move on a hoop. Three identical springs connect the masses and wrap around the hoop (see Fig. 3.22). Find the normal modes.

   (c) Now do the general case with $N$ identical masses and $N$ identical springs.

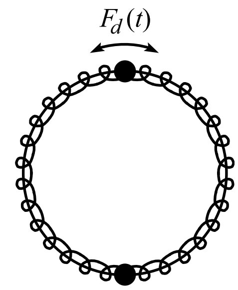


**Figure 3.20**

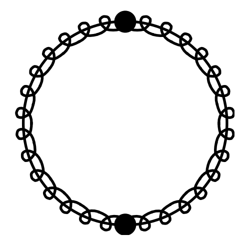

**Figure 3.21**

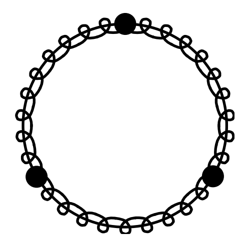

**Figure 3.22**

## 3.8 Solutions

1. **A limiting case**

   The expression "$a \to 0$" is sloppy, because $a$ has units of $[\text{time}]^{-2}$, and the number 0 has no units. The proper statement is that eq. (3.2) reduces to $x(t) = C + Dt$ when $\sqrt{a}\,t \ll 1$, or equivalently when $t \ll 1/\sqrt{a}$, which is now a comparison of quantities with the same units. The smaller $a$ is, the larger $t$ can be. Therefore, if "$a \to 0$," then $t$ can basically be anything.

   Under the condition $\sqrt{a}\,t \ll 1$, we can write $e^{\pm\sqrt{a}\,t} \approx 1 \pm \sqrt{a}\,t$. Therefore, eq. (3.2) becomes

$$\begin{aligned} x(t) &\approx A(1 + \sqrt{a}\,t) + B(1 - \sqrt{a}\,t) \\ &= (A + B) + \sqrt{a}(A - B)t \\ &\equiv C + Dt. \end{aligned} \tag{3.61}$$

   If $a$ is small but nonzero, then $t$ will eventually become large enough so that $\sqrt{a}\,t \ll 1$ doesn't hold, in which case the linear form in eq. (3.61) isn't valid.

2. **Average tension**

   Let the length of the pendulum be $\ell$. We know that the angle $\theta$ depends on time according to

$$\theta(t) = A\cos(\omega t), \tag{3.62}$$

   where $\omega = \sqrt{g/\ell}$. If $T$ is the tension in the string, then the radial $F = ma$ equation is $T - mg\cos\theta = m\ell\dot{\theta}^2$. Using eq. (3.62), this becomes

$$T = mg\cos\Big(A\cos(\omega t)\Big) + m\ell\Big(-\omega A\sin(\omega t)\Big)^2. \tag{3.63}$$

   Since $A$ is small, we can use the small-angle approximation $\cos\alpha \approx 1 - \alpha^2/2$, which gives

$$\begin{aligned} T &\approx mg\left(1 - \frac{1}{2}A^2\cos^2(\omega t)\right) + m\ell\omega^2 A^2\sin^2(\omega t) \\ &= mg + mgA^2\left(\sin^2(\omega t) - \frac{1}{2}\cos^2(\omega t)\right). \end{aligned} \tag{3.64}$$

   The average value of both $\sin^2\theta$ and $\cos^2\theta$ over one period is $1/2$,[7] so the average value of $T$ is

$$\overline{T} = mg + \frac{mgA^2}{4}\,, \tag{3.65}$$

   which is larger than $mg$, by $mgA^2/4$.

   Remark: It makes sense that $\overline{T} > mg$, because the average value of the vertical component of $T$ equals $mg$ (because the pendulum has no net rise or fall over a long period of time), and there is some non-zero contribution to the magnitude of $T$ from the horizontal component. ♣

[7] You can show this by doing the integrals, or by noting that the averages are equal and that they add up to 1.

3. **Maximum speed**

For the undamped case, the general form of $x$ is $x(t) = C\cos(\omega t + \phi)$. The initial condition $v(0) = 0$ tells us that $\phi = 0$, and then the initial condition $x(0) = x_0$ tells us that $C = x_0$. Therefore, $x(t) = x_0 \cos(\omega t)$, and so $v(t) = -\omega x_0 \sin(\omega t)$. This has a maximum magnitude of $\omega x_0$.

Now consider the overdamped case. Eq. (3.18) gives the position as

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}. \tag{3.66}$$

The initial conditions are

$$\begin{aligned} x(0) &= x_0 \quad \Longrightarrow \quad A + B = x_0, \\ v(0) &= 0 \quad \Longrightarrow \quad -(\gamma-\Omega)A - (\gamma+\Omega)B = 0. \end{aligned} \tag{3.67}$$

Solving these equations for $A$ and $B$, and then plugging the results into eq. (3.66), gives

$$x(t) = \frac{x_0}{2\Omega}\Big((\gamma+\Omega)e^{-(\gamma-\Omega)t} - (\gamma-\Omega)e^{-(\gamma+\Omega)t}\Big). \tag{3.68}$$

Taking the derivative to find $v(t)$, and using $\gamma^2 - \Omega^2 = \omega^2$, gives

$$v(t) = \frac{-\omega^2 x_0}{2\Omega}\Big(e^{-(\gamma-\Omega)t} - e^{-(\gamma+\Omega)t}\Big). \tag{3.69}$$

Taking the derivative one more time, we find that the maximum speed occurs at

$$t_{\text{max}} = \frac{1}{2\Omega}\ln\left(\frac{\gamma+\Omega}{\gamma-\Omega}\right). \tag{3.70}$$

Plugging this into eq. (3.69), and taking advantage of the logs in the exponentials, gives

$$\begin{aligned} v(t_{\text{max}}) &= \frac{-\omega^2 x_0}{2\Omega}\exp\left(-\frac{\gamma}{2\Omega}\ln\left(\frac{\gamma+\Omega}{\gamma-\Omega}\right)\right)\left(\sqrt{\frac{\gamma+\Omega}{\gamma-\Omega}} - \sqrt{\frac{\gamma-\Omega}{\gamma+\Omega}}\right) \\ &= -\omega x_0 \left(\frac{\gamma-\Omega}{\gamma+\Omega}\right)^{\gamma/2\Omega}. \end{aligned} \tag{3.71}$$

The desired ratio, $R$, of the maximum speeds in the two scenarios is therefore

$$R = \left(\frac{\gamma+\Omega}{\gamma-\Omega}\right)^{\gamma/2\Omega} \tag{3.72}$$

In the limit of strong damping ($\gamma \gg \omega$), we have $\Omega \equiv \sqrt{\gamma^2-\omega^2} \approx \gamma - \omega^2/2\gamma$. So the ratio becomes

$$R \approx \left(\frac{2\gamma}{\omega^2/2\gamma}\right)^{1/2} = \frac{2\gamma}{\omega}. \tag{3.73}$$

In the limit of critical damping ($\gamma \approx \omega$, $\Omega \approx 0$), we have, with $\Omega/\gamma \equiv \epsilon$,

$$R \approx \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/2\epsilon} \approx (1+2\epsilon)^{1/2\epsilon} \approx e, \tag{3.74}$$

in agreement with the result of Exercise 13. You can also show that in these two limits, $t_{\text{max}}$ equals $\ln(2\gamma/\omega)/\gamma$ and $1/\gamma \approx 1/\omega$, respectively.

4. **Exponential force**

Let's guess a particular solution to $\ddot{x} = e^{-bt}$ of the form $x(t) = Ce^{-bt}$. We find $C = 1/b^2$. And since the solution to the homogeneous equation $\ddot{x} = 0$ is $x(t) = At+B$, the complete solution for $x$ is

$$x(t) = \frac{e^{-bt}}{b^2} + At + B. \tag{3.75}$$

The initial condition $x(0) = 0$ gives $B = -1/b^2$. And the initial condition $v(0) = 0$ applied to $v(t) = -e^{-bt}/b + A$ gives $A = 1/b$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}\,. \tag{3.76}$$

5. **Driven oscillator**

Plugging $x(t) = A\cos\omega_d t + B\sin\omega_d t$ into eq. (3.29) gives

$$\begin{aligned}-\omega_d^2 A\cos\omega_d t - \omega_d^2 B\sin\omega_d t \\ - 2\gamma\omega_d A\sin\omega_d t + 2\gamma\omega_d B\cos\omega_d t \\ + \omega^2 A\cos\omega_d t + \omega^2 B\sin\omega_d t &= F\cos\omega_d t.\end{aligned} \tag{3.77}$$

If this is to be true for all $t$, the coefficients of $\cos\omega_d t$ on both sides must be equal. And likewise for $\sin\omega_d t$. Therefore,

$$\begin{aligned}-\omega_d^2 A + 2\gamma\omega_d B + \omega^2 A &= F, \\ -\omega_d^2 B - 2\gamma\omega_d A + \omega^2 B &= 0.\end{aligned} \tag{3.78}$$

Solving this system of equations for $A$ and $B$ gives

$$A = \frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\,, \qquad B = \frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\,, \tag{3.79}$$

in agreement with eq. (3.31).

6. **Unequal masses**

Let $x_1$ and $x_2$ be the positions of the left and right masses, respectively, relative to their equilibrium positions. The forces on the two masses are $-kx_1 + k(x_2 - x_1)$ and $-kx_2 - k(x_2 - x_1)$, respectively, so the $F = ma$ equations are

$$\begin{aligned}\ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ 2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0.\end{aligned} \tag{3.80}$$

The appropriate linear combinations of these equations aren't obvious, so we'll use the determinant method. Letting $x_1 = A_1 e^{i\alpha t}$ and $x_2 = A_2 e^{i\alpha t}$, we see that for there to be a nontrivial solution for $A$ and $B$, we must have

$$\begin{aligned}0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= 2\alpha^4 - 6\alpha^2\omega^2 + 3\omega^4.\end{aligned} \tag{3.81}$$

The roots of this equation are

$$\alpha = \pm\omega\sqrt{\frac{3+\sqrt{3}}{2}} \equiv \pm\alpha_1, \qquad \text{and} \qquad \alpha = \pm\omega\sqrt{\frac{3-\sqrt{3}}{2}} \equiv \pm\alpha_2. \tag{3.82}$$

If $\alpha^2 = \alpha_1^2$, then the normal mode is proportional to $(\sqrt{3}+1, -1)$. And if $\alpha^2 = \alpha_2^2$, then the normal mode is proportional to $(\sqrt{3}-1, 1)$. So the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \phi_1), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \phi_2), \end{aligned} \tag{3.83}$$

Note that these two vectors are not orthogonal (there is no need for them to be). You can show that the normal coordinates associated with these normal modes are $x_1 - (\sqrt{3}-1)x_2$ and $x_1 + (\sqrt{3}+1)x_2$, respectively, because these are the combinations that make the $\alpha_2$ and $\alpha_1$ frequencies disappear, respectively.

7. **Driven mass on a circle**

Label two diametrically opposite points as the equilibrium positions. Let the distances from the masses to these points be $x_1$ and $x_2$ (measured counterclockwise). If the driving force acts on mass "1", then the $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 + 2k(x_1 - x_2) &= F_d \cos\omega_d t, \\ m\ddot{x}_2 + 2k(x_2 - x_1) &= 0. \end{aligned} \tag{3.84}$$

To solve these equations, we can treat the driving force as the real part of $F_d e^{i\omega_d t}$ and try solutions of the form $x_1(t) = A_1 e^{i\omega_d t}$ and $x_2(t) = A_2 e^{i\omega_d t}$, and then solve for $A_1$ and $A_2$. Or we can try some trig functions. If we take the latter route, we will quickly find that the solutions can't involve any sine terms (this is due to the fact that there are no first derivatives of the $x$'s in eq. (3.84)). Therefore, the trig functions must look like $x_1(t) = A_1 \cos\omega_d t$ and $x_2(t) = A_2 \cos\omega_d t$. Using either of the two methods, eqs. (3.84) become

$$\begin{aligned} -\omega_d^2 A_1 + 2\omega^2(A_1 - A_2) &= F, \\ -\omega_d^2 A_2 + 2\omega^2(A_2 - A_1) &= 0, \end{aligned} \tag{3.85}$$

where $\omega \equiv \sqrt{k/m}$ and $F \equiv F_d/m$. Solving for $A_1$ and $A_2$, we find that the desired particular solution is

$$x_1(t) = \frac{-F(2\omega^2 - \omega_d^2)}{\omega_d^2(4\omega^2 - \omega_d^2)} \cos\omega_d t, \qquad x_2(t) = \frac{-2F\omega^2}{\omega_d^2(4\omega^2 - \omega_d^2)} \cos\omega_d t. \tag{3.86}$$

The most general solution is the sum of this particular solution and the homogeneous solution found in eq. (3.91) in Problem 8 below.

REMARKS: If $\omega_d = 2\omega$, the amplitudes of the motions go to infinity. This makes sense, considering that there is no damping, and that the natural frequency of the system (calculated in Problem 8) is $2\omega$.

If $\omega_d = \sqrt{2}\omega$, then the mass that is being driven doesn't move. What is going on here is that the driving force balances the force that the mass feels from the springs due to the other mass's motion. And indeed, you can show that $\sqrt{2}\omega$ is the frequency that one mass moves at if the other mass is at rest (and thereby acts essentially like a brick wall). Note that $\omega_d = \sqrt{2}\omega$ is the cutoff between the masses moving in the same direction or in opposite directions.

If $\omega_d \to \infty$, then both motions go to zero. But $x_2$ fourth-order small, whereas $x_1$ is only second-order small.

If $\omega_d \to 0$, then $A_1 \approx A_2 \approx -F/2\omega_d^2$. This is very large. The driving force basically spins the masses around in one direction, and then reverses and spins them around in the other direction. We essentially have the driving force acting on a mass $2m$, and two integrations of $F_d \cos\omega_d t = (2m)\ddot{x}$ shows that the amplitude of the motion is $F/2\omega_d^2$, as above. ♣

8. **Springs on a circle**

(a) Label two diametrically opposite points as the equilibrium positions. Let the distances from the masses to these points be $x_1$ and $x_2$ (measured counterclockwise). Then the $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 + 2k(x_1 - x_2) &= 0, \\ m\ddot{x}_2 + 2k(x_2 - x_1) &= 0. \end{aligned} \tag{3.87}$$

The determinant method works here, but let's just do it the easy way. Adding the equations gives

$$\ddot{x}_1 + \ddot{x}_2 = 0, \tag{3.88}$$

and subtracting them gives

$$(\ddot{x}_1 - \ddot{x}_2) + 4\omega^2(x_1 - x_2) = 0. \tag{3.89}$$

The normal coordinates are therefore

$$\begin{aligned} x_1 + x_2 &= At + B, \\ x_1 - x_2 &= C\cos(2\omega t + \phi). \end{aligned} \tag{3.90}$$

Solving these two equations for $x_1$ and $x_2$, and writing the results in vector form, gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B) + C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi), \tag{3.91}$$

where the constants $A$, $B$, and $C$ are defined to be half of what they were in eq. (3.90). The normal modes are therefore

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi). \end{aligned} \tag{3.92}$$

The first mode has frequency zero. It corresponds to the masses sliding around the circle, equally spaced, at constant speed. The second mode has both masses moving to the left, then both to the right, back and forth.

(b) Label three equally spaced points as the equilibrium positions. Let the distances from the masses to these points be $x_1$, $x_2$, and $x_3$ (measured counterclockwise). Then the $F = ma$ equations are, as you can show,

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) + k(x_1 - x_3) &= 0, \\ m\ddot{x}_2 + k(x_2 - x_3) + k(x_2 - x_1) &= 0, \\ m\ddot{x}_3 + k(x_3 - x_1) + k(x_3 - x_2) &= 0. \end{aligned} \tag{3.93}$$

The sum of all three of these equations definitely gives something nice. Also, differences between any two of the equations gives something useful. But let's use the determinant method to get some practice. Trying solutions of the form $x_1 = A_1 e^{i\alpha t}$, $x_2 = A_2 e^{i\alpha t}$, and $x_3 = A_3 e^{i\alpha t}$, we obtain the matrix equation,

$$\begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (3.94)$$

Setting the determinant equal to zero yields a cubic equation in $\alpha^2$. But it is a nice cubic equation, with $\alpha^2 = 0$ as a solution. The other solution is the double root $\alpha^2 = 3\omega^2$.

The $\alpha = 0$ root corresponds to $A_1 = A_2 = A_3$. That is, it corresponds to the vector $(1, 1, 1)$. This $\alpha = 0$ case is the one case where our exponential solution isn't really an exponential. But $\alpha^2$ equalling zero in eq. (3.94) basically tells us that we are dealing with a function whose second derivative is zero, that is, a linear function $At + B$. Therefore, the normal mode is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (At + B). \qquad (3.95)$$

This mode has frequency zero, and corresponds to the masses sliding around the circle, equally spaced, at constant speed.

The two $\alpha^2 = 3\omega^2$ roots correspond to a two-dimensional subspace of normal modes. You can show that any vector of the form $(a, b, c)$ with $a + b + c = 0$ is a normal mode with frequency $\sqrt{3}\omega$. We will arbitrarily pick the vectors $(0, 1, -1)$ and $(1, 0, -1)$ as basis vectors for this space. We can then write the normal modes as linear combinations of the vectors

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= C_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_1), \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_2). \end{aligned} \qquad (3.96)$$

REMARKS: This is very similar to the example in Section 3.5 with two masses and three springs oscillating between two walls. The way we've written these modes, the first one has the first mass stationary (so there could be a wall there, for all the other two masses know). Similarly for the second mode. Hence the $\sqrt{3}\omega$ result here, as in the example.

The normal coordinates in this problem are $x_1 + x_2 + x_3$ (obtained by adding the three equations in (3.93)), and also any combination of the form $ax_1 + bx_2 + cx_3$, where $a + b + c = 0$ (obtained by taking $a$ times the first eq. in (3.93), plus $b$ times the second, plus $c$ times the third). The three normal coordinates that correspond to the mode in eq. (3.95) and the two modes we chose in eq. (3.96) are, respectively, $x_1 + x_2 + x_3$, $-2x_1 + x_2 + x_3$, and $x_1 - 2x_2 + x_3$, because each of these combinations gets no contribution from the other two modes. ♣

(c) In part (b), when we set the determinant of the matrix in eq. (3.94) equal to zero, we were essentially finding the eigenvectors and eigenvalues[8] of the matrix,

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3I - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.97}$$

We haven't bothered writing the common factor $\omega^2$ here, because it doesn't affect the eigenvectors. As an exercise, you can show that for the general case of $N$ springs and masses on a circle, the above matrix becomes the $N \times N$ matrix,

$$3I - \begin{pmatrix} 1 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & 0 \\ 0 & 0 & 1 & 1 & & 0 \\ & \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \equiv 3I - M. \tag{3.98}$$

In the matrix $M$, the three consecutive 1's keep shifting to the right, and they wrap around cyclicly. We must now find the eigenvectors of $M$, which will require being a little clever.

We can guess the eigenvectors and eigenvalues of $M$ if we take a hint from its cyclic nature. A particular set of things that are rather cyclic are the $N$th roots of 1. If $\eta$ is an $N$th root of 1, you can verify that $(1, \eta, \eta^2, \dots, \eta^{N-1})$ is an eigenvector of $M$ with eigenvalue $\eta^{-1} + 1 + \eta$. (This general method works for any matrix where the entries keep shifting to the right. The entries don't have to be equal.) The eigenvalues of the entire matrix in eq. (3.98) are therefore $3 - (\eta^{-1} + 1 + \eta) = 2 - \eta^{-1} - \eta$.

There are $N$ different $N$th roots of 1, namely $\eta_n = e^{2\pi i n/N}$, for $0 \le n < N$. So the $N$ eigenvalues are

$$\begin{aligned} \lambda_n = 2 - \left(e^{-2\pi i n/N} + e^{2\pi i n/N}\right) &= 2 - 2\cos(2\pi n/N) \\ &= 4\sin^2(\pi n/N). \end{aligned} \tag{3.99}$$

The corresponding eigenvectors are

$$V_n = \left(1, \eta_n, \eta_n^2, \dots, \eta_n^{N-1}\right). \tag{3.100}$$

Since the numbers $n$ and $N - n$ yield the same value for $\lambda_n$ in eq. (3.99), the eigenvalues come in pairs (except for $n = 0$, and $n = N/2$ if $N$ is even). This is fortunate, because we may then form real linear combinations of the two corresponding complex eigenvectors given in eq. (3.100). We see that the vectors

$$V_n^+ \equiv \frac{1}{2}(V_n + V_{N-n}) = \begin{pmatrix} 1 \\ \cos(2\pi n/N) \\ \cos(4\pi n/N) \\ \vdots \\ \cos(2(N-1)\pi n/N) \end{pmatrix} \tag{3.101}$$

[8] An eigenvector $v$ of a matrix $M$ is a vector that gets taken into a multiple of itself when acted upon by $M$. That is, $Mv = \lambda v$, where $\lambda$ is some number (the eigenvalue). This can be rewritten as $(M - \lambda I)v = 0$, where $I$ is the identity matrix. By our usual reasoning about invertible matrices, a nonzero vector $v$ exists only if $\lambda$ satisfies $\det|M - \lambda I| = 0$.

and

$$V_n^- \equiv \frac{1}{2i}(V_n - V_{N-n}) = \begin{pmatrix} 0 \\ \sin(2\pi n/N) \\ \sin(4\pi n/N) \\ \vdots \\ \sin(2(N-1)\pi n/N) \end{pmatrix} \tag{3.102}$$

both have eigenvalue $\lambda_n = \lambda_{N-n}$. Referring back to the $N = 3$ case in eq. (3.94), we see that we must take the square root of the eigenvalues and then multiply by $\omega$ to obtain the frequencies (because it was an $\alpha^2$ that appeared in the matrix, and because we dropped the factor of $\omega^2$). The frequencies corresponding to the above two normal modes are therefore, using eq. (3.99),

$$\omega_n = \omega\sqrt{\lambda_n} = 2\omega\sin(\pi n/N). \tag{3.103}$$

Remark: Let's check our results for $N = 3$. If $n = 0$, we find $\lambda_0 = 0$, and $V_0 = (1, 1, 1)$. If $n = 1$, we find $\lambda_1 = 3$, and $V_1^+ = (1, -1/2, -1/2)$ and $V_1^- = (0, 1/2, -1/2)$. These two vectors span the same space we found in eq. (3.96). And $\sqrt{\lambda_1} = \sqrt{3}$, in agreement with eq. (3.96). You can also find the vectors for $N = 4$. These are fairly intuitive, so try to write them down first without using the above results. ♣

# Chapter 4

# Conservation of Energy and Momentum



Conservation laws are extremely important in physics. They are enormously helpful, both quantitatively and qualitatively, in figuring out what is going on in a physical system.

When we say that something is "conserved", we mean that it is constant over time. If a certain quantity is conserved, for example, while a ball rolls around on a hill, or while a group of particles interact, then the possible final motions are greatly restricted. If we can write down enough conserved quantities (which we are generally able to do, at least for the problems in this book), then we can restrict the final motions down to just one possibility, and so we have solved our problem. Conservation of energy and momentum are two of the main conservation laws in physics. A third, conservation of angular momentum, is discussed in Chapters 6-8.

It should be noted that it is not *necessary* to use conservation of energy and momentum when solving a problem. We will derive these conservation laws from Newton's laws. Therefore, if you felt like it, you could always simply start with first principles and use $F = ma$, etc. You would, however, soon grow weary of this approach. The point of conservation laws is that they make your life easier, and they provide a means for getting a good idea of the overall behavior of a given system.

## 4.1 Conservation of energy in 1-D

Consider a force, in just one dimension for now, that depends only on position. That is, $F = F(x)$. If we write $a$ as $v\,dv/dx$, then $F = ma$ becomes

$$mv\frac{dv}{dx} = F(x). \tag{4.1}$$

Separating variables and integrating gives $mv^2/2 = E + \int_{x_0}^{x} F(x')\,dx'$, where $E$ is a constant of integration, dependent on the choice of $x_0$. (We're simply following the procedure in Section 2.3 here, for a function that depends only on $x$.) If we now define the *potential energy*, $V(x)$, as

$$V(x) \equiv -\int_{x_0}^{x} F(x')\,dx', \tag{4.2}$$

then we may write

$$\frac{1}{2}mv^2 + V(x) = E. \tag{4.3}$$

We define the first term here to be the kinetic energy. Since this equation is true at all points in the particle's motion, the sum of the kinetic energy and potential energy is a constant. If a particle loses (or gains) potential energy, then its speed increases (or decreases).

In Boston, lived Jack as did Jill,
Who gained $mgh$ on a hill.
In their liquid pursuit,
Jill exclaimed with a hoot,
"I think we've just climbed a landfill!"

While noting, "Oh, this is just grand,"
Jack tripped on some trash in the sand.
He changed his potential
To kinetic, torrential,
But not before grabbing Jill's hand.

Both $E$ and $V(x)$ depend, of course, on the arbitrary choice of $x_0$ in eq. (4.2). What this means is that $E$ and $V(x)$ have no meaning by themselves. Only differences in $E$ and $V(x)$ are relevant, because these differences are independent of the choice of $x_0$. For example, it makes no sense to say that the gravitational potential energy of an object at height $y$ equals $-\int F\,dy = -\int(-mg)\,dy = mgy$. We have to say that $mgy$ is the potential energy *with respect to the ground* (if your $x_0$ is at ground level). If we wanted to, we could say that the potential energy is $mgy + 7mg$ with respect to a point 7 meters below the ground. This is perfectly correct, although a little unconventional.[1]

If we take the difference between eq. (4.3) evaluated at two points, $x_1$ and $x_2$, then we obtain

$$\begin{aligned} \frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1) &= V(x_1) - V(x_2) \\ &= \int_{x_1}^{x_2} F(x')\,dx'. \end{aligned} \tag{4.4}$$

Here it is clear that only differences in energies matter. If we define the integral here to be the *work* done on the particle as it moves from $x_1$ to $x_2$, then we have produced the *work-energy theorem*,

[1] It gets to be a pain to keep repeating "with respect to the ground." Therefore, whenever anyone talks about gravitational potential energy in an experiment on the surface of the earth, it is understood that the ground is the reference point. If, on the other hand, the experiment reaches out to distances far from the earth, then $r = \infty$ is understood to be the reference point, for reasons of convenience we will shortly see.

**Theorem 4.1** *The change in a particle's kinetic energy between points $x_1$ and $x_2$ is equal to the work done on the particle between $x_1$ and $x_2$.*

If the force points in the same direction as the motion (that is, if the $F(x)$ and the $dx$ in eq. (4.4) have the same sign), then the work is positive and the speed increases. If the force points in the direction opposite to the motion, then the work is negative and the speed decreases.

Having chosen a reference point $x_0$ for the potential energy, if we draw the $V(x)$ curve and also the constant $E$ line (see Fig. 4.1), then the difference between them gives the kinetic energy. The places where $V(x) > E$ are the regions where the particle cannot go. The places where $V(x) = E$ are the "turning points" where the particle stops and changes direction. In the figure, the particle is trapped between $x_1$ and $x_2$, and oscillates back and forth. The potential $V(x)$ is extremely useful this way, because it makes clear the general properties of the motion.

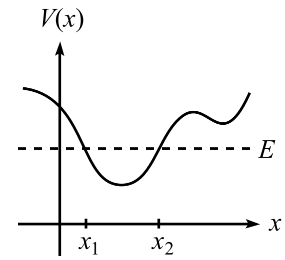


**Figure 4.1**

Remark: It may seem silly to introduce a specific $x_0$ as a reference point, considering that it is only eq. (4.4) (which makes no mention of $x_0$) that has any meaning. It's sort of like taking the difference between 17 and 8 by first finding their sizes relative to 5, namely 12 and 3, and then subtracting 3 from 12 to obtain 9. However, since integrals are harder to do than simple subtractions, it is advantageous to do the integral once and for all and thereby label all positions with a definite number $V(x)$, and to then take differences between the $V$'s when needed. ♣

Note that eq. (4.2) implies

$$F(x) = -\frac{dV(x)}{dx}\,. \tag{4.5}$$

Given $V(x)$, it is easy to take its derivative to obtain $F(x)$. But given $F(x)$, it may be difficult (or impossible) to perform the integration in eq. (4.2) and write $V(x)$ in closed form. But this is not of much concern. The function $V(x)$ is well-defined (assuming that the force is a function of $x$ only), and if needed it can be computed numerically to any desired accuracy.

---

**Example 1 (Gravitational potential energy):** Consider two point masses, $M$ and $m$, separated by a distance $r$. Newton's law of gravitation says that the force between them is attractive and has magnitude $GMm/r^2$. The potential energy of the system at separation $r$, measured relative to separation $r_0$, is

$$V(r) - V(r_0) = -\int_{r_0}^{r} \frac{-GMm}{r'^2}\,dr' = \frac{-GMm}{r} + \frac{GMm}{r_0}\,. \tag{4.6}$$

A convenient choice for $r_0$ is $\infty$, because this makes the second term vanish. It will be understood from now on that this $r_0 = \infty$ reference point has been chosen. Therefore (see Fig. 4.2),

$$V(r) = \frac{-GMm}{r}\,. \tag{4.7}$$

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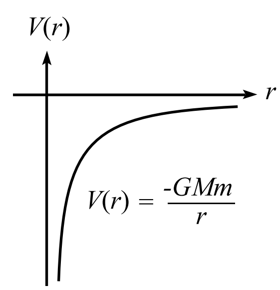


**Figure 4.2**

**Example 2 (Gravity near the earth):** What is the gravitational potential energy of a mass $m$ at height $y$, relative to the ground? We know, of course, that it is $mgy$, but let's do it the hard way. If $M$ is the mass of the earth and $R$ is its radius, then (assuming $y \ll R$)

$$\begin{aligned} V(R+y) - V(R) &= \frac{-GMm}{R+y} - \frac{-GMm}{R} \\ &= \frac{-GMm}{R}\left(\frac{1}{1+y/R} - 1\right) \\ &\approx \frac{-GMm}{R}\Big((1-y/R) - 1\Big) \\ &= \frac{GMmy}{R^2}, \end{aligned} \tag{4.8}$$

where we have used the Taylor series approximation for $1/(1+\epsilon)$ to obtain the third line. We have also used the fact that a sphere can be treated like a point mass, as far as gravity is concerned. We'll prove this in Section 4.4.1.

Using $g \equiv GM/R^2$, we see that the potential energy difference in eq. (4.8) equals $mgy$. We have, of course, simply gone around in circles here. We integrated in eq. (4.6), and then we basically differentiated in eq. (4.8) by taking the difference between the forces. But it's good to check that everything works out.

---

REMARK: A good way to visualize a potential $V(x)$ is to imagine a ball sliding around in a valley or on a hill. For example, the potential of a typical spring is $V(x) = kx^2/2$ (which produces the Hooke's-law force, $F(x) = -dV/dx = -kx$), and we can get a decent idea of what is going on if we imagine a valley with height given by $y = x^2/2$. The gravitational potential of the ball is then $mgy = mgx^2/2$. Choosing $mg = k$ gives the desired potential. If we then look at the projection of the ball's motion on the $x$-axis, it seems like we have constructed a setup identical to the original spring.

*However*, although this analogy helps in visualizing the basic properties of the motion, the two setups are *not* the same. The details of this fact are left for Problem 5, but the following observation should convince you that they are indeed different. Let the ball be released from rest in both setups at a large value of $x$. Then the force, $kx$, due to the spring is very large. But the force in the $x$-direction on the particle in the valley is only a fraction of $mg$ (namely $mg \sin\theta\cos\theta$, where $\theta$ is the angle of the ground). ♣

### Conservative forces

Given any force (it can depend on $x$, $v$, $t$, and/or whatever), the work it does on a particle is defined by $W \equiv \int F\,dx$. If the particle starts at $x_1$ and ends up at $x_2$, then no matter how it gets there (it can speed up or slow down, or reverse direction a few times, perhaps due to the influence of another force), we can calculate the work done by the given force and equate the result with the change in kinetic energy, via

$$W \equiv \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} m\left(\frac{v\,dv}{dx}\right) dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2. \tag{4.9}$$

For some forces, the work done is independent of how the particle moves. A force that depends only on position (in one dimension) has this property, because the integral in eq. (4.4) depends only on the endpoints. The $W = \int F\,dx$ integral is simply the area under the $F$ vs. $x$ graph, and this area is independent of how the particle goes from $x_1$ to $x_2$.

For other forces, the work done depends on how the particle moves. Such is the case for forces that depend on $t$ or $v$, because it then matters *when* or *how quickly* the particle goes from $x_1$ to $x_2$. An common example of such a force is friction. If you slide a brick across a table from $x_1$ to $x_2$, then the work done by friction equals $-\mu mg|\Delta x|$. But if you slide the brick by wiggling it back and forth for an hour before you finally reach $x_2$, then the amount of negative work done by friction will be very large. Since friction always opposes the motion, the contributions to the $W = \int F\,dx$ integral are always negative, so there is never any cancellation. The result is therefore a large negative number.

The issue with friction is that the $\mu mg$ force isn't a function only of position, because at a given location the friction can point to the right or to the left, depending on which way the particle is moving. Friction is therefore a function of velocity. True, it's a function only of the *sign* of the velocity, but that's enough to ruin the position-only dependence.

We now define a *conservative force* as one for which the work done on a particle between two given points is independent of how the particle makes the journey. From the preceding discussion, we know that a one-dimensional force is conservative if and only if it depends only on $x$ (or is constant).[2]

The point we're leading up to here is that although we can define the work done by any force, we can only talk about potential energy associated with a force if the force is conservative. This is true because we want to be able to label each value of $x$ with a unique number, $V(x)$, given by $V(x) = -\int_{x_0}^{x} F\,dx$. If this integral were dependent on the path, then it wouldn't be well-defined, so we wouldn't know what number to assign to $V(x)$. We therefore talk only about potential energies that are associated with conservative forces. In particular, it makes no sense to talk about the potential energy associated with a friction force.

### Work vs. potential energy

When you drop a ball, does its speed increase because the gravitational force is doing work on it, or because its gravitational potential energy is decreasing? Well, both (or more precisely, either). Work and potential energy are two different ways of talking about the same thing (at least for conservative forces). Either method of reasoning will give the correct result. However, be careful not to use *both* reasonings and "double count" the effect of gravity on the ball.

Which terminology you use depends on what you call your "system". Just as with $F = ma$ and free-body diagrams, it is important to label your system when dealing with work and energy.

The work-energy theorem stated in Theorem 4.1 was relevant to one particle. What if we are dealing with the work done on a system that is composed of various

[2]In two or three dimensions, however, we will see in Section 4.3.1 that a conservative force must satisfy another requirement, in addition to being dependent only on position.

parts? The general work-energy theorem says that the work done on a system by *external* forces equals the change in energy of the system. This energy many come in the form of kinetic energy, or internal potential energy, or heat (which is really just kinetic energy). For a point particle, there is no internal structure (so we'll assume it can't heat up), so this general form of the theorem reduces to Theorem 4.1. But to see what happens when a system has internal structure, consider the following example.

---

**Example (Raising a book):** You lift a book up at constant speed, so there is no change in kinetic energy. Let's see what the general work-energy theorem says for various choices of the system.

- System = (book): Both you and gravity are external forces, and there is no change in energy of the book as a system in itself. So the W-E theorem says

$$W_{\text{you}} + W_{\text{grav}} = 0 \qquad \Longleftrightarrow \qquad mgh + (-mgh) = 0. \tag{4.10}$$

- System = (book + earth): Now you are the only external force. The gravitational force between the earth and the book is an internal force which produces and internal potential energy. So the W-E theorem says

$$W_{\text{you}} = \Delta V_{\text{earth}-\text{book}} \qquad \Longleftrightarrow \qquad mgh = mgh. \tag{4.11}$$

- System = (book + earth + you): There is now no external force. The internal energy of the system changes because the earth-book gravitational potential energy increases, and also because *your* potential energy decreases. In order to lift the book, you have to burn some calories from the dinner you ate. So the W-E theorem says

$$0 = \Delta V_{\text{earth}-\text{book}} + \Delta V_{\text{you}} \qquad \Longleftrightarrow \qquad 0 = mgh + (-mgh). \tag{4.12}$$

  Actually, a human body isn't 100% efficient, so what really happens here is that your potential energy decreases by more than $mgh$, but heat is produced. The sum of these two changes in energy equals $-mgh$.

---

The moral of all this is that you can look at a setup in various ways. Potential energy in one way might be work in another. In practice, it is usually more convenient to work in terms of potential energy. So for a dropped ball, people usually consider gravity to be an internal force in the earth-ball system, as opposed to an external force on the ball system.

## 4.2 Small Oscillations

Consider an object in one dimension, subject to the potential $V(x)$. Let the object initially be at rest at a local minimum of $V(x)$, and then let it be given a small kick so that it moves back and forth around the equilibrium point. What can we say about this motion? Is it harmonic? Does the frequency depend on the amplitude?

It turns out that for small amplitudes, the motion is indeed simple harmonic motion, and the frequency can easily be found, given $V(x)$. To see this, expand $V(x)$ in a Taylor series around the equilibrium point, $x_0$.

$$V(x) = V(x_0)+V'(x_0)(x-x_0)+\frac{1}{2!}V''(x_0)(x-x_0)^2+\frac{1}{3!}V'''(x_0)(x-x_0)^3+\cdots. \quad (4.13)$$

This looks like a bit of a mess, but we can simplify it greatly. $V(x_0)$ is an irrelevant additive constant. We can ignore it because only differences in energy matter (or equivalently, because $F = -dV/dx$). And $V'(x_0) = 0$, by definition of the equilibrium point. So that leaves us with the $V''(x_0)$ and higher-order terms. But for sufficiently small displacements, these higher-order terms are negligible compared to the $V''(x_0)$ term, because they are suppressed by additional powers of $(x - x_0)$. So we are left with[3]

$$V(x) \approx \frac{1}{2}V''(x_0)(x-x_0)^2. \quad (4.14)$$

But this looks exactly like the Hooke's-law potential, $V(x) = (1/2)k(x-x_0)^2$, provided that we let $V''(x_0)$ be our "spring constant," $k$. The frequency of small oscillations, $\omega = \sqrt{k/m}$, therefore equals

$$\omega = \sqrt{\frac{V''(x_0)}{m}}\,. \quad (4.15)$$

---

**Example:** A particle moves under the influence of the potential $V(x) = A/x^2 - B/x$. Find the frequency of small oscillations around the equilibrium point. This potential is relevant to planetary motion, as we will see in Chapter 6.

**Solution:** The first thing we need to do is calculate the equilibrium point, $x_0$. We have

$$V'(x) = -\frac{2A}{x^3} + \frac{B}{x^2}\,. \quad (4.16)$$

Therefore, $V'(x) = 0$ when $x = 2A/B \equiv x_0$. The second derivative of $V(x)$ is

$$V''(x) = \frac{6A}{x^4} - \frac{2B}{x^3}\,. \quad (4.17)$$

Plugging in $x_0 = 2A/B$, we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{B^4}{8mA^3}}\,. \quad (4.18)$$

---

Eq. (4.15) is an important result, because *any* function $V(x)$ looks basically like a parabola (see Fig. 4.3) in a small enough region around a minimum (except in the special case where $V''(x_0) = 0$).

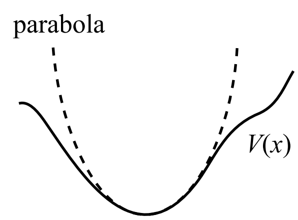


**Figure 4.3**

[3]Even if $V'''(x_0)$ is much larger than $V''(x_0)$, we can always pick $(x - x_0)$ small enough so that the $V'''(x_0)$ term is negligible. The one case where this is not true is when $V''(x_0) = 0$. But the result in eq. (4.15) is still valid in this case. The frequency $\omega$ just happens to be zero.

A potential may look quite erratic,
And its study may seem problematic.
But down near a min,
You can say with a grin,
"It behaves like a simple quadratic!"

## 4.3 Conservation of energy in 3-D

The concepts of work and potential energy in three dimensions are slightly more complicated than in one dimension, but the general ideas are the same. As in the 1-D case, we start with Newton's second law, which now takes the vector form, $\mathbf{F} = m\mathbf{a}$. And as in the 1-D case, we will deal only with forces that depend only on position, that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$. This vector equation is shorthand for three equations analogous to eq. (4.1), namely $mv_x(dv_x/dx) = F_x$, and likewise for $y$ and $z$. Multiplying through by $dx$, etc., in these three equations, and then adding them together gives

$$F_x\,dx + F_y\,dy + F_z\,dz = m(v_x\,dv_x + v_y\,dv_y + v_z\,dv_z). \tag{4.19}$$

Integrating from the point $(x_0, y_0, z_0)$ to the point $(x, y, z)$ yields

$$E + \int_{x_0}^{x} F_x\,dx + \int_{y_0}^{y} F_y\,dy + \int_{z_0}^{z} F_z\,dz = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}mv^2, \tag{4.20}$$

where $E$ is a constant of integration.[4] Note that the integrations on the left-hand side depend on what path in 3-D space the particle takes in going from $(x_0, y_0, z_0)$ to $(x, y, z)$. We will address this issue below.

With $d\mathbf{r} \equiv (dx, dy, dz)$, the left-hand side of eq. (4.19) is equal to $\mathbf{F} \cdot d\mathbf{r}$. Hence, eq. (4.20) may be written as

$$\frac{1}{2}mv^2 - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = E. \tag{4.21}$$

Therefore, if we define the potential energy, $V(\mathbf{r})$, as

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.22}$$

then we may write

$$\frac{1}{2}mv^2 + V(\mathbf{r}) = E. \tag{4.23}$$

In other words, the sum of the kinetic energy and potential energy is constant.

[4]Technically, we should put primes on the integration variables so that we don't confuse them with the limits of integration, but this gets too messy.

### 4.3.1 Conservative forces in 3-D

For a force that depends only on position (as we have been assuming), there is one complication that arises in 3-D that we didn't have to worry about in 1-D. In 1-D, there is only one route that goes from $x_0$ to $x$. The motion itself may involve speeding up or slowing down, or backtracking, but the path is always restricted to be along the line containing $x_0$ and $x$. But in 3-D, there is an infinite number of routes that go from $\mathbf{r}_0$ to $\mathbf{r}$. In order for the potential, $V(\mathbf{r})$, to have any meaning and to be of any use, it must be well-defined. That is, it must be path-independent. As in the 1-D case, we call the force associated with such a potential a *conservative force*. Let's now see what types of 3-D forces are conservative.

**Theorem 4.2** *Given a force* $\mathbf{F}(\mathbf{r})$*, a necessary and sufficient condition for the potential,*

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.24}$$

*to be well-defined (that is, to be path-independent) is that the curl of* $\mathbf{F}$ *is zero (that is,* $\nabla \times \mathbf{F} = \mathbf{0}$*).*[5]

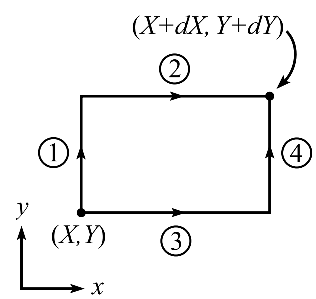

**Figure 4.4**

**Proof:** First, let us show that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path-independence. In other words, "If $V(\mathbf{r})$ is path-independent, then $\nabla \times \mathbf{F} = \mathbf{0}$."

Consider the infinitesimal rectangle shown in Fig. 4.4. This rectangle lies in the $x$-$y$ plane, so in the present analysis we will suppress the $z$-component of all coordinates, for convenience. If the potential is path-independent, then the work done in going from $(X, Y)$ to $(X + dX, Y + dY)$, which equals the integral $\int \mathbf{F} \cdot d\mathbf{r}$, must be path-independent. In particular, the integral along the segments "1" and "2" must equal the integral along the segments "3" and "4". That is, $\int_1 F_y\,dy + \int_2 F_x\,dx = \int_3 F_x\,dx + \int_4 F_y\,dy$. Therefore, a necessary condition for path-independence is

$$\begin{aligned} \int_2 F_x\,dx - \int_3 F_x\,dx = \int_4 F_y\,dy - \int_1 F_y\,dy \qquad &\Longrightarrow \\ \int_X^{X+dX} \Big(F_x(x, Y + dY) - F_x(x, Y)\Big)\,dx & \\ = \int_Y^{Y+dY} \Big(F_y(X + dX, y) - F_y(X, y)\Big)\,dy. & \end{aligned} \tag{4.25}$$

Now,

$$F_x(x, Y + dY) - F_x(x, Y) \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(x,Y)} \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(X,Y)}. \tag{4.26}$$

The first approximation holds due to the definition of the partial derivative. The second approximation holds because our rectangle is small enough so that $x$ is

[5]If you haven't seen curl before, it's defined below in eq. (4.30). But there is actually no need to be familiar with the definition of curl here, because it is, after all, just a definition. The important result that we will be deriving is the equality to the right of the "≡" sign in eq. (4.30).

essentially equal to $X$. Any errors due to this approximation will be second-order small, because we already have one factor of $dY$ in our term.

A similar treatment works for the $F_y$ terms, so eq. (4.25) becomes

$$\int_X^{X+dX} dY \frac{\partial F_x(x,y)}{\partial y}\bigg|_{(X,Y)} dx = \int_Y^{Y+dY} dX \frac{\partial F_y(x,y)}{\partial x}\bigg|_{(X,Y)} dy. \tag{4.27}$$

The integrands here are constants, so we can quickly perform the integrals to obtain

$$dX dY \left(\frac{\partial F_x(x,y)}{\partial y} - \frac{\partial F_y(x,y)}{\partial x}\right)\bigg|_{(X,Y)} = 0. \tag{4.28}$$

Cancelling the $dXdY$ factor, and noting that $(X, Y)$ is an arbitrary point, we see that if the potential is path-independent, then we must have

$$\frac{\partial F_x(x,y)}{\partial y} - \frac{\partial F_y(x,y)}{\partial x} = 0, \tag{4.29}$$

at any point $(x, y)$.

The preceding analysis also works, of course, for little rectangles in the $x$-$z$ and $y$-$z$ planes. We therefore obtain two other analogous conditions for the potential to be well-defined. All three conditions may be concisely written as

$$\nabla \times \mathbf{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0. \tag{4.30}$$

We have therefore shown that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path independence. Let us now show that it is sufficient. In other words, "If $\nabla \times \mathbf{F} = \mathbf{0}$, then $V(\mathbf{r})$ is path-independent."

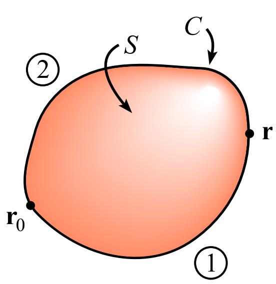

**Figure 4.5**

The proof of sufficiency follows immediately from Stokes' theorem (but see the remark below for another proof), which states that (see Fig. 4.5)

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}. \tag{4.31}$$

Here, $C$ is an arbitrary closed curve, which we make pass through $\mathbf{r}_0$ and $\mathbf{r}$. $S$ is an arbitrary surface that has $C$ as its boundary. And $d\mathbf{A}$ has a magnitude equal to an infinitesimal piece of area on $S$ and a direction defined to be orthogonal to $S$.

Eq. (4.31) implies that if $\nabla \times \mathbf{F} = 0$ everywhere, then $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any closed curve. But Fig. 4.5 shows that traversing the loop $C$ entails traversing path "1" in the "forward" direction, and then traversing path "2" in the "backward" direction. Hence, $\int_1 \mathbf{F} \cdot d\mathbf{r} - \int_2 \mathbf{F} \cdot d\mathbf{r} = 0$, where both integrals run from $\mathbf{r}_0$ to $\mathbf{r}$. Therefore, any two paths from $\mathbf{r}_0$ to $\mathbf{r}$ give the same integral, as we wanted to show. ■

Remarks:

1. If you don't like invoking Stokes' theorem, then you can just back up a step and prove it from scratch. Here's the rough idea of the proof. For simplicity, pick a path confined to the $x$-$y$ plane (the general case proceeds in the same manner). For the

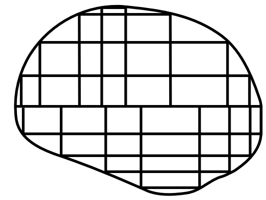

**Figure 4.6**

purposes of the $dx$ and $dy$ integrations, any path can be approximated by a series of little segments parallel to the coordinate axes (see Fig. 4.6).

Now imagine integrating $\int \mathbf{F} \cdot d\mathbf{r}$ over every little rectangle in the figure (in a counterclockwise direction). The result can be viewed in two ways: (1) From the above analysis leading to eq. (4.28), each integral gives the curl times the area of the rectangle. So whole integral gives $\int_S (\nabla \times \mathbf{F})\, dA$. (2) Each interior line gets counted twice (in opposite directions) in the whole integration, so these contributions cancel. We are therefore left with the integral over only the edge segments, which gives $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$. Equating these two ways of looking at the integration gives Stokes' theorem in eq. (4.31).

2. Another way to show that $\nabla \times \mathbf{F} = 0$ is a necessary condition for path-independence (that is, "If $V(\mathbf{r})$ is path-independent, then $\nabla \times \mathbf{F} = 0$.") is the following. If $V(\mathbf{r})$ is path-independent (and therefore well-defined), then it is legal to write down the differential form of eq. (4.22). This is

$$dV(\mathbf{r}) = -\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \equiv -(F_x\, dx + F_y\, dy + F_z\, dz). \tag{4.32}$$

But another expression for $dV$ is

$$dV(\mathbf{r}) = \frac{\partial V}{\partial x}\, dx + \frac{\partial V}{\partial y}\, dy + \frac{\partial V}{\partial z}\, dz. \tag{4.33}$$

The previous two equations must be equivalent for arbitrary $dx$, $dy$, and $dz$. So we have

$$\begin{aligned} (F_x, F_y, F_z) &= -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \\ \Longrightarrow \quad \mathbf{F}(\mathbf{r}) &= -\nabla V(\mathbf{r}). \end{aligned} \tag{4.34}$$

In other words, the force is simply the gradient of the potential. Therefore,

$$\nabla \times \mathbf{F} = -\nabla \times \nabla V(\mathbf{r}) = 0, \tag{4.35}$$

because the curl of a gradient is identically zero, as you can explicitly verify. ♣

---

**Example (Central force):** A *central force* is defined to be a force that points radially and whose magnitude depends only on $r$. That is, $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$. Show that a central force is a conservative force by explicitly showing that $\nabla \times \mathbf{F} = \mathbf{0}$.

**Solution:** $\mathbf{F}$ may be written as

$$\mathbf{F}(x, y, z) = F(r)\hat{\mathbf{r}} = F(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right). \tag{4.36}$$

Now, as you can verify,

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{x}{r}, \tag{4.37}$$

and similarly for $y$ and $z$. Therefore, the $z$ component of $\nabla \times \mathbf{F}$ equals (writing $F$ for $F(r)$, and $F'$ for $dF(r)/dr$)

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial (yF/r)}{\partial x} - \frac{\partial (xF/r)}{\partial y}$$

$$= \left(\frac{y}{r}F'\frac{\partial r}{\partial x} - yF\frac{1}{r^2}\frac{\partial r}{\partial x}\right) - \left(\frac{x}{r}F'\frac{\partial r}{\partial y} - xF\frac{1}{r^2}\frac{\partial r}{\partial y}\right)$$
$$= \left(\frac{yxF'}{r^2} - \frac{yxF}{r^3}\right) - \left(\frac{xyF'}{r^2} - \frac{xyF}{r^3}\right) = 0. \qquad (4.38)$$

Likewise for the $x$- and $y$-components.

## 4.4 Gravity

### 4.4.1 Gravity due to a sphere

The gravitational force on a point-mass $m$, located a distance $r$ from a point-mass $M$, is given by Newton's law of gravitation,

$$F(r) = \frac{-GMm}{r^2}\,, \qquad (4.39)$$

where the minus sign indicates an attractive force. What is the force if we replace the point mass $M$ by a sphere of radius $R$ and mass $M$? The answer (assuming that the sphere is spherically symmetric, that is, the density is a function only of $r$) is that it is still $-GMm/r^2$. A sphere acts just like a point mass at its center, for the purposes of gravity. This is an extremely pleasing result, to say the least. If it were not the case, then the universe would be a far more complicated place than it is. In particular, the motion of planets and such things would be much harder to describe.

To prove this result, it turns out to be much easier to calculate the potential energy due to a sphere, and to then take the derivative to obtain the force, rather than to calculate the force explicitly.[6] So this is the route we will take. It will suffice to demonstrate the result for a thin spherical shell, because a sphere is the sum of many such shells.

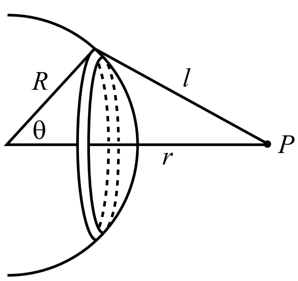


Figure 4.7

Our strategy for calculating the potential energy at a point $P$, due to a spherical shell, will be to slice the shell into rings as shown in Fig. 4.7. Let the radius of the shell be $R$. Let $P$ be a distance $r$ from the center of the shell, and let the ring make the angle $\theta$ shown.

The distance, $\ell$, from $P$ to the ring is a function of $R$, $r$, and $\theta$. It may be found as follows. In Fig. 4.8, segment $AB$ has length $R\sin\theta$, and segment $BP$ has length $r - R\cos\theta$. So the length $\ell$ in triangle $ABP$ is

$$\ell = \sqrt{(R\sin\theta)^2 + (r - R\cos\theta)^2} = \sqrt{R^2 + r^2 - 2rR\cos\theta}. \qquad (4.40)$$

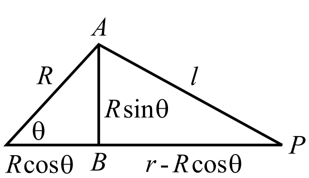


Figure 4.8

What we've done here is just prove the law of cosines.

The area of a ring between $\theta$ and $\theta + d\theta$ is its width (which is $R\,d\theta$) times its circumference (which is $2\pi R\sin\theta$). Letting $\sigma = M/(4\pi R^2)$ be the mass density of

[6]The reason for this is that the potential energy is a scalar quantity (just a number), whereas the force is a vector. If we tried to calculate the force, we would have to worry about forces pointing in all sorts of directions. With the potential energy, we simply have to add up a bunch of numbers.

the shell, we see that the potential energy of a mass $m$ at $P$ due to a thin ring is $-Gm\sigma(R\,d\theta)(2\pi R\sin\theta)/\ell$. This is true because the gravitational potential energy,

$$V(r) = \frac{-Gm_1m_2}{\ell}\,, \tag{4.41}$$

is a scalar quantity, so the contributions from the little mass pieces simply add. Every piece of the ring is the same distance from $P$, and this distance is all that matters; the direction from $P$ is irrelevant (unlike it would be with the force). The total potential energy at $P$ is therefore

$$\begin{aligned} V(r) &= -\int_0^\pi \frac{2\pi\sigma GR^2 m\sin\theta\,d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}} \\ &= -\frac{2\pi\sigma GRm}{r}\sqrt{R^2+r^2-2rR\cos\theta}\,\Big|_0^\pi. \end{aligned} \tag{4.42}$$

Note that the $\sin\theta$ in the numerator is what made this integral nice and doable. We must now consider two cases. If $r > R$, then we have

$$V(r) = -\frac{2\pi\sigma GRm}{r}\Big((r+R)-(r-R)\Big) = -\frac{G(4\pi R^2\sigma)m}{r} = -\frac{GMm}{r}\,, \tag{4.43}$$

which is the potential due to a point-mass $M$ located at the center of the shell, as desired. If $r < R$, then we have

$$V(r) = -\frac{2\pi\sigma GRm}{r}\Big((r+R)-(R-r)\Big) = -\frac{G(4\pi R^2\sigma)m}{R} = -\frac{GMm}{R}\,, \tag{4.44}$$

which is independent of $r$.

Having found $V(r)$, we can now find $F(r)$ by simply taking the negative of the gradient of $V$. The gradient is just $\hat{\mathbf{r}}(d/dr)$ here, because $V$ is a function only of $r$. Therefore,

$$\begin{aligned} F(r) &= -\frac{GMm}{r^2}\,, \qquad \text{if } r > R, \\ F(r) &= 0, \qquad \text{if } r < R. \end{aligned} \tag{4.45}$$

These forces are directed radially, of course. A sphere is the sum of many spherical shells, so if $P$ is outside a given sphere, then the force at $P$ is $-GMm/r^2$, where $M$ is the total mass of the sphere. This result will still hold even if the shells have different mass densities (but each one must have uniform density).

Newton looked at the data, numerical,
And then observations, empirical.
He said, "But, of course,
We get the same force
From a point mass and something that's spherical!"

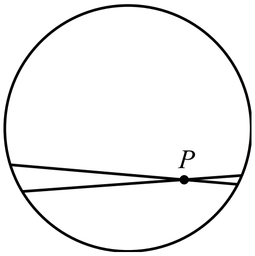


Figure 4.9

If $P$ is inside a given sphere, then the only relevant material is the mass inside a concentric sphere through $P$, because all the shells outside this region give zero force, from the second equation in eq. (4.45). The material "outside" of $P$ is, for the purposes of gravity, not there.

It is not obvious that the force inside a spherical shell is zero. Consider the point $P$ in Fig. 4.9. A piece of mass, $dm$, on the right side of the shell gives a larger force on $P$ than a piece of mass, $dm$, on the left side, due to the $1/r^2$ dependence. But from the figure, there is more mass on the left side than the right side. These two effects happen to exactly cancel, as you can show in Problem 9.

Note that the gravitational force between two spheres is the same as if they were replaced by two point-masses. This follows from two applications of our "point-mass" result.

### 4.4.2 Tides

The tides on the earth exist because the gravitational force from a point mass (or a spherical object, in particular the moon or the sun) is not uniform. The direction of the force is not constant (the force lines converge to the source), and the magnitude is not constant (it falls off like $1/r^2$). On the earth, these effects cause the oceans to bulge around the earth, producing the observed tides.

The study of tides is useful in part because tides are a very real phenomenon in the world, and in part because the following analysis gives us an excuse to make lots of approximations with Taylor series and such. Before considering the general case of tidal forces, let's look at two special cases.

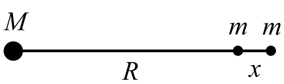


Figure 4.10

#### Longitudinal tidal force

In Fig. 4.10, two particles of mass $m$ are located at points $(R, 0)$ and $(R + x, 0)$, with $x \ll R$. A planet of mass $M$ is located at the origin. What is the difference between the gravitational forces acting on these two masses?

The difference in the forces is (using $x \ll R$ to make suitable approximations)

$$\begin{aligned} \frac{-GMm}{(R+x)^2} - \frac{-GMm}{R^2} &\approx \frac{-GMm}{R^2+2Rx} + \frac{GMm}{R^2} = \frac{GMm}{R^2}\left(\frac{-1}{1+2x/R}+1\right) \\ &\approx \frac{GMm}{R^2}\Big(-(1-2x/R)+1\Big) = \frac{2GMmx}{R^3}. \end{aligned} \tag{4.46}$$

This is, of course, simply the derivative of the force, times $x$. This difference points along the line joining the masses, and its effect is to pull the masses apart.

We see that this force difference is linear in the separation $x$, and inversely proportional to the *cube* of the distance from the source. This *force difference* is the important quantity (as opposed to the force on each mass) when we are dealing with the *relative* motion of objects in free-fall around a given mass (for example, circular orbiting motion, or radial falling motion). This force difference is referred to as the "tidal force."

Consider two people, $A$ and $B$, both of mass $m$, in radial free-fall toward a planet. Imagine that they are connected by a string, and enclosed in a windowless box. Neither can feel the gravitational force acting on him (for all they know, they are floating freely in space). But they each feel a tension in the string equal to $T = GMmx/R^3$ (neglecting higher-order terms in $x/R$), pulling in opposite directions. The difference in these tension forces is $2T$, which exactly cancels the difference in the gravitational force, thereby allowing the separation to remain fixed.

How do $A$ and $B$ view the situation? They will certainly feel the tension force. They will therefore conclude that there must be some other mysterious "tidal force" that opposes the tension, yielding a total net force of zero, as measured in their windowless box.

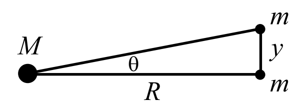

**Figure 4.11**

**Transverse tidal force**

In Fig. 4.11, two particles of mass $m$ are located at points $(R, 0)$ and $(R, y)$, with $y \ll R$. A planet of mass $M$ is located at the origin. What is the difference between the gravitational forces acting on these two masses?

Both masses are the same distance $R$ from the origin, up to second-order effects in $y/R$ (using the Pythagorean theorem), so the magnitudes of the forces on them are essentially the same. The direction is the only thing that is different, to first order in $y/R$. The difference in the forces is the $y$-component of the force on the top mass. The magnitude of this component is

$$\frac{GMm}{R^2}\sin\theta \approx \frac{GMm}{R^2}\left(\frac{y}{R}\right) = \frac{GMmy}{R^3}\,. \tag{4.47}$$

This difference points along the line joining the masses, and its effect is to pull the masses together. As in the longitudinal case, the transverse tidal force is linear in the separation $y$, and inversely proportional to the cube of the distance from the source.

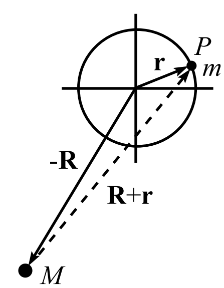

**Figure 4.12**

**General tidal force**

We will now calculate the tidal force at an arbitrary point on a circle of radius $r$ centered at the origin (this circle represents the earth), due to a mass $M$ located at the vector $-\mathbf{R}$; see Fig. 4.12. We will calculate the tidal force relative to the origin. Note that the vector from $M$ to a point $P$ on the circle is $\mathbf{R}+\mathbf{r}$. And as usual, assume $|\mathbf{r}| \ll |\mathbf{R}|$.

The attractive gravitational force may be written as $\mathbf{F}(\mathbf{x}) = -GMm\mathbf{x}/|\mathbf{x}|^3$, where $\mathbf{x}$ is the vector from $M$ to the point in question. The cube is in the denominator because the vector in the numerator contains one power of the distance. In the present case we have $\mathbf{x} = \mathbf{R}+\mathbf{r}$, so the desired difference between the force on a mass $m$ at point $P$ and the force on a mass $m$ at the origin is the tidal force $\mathbf{F}_{\rm t}(\mathbf{r})$ given by

$$\frac{\mathbf{F}_{\rm t}(\mathbf{r})}{GMm} = \frac{-(\mathbf{R}+\mathbf{r})}{|\mathbf{R}+\mathbf{r}|^3} - \frac{-\mathbf{R}}{|\mathbf{R}|^3}\,. \tag{4.48}$$

This is the exact expression for the tidal force. However, it is completely useless.[7]

[7] This reminds me of a joke about two people lost in a hot-air balloon.

Let us therefore make some approximations in eq. (4.48) and transform it into something technically incorrect (as approximations tend to be), but far more useful.

The first thing we need to do is rewrite the $|\mathbf{R}+\mathbf{r}|$ term. We have (using $r \ll R$ and ignoring higher-order terms)

$$\begin{aligned} |\mathbf{R}+\mathbf{r}| &= \sqrt{(\mathbf{R}+\mathbf{r})\cdot(\mathbf{R}+\mathbf{r})} \\ &= \sqrt{R^2+r^2+2\mathbf{R}\cdot\mathbf{r}} \\ &\approx R\sqrt{1+2\mathbf{R}\cdot\mathbf{r}/R^2} \\ &\approx R\left(1+\frac{\mathbf{R}\cdot\mathbf{r}}{R^2}\right). \end{aligned} \tag{4.49}$$

Therefore (again using $r \ll R$),

$$\begin{aligned} \frac{\mathbf{F}_{\mathrm{t}}(\mathbf{r})}{GMm} &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+\mathbf{R}\cdot\mathbf{r}/R^2)^3}+\frac{\mathbf{R}}{R^3} \\ &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+3\mathbf{R}\cdot\mathbf{r}/R^2)}+\frac{\mathbf{R}}{R^3} \\ &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3}\left(1-\frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\right)+\frac{\mathbf{R}}{R^3}. \end{aligned} \tag{4.50}$$

Letting $\hat{\mathbf{R}} \equiv \mathbf{R}/R$, we finally have (once again using $r \ll R$)

$$\mathbf{F}_{\mathrm{t}}(\mathbf{r}) \approx \frac{GMm(3\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\mathbf{r})-\mathbf{r})}{R^3}. \tag{4.51}$$

This is the general expression for the tidal force. We can put it in a simpler form if we let $M$ lie on the negative $x$-axis, which can arrange for with a rotation of the axes. We then have $\hat{\mathbf{R}} = \hat{\mathbf{x}}$, and so $\hat{\mathbf{R}}\cdot\mathbf{r} = x$. Eq. (4.51) then tells us that the tidal force at the point $P = (x, y)$ equals

$$\mathbf{F}_{\mathrm{t}}(\mathbf{r}) \approx \frac{GMm}{R^3}\Big(3x\hat{\mathbf{x}} - (x\hat{\mathbf{x}}+y\hat{\mathbf{y}})\Big) = \frac{GMm}{R^3}\,(2x, -y). \tag{4.52}$$

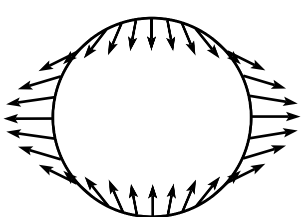

**Figure 4.13**

This reduces properly in the two special cases considered above. The tidal forces at various points on the circle are shown in Fig. 4.13.

If the earth were a rigid body, then the tidal force would have no effect on it. But the water in the oceans is free to slosh around. The water on the earth bulges along the line from the earth to the moon, and also along the line from the earth to the sun. As the earth rotates beneath the bulge, a person on the earth sees the bulge rotate relative to the earth. From Fig. 4.13, we see that this produces *two* high tides and two low tides per day. It's actually not exactly two per day, because the moon moves around the earth. But this motion is fairly slow, taking about a month, so it's a reasonable approximation for the present purposes to think of the moon as motionless.

Note that it is *not* the case that the moon *pushes* the water away on the far side of the earth. It pulls on that water, too; it just does so in a weaker manner than it pulls on the rigid part of the earth. Tides are a *comparative* effect.

REMARKS:

1. Consider two equal masses separated by a given distance on the earth. It turns out that the gravitational force from the sun on them is (much) larger than that from the moon, whereas the tidal force from the sun on them is (slightly) weaker than that from the moon. Quantitatively, the ratio of the gravitational forces is

$$\frac{F_{\rm S}}{F_{\rm M}} = \left(\frac{GM_{\rm S}}{R^2_{\rm E,S}}\right) \Big/ \left(\frac{GM_{\rm M}}{R^2_{\rm E,M}}\right) = \frac{6\cdot 10^{-3}\,{\rm m/s^2}}{3.4\cdot 10^{-5}\,{\rm m/s^2}} \approx 175. \tag{4.53}$$

And the ratio of the tidal forces is

$$\frac{F_{\rm t,S}}{F_{\rm t,M}} = \left(\frac{GM_{\rm S}}{R^3_{\rm E,S}}\right) \Big/ \left(\frac{GM_{\rm M}}{R^3_{\rm E,M}}\right) = \frac{4\cdot 10^{-14}\,{\rm s^{-2}}}{9\cdot 10^{-14}\,{\rm s^{-2}}} \approx 0.45. \tag{4.54}$$

2. Eq. (4.54) shows that the moon's tidal effect is roughly twice the sun's. This has an interesting implication about the densities of the moon and sun. Note that the tidal force from, say, the moon is proportional to

$$\left(\frac{GM_{\rm M}}{R^3_{\rm E,M}}\right) = \left(\frac{G\left(\frac{4}{3}\pi r^3_{\rm M}\right)\rho_{\rm M}}{R^3_{\rm E,M}}\right) \propto \rho_{\rm M}\left(\frac{r_{\rm M}}{R_{\rm E,M}}\right)^3 \approx \rho_{\rm M}\theta^3_{\rm M}, \tag{4.55}$$

where $\theta_{\rm M}$ is half of the angular size of the moon in the sky. Likewise for the sun's tidal force. But it just so happens that the angular sizes of the sun and the moon are essentially equal, as you can see by looking at them (preferably through some haze), or by noting that total solar eclipses barely exist. Therefore, the combination of eq. (4.54) and eq. (4.55) tells us that the moon's density is about twice the sun's. ♣

## 4.5 Momentum

### 4.5.1 Conservation of momentum

Newton's third law says that for every force there is an equal and opposite force. More precisely, if $\mathbf{F}_{ab}$ is the force that particle $a$ feels due to particle $b$, and if $\mathbf{F}_{ba}$ is the force that particle $b$ feels due to particle $a$, then $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$ at all times.

This law has important implications concerning momentum. Consider two particles that interact over a period of time. Assume that they are isolated from outside forces. From Newton's second law,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,, \tag{4.56}$$

we see that the total change in a particle's momentum equals the time integral of the force acting on it. That is,

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} \mathbf{F}\,dt. \tag{4.57}$$

This integral is called the *impulse*. If we now invoke the third law, $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$, we find

$$\begin{aligned}\mathbf{p}_a(t_2) - \mathbf{p}_a(t_1) &= \int_{t_1}^{t_2} \mathbf{F}_{ab} dt \\ &= -\int_{t_1}^{t_2} \mathbf{F}_{ba} dt \;=\; -\Big(\mathbf{p}_b(t_2) - \mathbf{p}_b(t_1)\Big).\end{aligned} \tag{4.58}$$

Therefore,

$$\mathbf{p}_a(t_2) + \mathbf{p}_b(t_2) = \mathbf{p}_a(t_1) + \mathbf{p}_b(t_1). \tag{4.59}$$

In other words, the total momentum of this isolated system is *conserved*. It does not depend on time. Note that eq. (4.59) is a vector equation, so it is really three equations, namely conservation of $p_x$, $p_y$, and $p_z$.

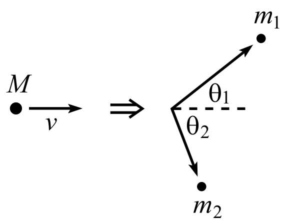


**Figure 4.14**

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**Example (Splitting mass):** A mass $M$ moves with speed $V$ in the $x$-direction. It explodes into two pieces that go off at angles $\theta_1$ and $\theta_2$, as shown in Fig. 4.14. What are the magnitudes of the momenta of the two pieces?

**Solution:** Let $P \equiv MV$ be the initial momentum, and let $p_1$ and $p_2$ be the final momenta. Conservation of momentum in the $x$- and $y$-directions gives, respectively,

$$\begin{aligned} p_1 \cos\theta_1 + p_2 \cos\theta_2 &= P, \\ p_1 \sin\theta_1 - p_2 \sin\theta_2 &= 0. \end{aligned} \tag{4.60}$$

Solving for $p_1$ and $p_2$, and using a trig addition formula, gives

$$p_1 = \frac{P\sin\theta_2}{\sin(\theta_1+\theta_2)}, \qquad \text{and} \qquad p_2 = \frac{P\sin\theta_1}{\sin(\theta_1+\theta_2)}. \tag{4.61}$$

Let's check a few limits. If $\theta_1 = \theta_2$, then $p_1 = p_2$, as it should. If, in addition, $\theta_1$ and $\theta_2$ are both small, then $p_1 = p_2 \approx P/2$, as they should. If, on the other hand, $\theta_1 = \theta_2 \approx 90°$, then $p_1$ and $p_2$ are both very large; the explosion must have provided a large amount of energy.

Note that with the given information, we can't determine what the masses of the two pieces are. To find these, we would need to know two more pieces on information, such as how much energy the explosion gave to the system, and what one of the masses or speeds is. Then we would have an equal number of equations and unknowns.

---

Remark: Newton's third law makes a statement about forces. But force is defined in terms of momentum via $F = dp/dt$. So the third law essentially *postulates* conservation of momentum. (The "proof" above in eq. (4.58) is hardly a proof. It involves one simple integration.) So you might wonder if momentum conservation is something you can *prove*, or if it's something you have to *assume* (as we have basically done).

The difference between a postulate and a theorem is rather nebulous. One person's postulate might be another person's theorem, and vice-versa. You have to start *somewhere* in your assumptions. We chose to start with the third law. In the Lagrangian formalism in Chapter 5, the starting point is different, and momentum conservation is deduced as a consequence of translational invariance (as we will see). So it looks more like a theorem in that formalism.

But one thing is certain. Momentum conservation of two particles *cannot* be proven from scratch for arbitrary forces, because it is not necessarily true. For example, if two charged particles interact in a certain way through the magnetic fields they produce, then the total momentum of the two particles might *not* be conserved. Where is the missing momentum? It is carried off in the electromagnetic field. The total momentum of the system is indeed conserved, but the fact of the matter is that the system consists of the two particles *plus* the electromagnetic field. Said in another way, each particle actually interacts with the electromagnetic field, and not the other particle. Newton's third law does not necessarily hold for particles subject to such a force. ♣

Let's now look at momentum conservation for a system of many particles. As above, let $\mathbf{F}_{ij}$ be the force that particle $i$ feels due to particle $j$. Then $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ at all times. Assume the particles are isolated from outside forces.

The change in the momentum of the $i$th particle from $t_1$ to $t_2$ is (we won't bother writing all the $t$'s in the expressions below)

$$\Delta \mathbf{p}_i = \int \left( \sum_j \mathbf{F}_{ij} \right) dt. \tag{4.62}$$

Therefore, the change in the total momentum of all the particles is

$$\Delta \mathbf{P} \equiv \sum_i \Delta \mathbf{p}_i = \int \left( \sum_i \sum_j \mathbf{F}_{ij} \right) dt. \tag{4.63}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ at all times, because for every term $\mathbf{F}_{ab}$, there is a term $\mathbf{F}_{ba}$, and $\mathbf{F}_{ab} + \mathbf{F}_{ba} = 0$. (And also, $\mathbf{F}_{aa} = 0$.) Therefore, the total momentum of an isolated system of particles is conserved.

### 4.5.2 Rocket motion

The application of momentum conservation becomes a little more exciting when the mass $m$ is allowed to vary. Such is the case with rockets, because most of their mass consists of fuel which is eventually ejected.

Let mass be ejected with speed $u$ relative to the rocket,[8] at a rate $dm/dt$. We'll define the quantity $dm$ to be negative, so during a time $dt$ the mass $dm$ gets *added* to the rocket's mass. (If you wanted, you could define $dm$ to be positive, and then *subtract* it from the rocket's mass. Either way is fine.) Also, we'll define $u$ to be positive, so the ejected particles *lose* a speed $u$ relative to the rocket. It may sound silly, but the hardest thing about rocket motion is picking a sign for these quantities and sticking with it.

Consider a moment when the rocket has mass $m$ and speed $v$. Then at a time $dt$ later (see Fig. 4.15), the rocket has mass $m+dm$ and speed $v+dv$, while the exhaust has mass $(-dm)$ and speed $v-u$ (which may be positive or negative, depending on the relative size of $v$ and $u$). There are no external forces, so the total momentum at each of these times must be equal. Therefore,

$$mv = (m + dm)(v + dv) + (-dm)(v - u). \tag{4.64}$$

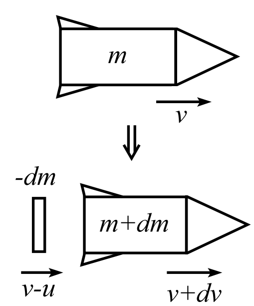


**Figure 4.15**

[8] Just to emphasize, $u$ is the speed with respect to the rocket. It wouldn't make much sense to say "relative to the ground," because the rocket's engine spits out the matter relative to itself, and the engine has no way of knowing how fast the rocket is moving with respect to the ground.

Ignoring the second-order term yields $m\,dv = -u\,dm$. Dividing by $m$ and integrating from $t_1$ to $t_2$ gives

$$\int_{v_1}^{v_2} dv = -\int_{m_1}^{m_2} u\frac{dm}{m} \qquad \Longrightarrow \qquad v_2 - v_1 = u\ln\frac{m_1}{m_2}\,. \tag{4.65}$$

For the case where the initial mass is $M$ and the initial speed is 0, we have $v = u\ln(M/m)$. And if $dm/dt$ happens to be constant (call it $-\eta$, where $\eta$ is positive), then $v(t) = u\ln[M/(M-\eta t)]$.

The log in the result in eq. (4.65) is not very encouraging. If the mass of the metal in the rocket is $m$, and if the mass of the fuel is $9m$, then the final speed is only $u\ln 10 \approx (2.3)u$. If the mass of the fuel is increased by a factor of 11 up to $99m$ (which is probably not even structurally possible, given the amount of metal required to hold it), then the final speed only doubles to $u\ln 100 = 2(u\ln 10) \approx (4.6)u$. How do you make a rocket go significantly faster? Exercise 33 deals with this question.

Remark: If you want, you can solve this rocket problem by using force instead of conservation of momentum. If a chunk of mass $(-dm)$ is ejected out the back, then its momentum changes by $u\,dm$ (which is negative). Since force equals the rate of change of momentum, the force on this chunk is $u\,dm/dt$. By Newton's third law, the remaining part of the rocket feels a force of $-u\,dm/dt$ (which is positive). This force accelerates the remaining part of the rocket, so $F = ma$ gives $-u\,dm/dt = m\,dv/dt$,[9] which is equivalent to the $m\,dv = -u\,dm$ result above.

We see that this rocket problem can be solved by using either force or conservation of momentum. In the end, these two strategies are really the same, because the latter was derived from $F = dp/dt$. But the philosophies behind the approaches are somewhat different. The choice of strategy depends on personal preference. In an isolated system such as a rocket, conservation of momentum is usually simpler. But in a problem involving an external force, $F = dp/dt$ is the way to go. You'll get lots of practice with $F = dp/dt$ in the problems for this section and also in Section 4.8.

Note that we used both $F = dp/dt$ and $F = ma$ in this second solution to the rocket problem. These are not equal if the mass of a particle changes. For further discussion on which expression to use in a given situation, see Appendix E. ♣

## 4.6 The CM frame

### 4.6.1 Definition

When talking about momentum, it is understood that a certain frame of reference has been picked. After all, the velocities of the particles have to be measured with respect to some coordinate system. Any inertial (that is, non-accelerating) frame is as good as any other, but we will see that there is one particular reference frame that is often advantageous to use.

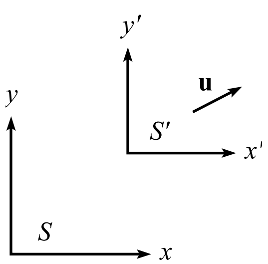

Figure 4.16

Consider a frame $S$ and another frame $S'$ that moves at constant velocity $\mathbf{u}$ with respect to $S$ (see Fig. 4.16). Given a system of particles, the velocity of the $i$th

[9] Whether we use $m$ or $m + dm$ here for the mass of the rocket doesn't matter. Any differences are of second order.

particle in $S$ is related to its velocity in $S'$ by

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{4.66}$$

It is then easy to see that if momentum is conserved during a collision in frame $S'$, then it is also conserved in frame $S$. This is true because both the initial and final momenta of the system in $S$ are increased by the same amount $(\sum m_i)\mathbf{u}$, compared to what they are in $S'$.[10]

Let us therefore consider the unique frame in which the total momentum of a system of particles is zero. This is called the *center of mass frame*, or CM frame. If the total momentum is $\mathbf{P} \equiv \sum m_i \mathbf{v}_i$ in frame $S$, then the CM frame $S'$ is the frame that moves with velocity

$$\mathbf{u} = \frac{\mathbf{P}}{M} \equiv \frac{\sum m_i \mathbf{v}_i}{M} \tag{4.67}$$

with respect to $S$, where $M \equiv \sum m_j$ is the total mass. This is true because we can use eq. (4.66) to write

$$\begin{aligned} \mathbf{P}' &= \sum m_i \mathbf{v}_i' \\ &= \sum m_i \left( \mathbf{v}_i - \frac{\mathbf{P}}{M} \right) \\ &= \mathbf{P} - \mathbf{P} = \mathbf{0}. \end{aligned} \tag{4.68}$$

The CM frame is extremely useful. Physical processes are generally much more symmetrical in this frame, and this makes the results more transparent.

The CM frame is also sometimes called the "zero-momentum" frame. But the "center of mass" name is commonly used because the center of mass of the particles does not move in the CM frame, defined by the velocity in eq. (4.67). The position of the center of mass is given by

$$\mathbf{R}_{\text{CM}} \equiv \frac{\sum m_i \mathbf{r}_i}{M}\,. \tag{4.69}$$

This is the location of the pivot upon which a rigid system would balance, as we will see in Chapter 7. The fact that the CM doesn't move in the CM frame follows from the fact that the derivative of $\mathbf{R}_{\text{CM}}$ is simply the velocity of the CM frame in eq. (4.67). The center of mass may therefore be chosen as the origin of the CM frame.

Along with the CM frame, the other frame that people generally work with is the *lab frame*. There is nothing at all special about this frame. It is simply the frame (assumed to be inertial) in which the conditions of the problem are given. Any inertial frame can be called the "lab frame." Solving problems often involves switching back and forth between the lab and CM frames. For example, if the final answer is requested in the lab frame, then you may want to transform the given

[10] Alternatively, nowhere in our earlier derivation of momentum conservation did we say what frame we were using. We only assumed that the frame was not accelerating. If it were accelerating, then $\mathbf{F}$ would *not* equal $m\mathbf{a}$. We will see in Chapter 9 how $\mathbf{F} = m\mathbf{a}$ is modified in a non-inertial frame. But no need to worry about that here.

information from the lab frame into the CM frame where things are more obvious, and then transform back to the lab frame to give the answer.

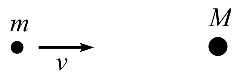


**Figure 4.17**

**Example (Two masses in 1-D):** A mass $m$ with speed $v$ approaches a stationary mass $M$ (see Fig. 4.17). The masses bounce off each other without any loss in total energy. What are the final velocities of the particles? Assume all motion takes place in 1-D.

**Solution:** Doing this problem in the lab frame would require a potentially messy use of conservation of energy (see the example in Section 4.7.1). But if we work in the CM frame, things are much easier.

The total momentum in the lab frame is $mv$, so the CM frame moves to the right with speed $mv/(m+M) \equiv u$ with respect to the lab frame. Therefore, in the CM frame, the velocities of the two masses are

$$v_m = v - u = \frac{Mv}{m+M}, \qquad \text{and} \qquad v_M = -u = -\frac{mv}{m+M}. \tag{4.70}$$

As a double-check, the difference in the velocities is $v$, and the ratio of the speeds is $M/m$.

The important point to realize now is that in the CM frame, the two particles must simply reverse their velocities after the collision (provided that they do indeed hit each other). This is true because the speeds must still be in the ratio $M/m$ after the collision, in order for the total momentum to remain zero. Therefore, the speeds must either both increase or both decrease. But if they do either of these, then energy is not conserved.[11]

If we now go back to the lab frame by adding the CM velocity of $mv/(m+M)$ to the two new velocities of $-Mv/(m+M)$ and $mv/(m+M)$, we obtain final lab velocities of

$$v_m = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}. \tag{4.71}$$

Remark: If $m = M$, then we see that the left mass stops, and the right mass picks up a speed of $v$. If $M \gg m$, then the left mass bounces back with speed $\approx v$, and the right mass hardly moves. If $m \gg M$, then the left mass keeps plowing along at speed $\approx v$, and the right mass picks up a speed of $\approx 2v$. This $2v$ is an interesting result (it is clearer if you consider things in the frame of the heavy mass $m$, which is essentially the CM frame), and it leads to some neat effects, as in Problem 22.

### 4.6.2 Kinetic energy

Given a system of particles, the relationship between the total kinetic energy in two different frames is generally rather messy and unenlightening. But if one of the frames is the CM frame, then the relationship turns out to be quite nice.

[11] So we *did* have to use conservation of energy in this CM-frame solution. But it was far less messy than it would have been in the lab frame.

Let $S'$ be the CM frame, which moves at constant velocity $\mathbf{u}$ with respect to another frame $S$. Then the velocities of the particles in the two frames are related by

$$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{u}. \tag{4.72}$$

The kinetic energy in the CM frame is

$$\mathrm{KE}_{\mathrm{CM}} = \frac{1}{2}\sum m_i |\mathbf{v}'_i|^2. \tag{4.73}$$

And the kinetic energy in frame $S$ is

$$\begin{aligned}
\mathrm{KE}_{\mathrm{S}} &= \frac{1}{2}\sum m_i |\mathbf{v}'_i + \mathbf{u}|^2 \\
&= \frac{1}{2}\sum m_i (\mathbf{v}'_i \cdot \mathbf{v}'_i + 2\mathbf{v}'_i \cdot \mathbf{u} + \mathbf{u}\cdot\mathbf{u}) \\
&= \frac{1}{2}\sum m_i |\mathbf{v}'_i|^2 + \mathbf{u}\cdot\left(\sum m_i \mathbf{v}'_i\right) + \frac{1}{2}|\mathbf{u}|^2 \sum m_i \\
&= \mathrm{KE}_{\mathrm{CM}} + \frac{1}{2}Mu^2,
\end{aligned} \tag{4.74}$$

where $M$ is the total mass of the system, and where we have used $\sum_i m_i \mathbf{v}'_i = 0$, by definition of the CM frame. Therefore, the KE in any frame equals the KE in the CM frame, plus the kinetic energy of the whole system treated like a point mass $M$ located at the CM (which moves with velocity $\mathbf{u}$). An immediate corollary of this fact is that if the KE is conserved in a collision in one frame, then it is conserved in any other frame.

## 4.7 Collisions

There are two basic types of collisions among particles, namely *elastic* ones (in which kinetic energy is conserved), and *inelastic* ones (in which kinetic energy is lost). In any collision, the *total* energy is conserved, but in inelastic collisions some of this energy goes into the form of heat (that is, relative motion of the atoms inside the particles) instead of showing up in the net translational motion of the particle.

We'll deal mainly with elastic collisions here, although some situations are inherently inelastic, as we'll discuss in Section 4.8. For inelastic collisions where it is stated that a certain fraction, say 20%, of the kinetic energy is lost, only a trivial modification of the following procedure is required.

To solve any elastic collision problem, we simply have to write down the conservation of energy and momentum equations, and then solve for whatever variables we want to find.

### 4.7.1 1-D motion

Let's first look at one-dimensional motion. To see the general procedure, we'll solve the example from Section 4.6.1 again.


**Figure 4.18**

**Example (Two masses in 1-D, again):** A mass $m$ with speed $v$ approaches a stationary mass $M$ (see Fig. 4.18). The masses bounce off each other elastically. What are the final velocities of the particles? Assume all motion takes place in 1-D.

**Solution:** Let $v'$ and $V'$ be the final velocities of the masses.[12] Then conservation of momentum and energy give, respectively,

$$\begin{aligned} mv + 0 &= mv' + MV', \\ \frac{1}{2}mv^2 + 0 &= \frac{1}{2}mv'^2 + \frac{1}{2}MV'^2. \end{aligned} \tag{4.75}$$

We must solve these two equations for the two unknowns $v'$ and $V'$. Solving for $V'$ in the first equation and substituting into the second gives

$$\begin{aligned} mv^2 &= mv'^2 + M\frac{m^2(v-v')^2}{M^2}, \\ \Longrightarrow \quad 0 &= (m+M)v'^2 - 2mvv' + (m-M)v^2, \\ \Longrightarrow \quad 0 &= \Big((m+M)v' - (m-M)v\Big)(v'-v). \end{aligned} \tag{4.76}$$

One solution is $v' = v$, but this is not the one we are concerned with. It is of course a solution, because the initial conditions certainly satisfy conservation of energy and momentum with the initial conditions (a fine tautology indeed). If you want, you can view $v' = v$ as the solution where the particles miss each other. The fact that $v' = v$ is always a root can often save you a lot of quadratic-formula trouble.

The $v' = v(m-M)/(m+M)$ root is the one we want. Plugging this $v'$ back into the first of eqs. (4.75) to obtain $V'$ gives

$$v' = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad V' = \frac{2mv}{m+M}, \tag{4.77}$$

in agreement with eq. (4.71).

This solution was somewhat of a pain, because it involved a quadratic equation. The following theorem is extremely useful because it offers a way to avoid the hassle of quadratic equations when dealing with 1-D elastic collisions.

**Theorem 4.3** *In a 1-D elastic collision, the relative velocity of two particles after a collision is the negative of the relative velocity before the collision.*

**Proof:** Let the masses be $m$ and $M$. Let $v_i$ and $V_i$ be the initial velocities, and let $v_f$ and $V_f$ be the final velocities. Conservation of momentum and energy give

$$\begin{aligned} mv_i + MV_i &= mv_f + MV_f \\ \frac{1}{2}mv_i^2 + \frac{1}{2}MV_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2. \end{aligned} \tag{4.78}$$

[12]In Section 4.6, a primed denoted a reference frame, but we're now using a prime to denote "final."

Rearranging these yields

$$\begin{aligned} m(v_i - v_f) &= M(V_f - V_i). \\ m(v_i^2 - v_f^2) &= M(V_f^2 - V_i^2) \end{aligned} \tag{4.79}$$

Dividing the second equation by the first gives $v_i + v_f = V_i + V_f$. Therefore,

$$v_i - V_i = -(v_f - V_f), \tag{4.80}$$

as we wanted to show. Note that in taking the quotient of these two equations, we have lost the $v_f = v_i$ and $V_f = V_i$ solution. But as stated in the above example, this is the trivial solution. ■

This is a splendid theorem. It has the quadratic energy-conservation statement built into it. Hence, using this theorem along with momentum conservation (both of which are linear statements) gives the same information as the standard combination of eqs. (4.78).

Note that the theorem is quite obvious in the CM frame (as we argued in the example in Section 4.6.1). Therefore, it is true in any frame, because it involves only differences in velocities.

### 4.7.2 2-D motion

Let's now look at the more general case of two-dimensional motion. 3-D motion is just more of the same, so we'll confine ourselves to 2-D. Everything is basically the same as in 1-D, except that there is one more momentum equation, and one more variable to solve for. This is best seen through an example.

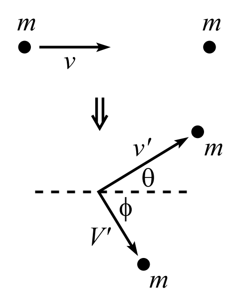


**Figure 4.19**

**Example (Billiards):** A billiard ball with speed $v$ approaches an identical stationary one. The balls bounce off each other elastically, in such a way that the incoming one gets deflected by an angle $\theta$ (see Fig. 4.19). What are the final speeds of the balls? What is the angle, $\phi$, at which the stationary ball is ejected?

**Solution:** Let $v'$ and $V'$ be the final speeds of the balls. Then conservation of $p_x$, $p_y$, and $E$ give, respectively,

$$\begin{aligned} mv &= mv'\cos\theta + mV'\cos\phi, \\ mv'\sin\theta &= mV'\sin\phi, \\ \frac{1}{2}mv^2 &= \frac{1}{2}mv'^2 + \frac{1}{2}mV'^2. \end{aligned} \tag{4.81}$$

We must solve these three equations for the three unknowns $v'$, $V'$, and $\phi$. There are various ways to do this. Here is one. Eliminate $\phi$ by adding the squares of the first two equations (after putting the $mv'\cos\theta$ on the left-hand side) to obtain

$$v^2 - 2vv'\cos\theta + v'^2 = V'^2. \tag{4.82}$$

Now eliminate $V'$ by combining this with the third equation to obtain[13]

$$v' = v\cos\theta. \tag{4.83}$$

The third equation then implies

$$V' = v\sin\theta. \tag{4.84}$$

The second equation then gives $m(v\cos\theta)\sin\theta = m(v\sin\theta)\sin\phi$, which implies $\cos\theta = \sin\phi$, or

$$\phi = 90^\circ - \theta. \tag{4.85}$$

In other words, the balls bounce off at right angles with respect to each other. This fact is well known to pool players. Problem 18 gives another (cleaner) way to demonstrate this result.

---

As we saw in the 1-D example in Section 4.6.1, collisions are often much easier to deal with in the CM frame. Using the same reasoning (conservation of $p$ and $E$) as in that example, we conclude that in 2-D (or 3-D), the final speeds of two elastically colliding particles must be the same as the initial speeds. The only degree of freedom is the angle of the line containing the final (oppositely directed) velocities. This simplicity in the CM frame invariably provides for a cleaner solution than the lab frame would yield. A good example of this is Exercise 43, which gives yet another way to derive the above right-angle billiard result.

## 4.8 Inherently inelastic processes

There is a nice class of problems where the system has inherently inelastic properties, even if it doesn't appear so at first glance. In such a problem, no matter how you try to set it up, there will be inevitable kinetic energy loss that shows up in the form of heat. Total energy is conserved, of course; heat is simply another form of energy. But the point is that if you try to write down a bunch of $(1/2)mv^2$'s and conserve their sum, then you're going to get the wrong answer. The following example is the classic illustration of this type of problem.

---

**Example (Sand on conveyor belt):** Sand drops vertically at a rate $\sigma$ kg/s onto a moving conveyor belt.

(a) What force must you apply to the belt in order to keep it moving at a constant speed $v$?

(b) How much kinetic energy does the sand gain per unit time?

(c) How much work do you do per unit time?

(d) How much energy is lost to heat per unit time?

[13] Another solution is $v' = 0$. In this case, $\phi$ must equal zero, and $\theta$ is not well-defined. We simply have the 1-D motion of the example in Section 4.6.1.

**Solution:**

(a) Your force equals the rate of change of momentum. If we let $m$ be the combined mass of the conveyor belt plus the sand on the belt, then

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = 0 + \sigma v, \tag{4.86}$$

where we have used the fact that $v$ is constant.

(b) The kinetic energy gained per unit time is

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = \frac{dm}{dt}\left(\frac{v^2}{2}\right) = \frac{\sigma v^2}{2}\,. \tag{4.87}$$

(c) The work done by your force per unit time is

$$\frac{d(\text{Work})}{dt} = \frac{F\,dx}{dt} = Fv = \sigma v^2, \tag{4.88}$$

where we have used eq. (4.86).

(d) If work is done at a rate $\sigma v^2$, and kinetic energy is gained at a rate $\sigma v^2/2$, then the "missing" energy must be lost to heat at a rate $\sigma v^2 - \sigma v^2/2 = \sigma v^2/2$.

---

In this example, it turned out that exactly the same amount of energy was lost to heat as was converted into kinetic energy of the sand. There is an interesting and simple way to see why this is true. In the following explanation, we'll just deal with one particle of mass $M$ that falls onto the conveyor belt, for simplicity.

In the lab frame, the mass simply gains a kinetic energy of $Mv^2/2$ by the time it finally comes to rest with respect to the belt, because the belt moves at speed $v$.

Now look at things in the conveyor belt's reference frame. In this frame, the mass comes flying in with an initial kinetic energy of $Mv^2/2$, and then it eventually slows down and comes to rest on the belt. Therefore, all of the $Mv^2/2$ energy is converted to heat. And since the heat is the same in both frames, this is the amount of heat in the lab frame, too.

We therefore see that in the lab frame, the equality of the heat loss and the gain in kinetic energy is a consequence of the obvious fact that the belt moves at the same rate with respect to the lab (namely $v$) as the lab moves with respect to the belt (also $v$).

In the solution to the above example, we did not assume anything about the nature of the friction force between the belt and the sand. The loss of energy to heat is an unavoidable result. You might think that if the sand comes to rest on the belt very "gently" (over a long period of time), then you can avoid the heat loss. This is not the case. In that scenario, the smallness of the friction force is compensated by the fact that the force must act over a very large distance. Likewise, if the sand comes to rest on the belt very abruptly, then the largeness of the friction force is compensated by the smallness of the distance over which it acts. No matter how you set things up, the work done by the friction force is the same nonzero quantity.

In other problems such as the following one, it is fairly clear that the process is inelastic. But the challenge is to correctly use $F = dp/dt$ instead of $F = ma$, which will get you into trouble because the mass is changing.

---

**Example (Chain on a scale):** A chain of length $L$ and mass density $\sigma$ kg/m is held such that it hangs vertically just above a scale. It is then released. What is the reading on the scale, as a function of the height of the top of the chain?

**First solution:** Let $y$ be the height of the top of the chain, and let $F$ be the desired force applied by the scale. The net force on the whole chain is $F - (\sigma L)g$ (with upward taken to be positive). The momentum of the chain is $(\sigma y)\dot{y}$. Note that this is negative, because $\dot{y}$ is negative. Equating the net force with the change in momentum gives

$$\begin{aligned} F - \sigma L g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{4.89}$$

The part of the chain that is still above the scale is in free fall. Therefore, $\ddot{y} = -g$. And $\dot{y} = \sqrt{2g(L-y)}$, which is the usual result for a falling object. Putting these into eq. (4.89) gives

$$\begin{aligned} F &= \sigma L g - \sigma y g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4.90}$$

This answer has the expected property of equaling zero when $y = L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading will suddenly drop down to the weight of the chain, namely $(\sigma L)g$.

**Second solution:** The normal force from the scale is responsible for doing two things. It holds up the part of the chain that already lies on the scale, and it also changes the momentum of the atoms that are suddenly brought to rest when they hit the scale. The first of these two parts of the force is simply the weight of the chain already on the scale, which is $F_{\text{weight}} = \sigma(L-y)g$.

To find the second part of the force, we need to find the change in momentum, $dp$, of the part of the chain that hits the scale during a given time $dt$. The amount of mass that hits the scale in a time $dt$ is $dm = \sigma|dy| = \sigma|\dot{y}|\,dt = -\sigma\dot{y}\,dt$. This mass initially has velocity $\dot{y}$, and then it is abruptly brought to rest. Therefore, the change in its momentum is $dp = 0 - (dm)\dot{y} = \sigma\dot{y}^2\,dt$. The force required to cause this change in momentum is thus

$$F_{dp/dt} = \frac{dp}{dt} = \sigma\dot{y}^2. \tag{4.91}$$

But as in the first solution, we have $\dot{y} = \sqrt{2g(L-y)}$. Therefore, the total force from the scale is

$$\begin{aligned} F &= F_{\text{weight}} + F_{dp/dt} \\ &= \sigma(L-y)g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4.92}$$

Note that $F_{dp/dt} = 2F_{\text{weight}}$ (until the chain is completely on the scale), independent of $y$.

---

Many other problems of this sort are included in the exercises and problems for this chapter.

## 4.9 Exercises

*Section 4.1: Conservation of energy in 1-D*

1. **Cart in a valley**

   A cart containing sand starts at rest and then rolls, without any energy loss to friction, down into a valley and then up a hill on the other side. Let the initial height be $h_1$, and let the final height attained on the other side be $h_2$. If the cart leaks sand along the way, how does $h_2$ compare to $h_1$?

2. **Walking on a escalator**

   An escalator moves downward at constant speed. You walk up the escalator at this same speed, so that you remain at rest with respect to the ground. Are you doing any work?

3. **Heading to infinity** $*$

   A particle moves away from the origin under the influence of a potential $V(x) = -A|x|^n$. For what values of $n$ will it reach infinity in a finite time?

4. **Work in different frames** $*$

   An object, initially at rest, is subject to a force that causes it to undergo constant acceleration $a$ for a time $t$. Verify explicitly that $W = \Delta K$ in (a) the lab frame, and (b) a frame moving to the left at speed $V$.

5. **Constant $\dot{x}$** $**$

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. Assume that at position $(x, y) = (0, 0)$, the wire is horizontal and the bead passes this point with a given speed $v_0$ to the right. What should the shape of the wire be (that is, what is $y$ as a function of $x$) so that the horizontal speed remains $v_0$ at all times? One solution is simply $y = 0$. Find the other.[14]

6. **Spring energy**

   Using the explicit form of the position of a mass on the end of a spring, $x(t) = A\cos(\omega t + \phi)$, verify that the total energy is conserved.

7. **Hanging spring** $*$

   A massless spring with spring-constant $k$ hangs vertically from a ceiling, initially at its relaxed length. A mass $m$ is then attached to the bottom and is released.

   (a) Calculate the total potential energy of the system, as a function of the height $y$ (which is negative), relative to the initial position. Make a rough plot of $V(y)$.

[14] Solve this exercise in the spirit of Problem 6, that is, by solving a differential equation. Once you get the answer, you'll see that you could have just written it down without any calculations, based on your knowledge of a certain kind of physical motion.

(b) Find $y_0$, the point at which the potential energy is minimum.

(c) Rewrite the potential energy as a function of $z \equiv y - y_0$. Explain why your result shows that a hanging spring can be considered to be a spring in a world without gravity, provided that the new equilibrium point, $y_0$, is taken to be the "relaxed" length of the spring.

8. **Removing the friction** $**$

A block of mass $m$ is supported by a spring on an inclined plane as shown in Fig. 4.20. The spring constant is $k$, the plane's angle of inclination is $\theta$, and the coefficient of friction between the block and the plane is $\mu$.

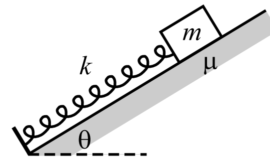


**Figure 4.20**

(a) You move the block down the plane, compressing the spring. What is the maximum compression length of the spring (relative to the relaxed length it has when nothing is attached to it) that allows the block to remain at rest when you let go of it?

(b) Assume that the block is at the maximum compression you found in part (a). At a given instant, you somehow cause the plane to become frictionless, and the block springs up along the plane. What must the relation between $\theta$ and the original $\mu$ be, so that the block reaches its maximum height when the spring is at its relaxed length?

9. **Spring and friction** $**$

A spring with spring-constant $k$ stands vertically, and a mass $m$ is placed on top of it. The mass is gradually lowered to its equilibrium position. With the spring held at this compression length, the system is rotated to a horizontal position. The left end of the spring is attached to a wall, and the mass is placed on a table with coefficient of kinetic friction $\mu = 1/8$; see Fig. 4.21. The mass is released.

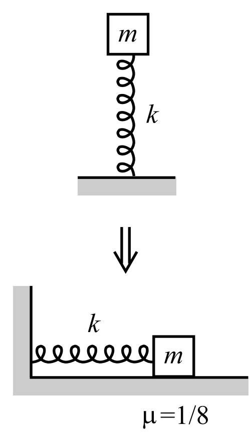


**Figure 4.21**

(a) What is the initial compression of the spring?

(b) How much does the maximal stretch (or compression) of the spring decrease after each half-oscillation? *Hint*: I wouldn't try to solve this by using $F = ma$.

(c) How many times does the mass oscillate back and forth before coming to rest?

10. **Over the pipe** $**$

A frictionless cylindrical pipe with radius $r$ is positioned with its axis parallel to the ground, at height $h$. What is the minimum initial speed at which a ball must be thrown (from ground level) in order to make it over the pipe? Consider two cases: (a) the ball is allowed to touch the pipe, and (b) the ball is not allowed to touch the pipe.

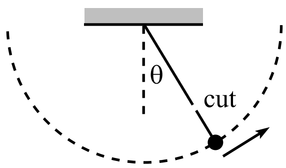

**Figure 4.22**

11. **Pendulum projectile** $*$

A pendulum is held with its string horizontal and is then released. The mass swings down, and then on its way back up, the string is cut when it makes an angle of $\theta$ with the vertical; see Fig. 4.22. What should $\theta$ be, so that the mass travels the largest horizontal distance by the time it returns to the height it had when the string was cut?

12. **Bead on a hoop** $**$

A bead is initially at rest at the top of a fixed frictionless hoop of radius $R$, which lies in a vertical plane. The bead is given a tiny kick so that it slides down and around the hoop. At what points on the hoop does the bead exert a maximum horizontal force on the hoop?

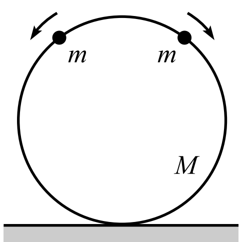

**Figure 4.23**

13. **Beads on a hoop** $**$

Two beads of mass $m$ are initially at rest at the top of a frictionless hoop of mass $M$ and radius $R$, which stands vertically on the ground. The beads are given tiny kicks, and they slide down the hoop, one to the right and one to the left, as shown in Fig. 4.23. What is the largest value of $m/M$ for which the hoop will never rise up off the ground?

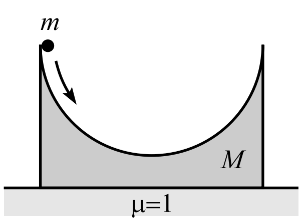

**Figure 4.24**

14. **Stationary bowl** $***$

A hemispherical bowl of mass $M$ rests on a table. The inside surface of the bowl in frictionless, while the coefficient of friction between the bottom of the bowl and the table is $\mu = 1$. A particle of mass $m$ is released from rest at the top of the bowl and slides down into it, as shown in Fig. 4.24. What is the largest value of $m/M$ for which the bowl will never slide on the table? *Hint*: The angle you will be concerned with is *not* $45°$.

15. **Roller coaster** $*$

A roller coaster car starts at rest and coasts down a frictionless track. It encounters a vertical loop of radius $R$. How much higher than the top of the loop must the car start if it to remain in contact with the track at all times?

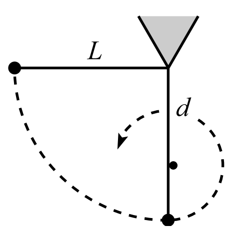

**Figure 4.25**

16. **Pendulum and peg** $*$

A pendulum of length $L$ is initially held horizontal, and is then released. The string runs into a peg a distance $d$ below the pivot, as shown in Fig. 4.25. What is the smallest value of $d$ for which the string remains taught at all times?

17. **Unwinding string** $**$

A mass is connected to one end of a massless string, the other end of which is connected to a very thin frictionless vertical pole. The string is initially wound completely around the pole, in a very large number of little horizontal circles, with the mass touching the pole. The mass is released, and the string gradually unwinds. What angle does the string make with the pole at the moment it becomes completely unwound?

18. **Leaving the hemisphere** ∗∗∗∗

A point particle of mass $m$ sits at rest on top of a frictionless hemisphere of mass $M$, which rests on a frictionless table. The particle is given a tiny kick and slides down the hemisphere. At what angle $\theta$ (measured from the top of the hemisphere) does the particle lose contact with the hemisphere?

In answering this question for $m \neq M$, it is sufficient for you to produce an equation that $\theta$ must satisfy (it will be a cubic). However, for the special case of $m = M$, this equation can be solved without too much difficulty; find the angle in this case.

*Section 4.4: Gravity*

19. **Projectile between planets** ∗

Two planets of mass $M$ and radius $R$ are at rest with respect to each other, with their centers a distance $4R$ apart. You wish to fire a projectile from the surface of one planet to the other. What is the minimum initial speed for which this is possible?

20. **Spinning quickly** ∗

Consider a planet with uniform mass density $\rho$. If the planet rotates too fast, it will fly apart. Show that the minimum period of rotation is given by

$$T = \sqrt{\frac{3\pi}{G\rho}}\,.$$

What is the minimum $T$ if $\rho = 5.5\,\text{g/cm}^3$ (the average density of the earth)?

21. **Supporting a tube** ∗

Imagine the following unrealistic undertaking. Drill a narrow tube (with cross sectional area $A$) from the surface of the earth down to the center. Then line the cylindrical wall of the tube with a frictionless coating. Then fill the tube back up with the dirt (and magma, etc.) you originally removed. What force is necessary at the bottom of the tube of dirt (that is, at the center of the earth) to hold it up? Let the earth's radius be $R$, and assume a uniform mass density $\rho$.

22. **Force from a straight wire** ∗∗

A particle of mass $m$ is placed a distance $\ell$ away from an infinitely long straight wire with mass density $\rho\,\text{kg/m}$. Show that the force on the particle is $F = 2G\rho m/\ell$.

23. **Speedy travel** ∗∗

A straight tube is drilled between two points on the earth, as shown in Fig. 4.26. An object is dropped into the tube. What is the resulting motion? How long does it take to reach the other end? Ignore friction, and assume (erroneously) that the density of the earth is constant.

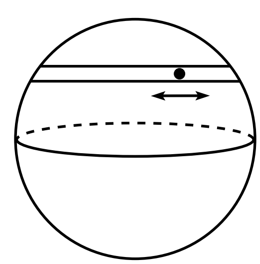

**Figure 4.26**

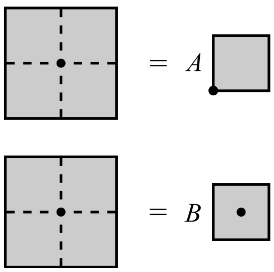

**Figure 4.27**

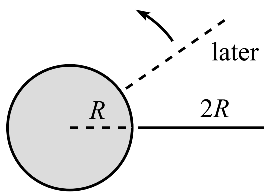

**Figure 4.28**

24. **Ratio of potentials** $*$

Consider the following two systems: (1) a mass $m$ is placed at the corner of a flat square sheet of mass $M$, and (2) a mass $m$ is placed at the center of a flat square sheet of mass $M$. What is the ratio of the potential energies of $m$ in the two systems? *Hint*: Find $A$ and $B$ in the suggestive relations in Fig. 4.27.

25. **Relative speed** $*$

Two particles with masses $m$ and $M$ are initially at rest, a very large (essentially infinite) distance apart. They are attracted to each other due to gravity. What is their relative speed when they are a distance $r$ apart?

26. **Orbiting stick** $**$

Consider a planet of mass $M$ and radius $R$. A very long stick of length $2R$ extends from just above the surface of the planet, to a radius $3R$. If initial conditions have been set up so that the stick moves in a circular orbit while always pointing radially (see Fig. 4.28), what is the period of this orbit? How does this period compare to the period of a satellite in a circular orbit of radius $2R$?

27. **Geosynchronous orbits** $**$

(a) Let the earth's radius be $R$, its average density be $\rho$, and its angular frequency of rotation be $\omega$. Show that if a satellite is to remain above the same point on the equator at all times, then it must travel in a circle of radius $\eta R$, where

$$\eta^3 = \frac{4\pi G\rho}{3\omega^2}\,. \tag{4.93}$$

What is the numerical value for $\eta$?

(b) Instead of a satellite, consider a long rope with uniform mass density extending radially from the surface of the earth out to a radius $\eta' R$.[15] Show that if the rope is to remain above the same point on the equator at all times, then $\eta'$ must be given by

$$\eta'^2 + \eta' = \frac{8\pi G\rho}{3\omega^2}\,. \tag{4.94}$$

What is the numerical value for $\eta'$? Where is the tension in the rope maximum? *Hint*: No messy calculations required.

28. **Spherical shell** $**$

(a) A spherical shell of mass $M$ has inner radius $R_1$ and outer radius $R_2$. A particle of mass $m$ is located a distance $r$ from the center of the shell. Calculate (and make a rough plot of) the force on $m$, as a function of $r$, for $0 \le r \le \infty$.

[15] Any proposed space elevator wouldn't have uniform mass density. But this simplifies problem still gives a good idea of the general features.

(b) If the mass $m$ is dropped from $r = \infty$ and falls down through the shell (assume that a tiny hole has been drilled in it), what will $m$'s speed be at the center of the shell? You can let $R_2 = 2R_1$ in this part of the problem, to keep things from getting too messy. Give your answer in terms of $R \equiv R_1$.

29. **Roche limit** $*$

A small spherical rock covered with sand falls in radially toward a planet. Let the planet have radius $R$ and density $\rho_{\text{p}}$, and let the rock have density $\rho_{\text{r}}$. It turns out that when the rock gets close enough to the planet, the tidal force ripping the sand off the rock will be larger than the gravitational force attracting the sand to the rock. The cutoff distance is called the Roche limit. Show that it is given by[16]

$$d = R\left(\frac{2\rho_{\text{p}}}{\rho_{\text{r}}}\right)^{1/3}. \tag{4.95}$$

30. **Maximal gravity** $***$

Given a point $P$ in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field at $P$?

*Section 4.5: Momentum*

31. **Sticking masses**

A mass $3m$ moving east at speed $v$ collides with a mass $2m$ moving northeast at speed $2v$. The masses stick together. What is the resulting speed and direction of the combined mass?

32. **Snow on a sled** $*$

A sled on which you are riding is given an initial push and slides across frictionless ice. Snow is falling vertically (in the frame of the ice) on the sled. Assume that the sled travels in tracks that constrain it to move in a straight line. Which of the following three strategies causes the sled to move the fastest? The slowest? Explain your reasoning.

(a) You sweep the snow off the sled so that it leaves the sled in the direction perpendicular to the sled's tracks, as seen by you in the frame of the sled.

(b) You sweep the snow off the sled so that it leaves the sled in the direction perpendicular to the sled's tracks, as seen by someone in the frame of the ice.

(c) You do nothing.

[16]For things orbiting circularly instead of falling radially inward, the cutoff distance is different, but only slightly. See the exercise in Chapter 9. The Roche limit gives the radial distance below which loose objects won't collect into larger blobs. Our moon (which is a sphere of rock and sand) lies outside the earth's Roche limit. But Saturn's rings (which consists of loose ice particles) lie inside its limit.

33. **Speedy rockets** ∗∗

Assume that it is impossible to build a structurally sound container that can hold fuel of more than, say, nine times its mass. It would then seem like the limit for the speed of a rocket is $u \ln 10$. How can you build a rocket that goes faster than this?

34. **Maximum $P$ and $E$ of rocket** ∗

A rocket ejects its exhaust at a given speed $u$. What is the mass of the rocket (including unused fuel) when its momentum is maximum? What is the mass when its energy is maximum?

35. **Leaky bucket** ∗∗∗

Consider the setup in Problem 16, but now let the sand leak at a rate $dm/dt = -bM$. In other words, the rate is constant with respect to time, not distance. We've factored out an $M$ here, just to make the calculations a little nicer.

(a) Find $v(t)$ and $x(t)$ for the times when the bucket contains a nonzero amount of sand.

(b) What is the maximum value of the bucket's kinetic energy, assuming it is achieved before it hits the wall?

(c) What is the maximum value of the magnitude of the bucket's momentum, assuming it is achieved before it hits the wall?

(d) For what value of $b$ does the bucket become empty right when it hits the wall?

36. **Throwing a brick** ∗∗∗

A brick is thrown from ground level, at an angle $\theta$ with respect to the (horizontal) ground. Assume that the long face of the brick remains parallel to the ground at all times, and that there is no deformation in the ground or the brick when the brick hits the ground. If the coefficient of friction between the brick and the ground is $\mu$, what should $\theta$ be so that the brick travels the maximum total horizontal distance before finally coming to rest? *Hint:* The brick slows down when it hits the ground. Think in terms of impulse.

*Section 4.7: Collisions*

37. **A 1-D collision** ∗

Consider the following one-dimensional collision. A mass $2m$ moves to the right, and a mass $m$ moves to the left, both with speed $v$. They collide elastically. Find their final lab-frame velocities. Solve this by:

(a) Working in the lab frame.

(b) Working in the CM frame.

38. **Perpendicular vectors** $*$

A mass $m$, moving with speed $v$, collides elastically with a stationary mass $2m$. Let their resulting velocities be $\vec{v}_1$ and $\vec{v}_2$, respectively. Show that $\vec{v}_2$ must be perpendicular to $\vec{v}_2 + 2\vec{v}_1$. *Hint*: See Problem 18.

39. **Maximum number of collisions** $**$

$N$ balls are constrained to move in one dimension. If you are allowed to pick the initial velocities, what is the maximum number of collisions you can arrange for the balls to have among themselves? Assume the collisions are elastic.

40. **Triangular room** $**$

A ball is thrown against a wall of a very long triangular room which has vertex angle $\theta$. The initial direction of the ball is parallel to the angle bisector (see Fig. 4.29). How many bounces does the ball make? Assume the walls are frictionless.

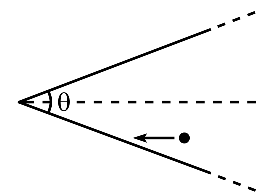


**Figure 4.29**

41. **Three pool balls** $*$

A pool ball with initial speed $v$ is aimed right between two other pool balls, as shown in Fig. 4.30. If the two right balls leave the collision at 30° with respect to the initial line of motion, find the final speeds of all three balls.

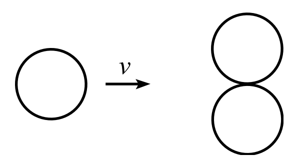


**Figure 4.30**

42. **Equal angles** $**$

(a) A mass $2m$ moving at speed $V_0$ collides elastically with a stationary mass $m$. If the two masses scatter at equal angles with respect to the incident direction, what is this angle?

(b) What is the largest number that the above "2" can be replaced with, if you want it to be possible for the masses to scatter at equal angles?

43. **Right angle in billiards** $**$

A billiard ball collides elastically with an identical stationary one. By looking at the collision in the CM frame, show that the angle between the resulting trajectories in the lab frame is 90°.

44. **Maximum $v_y$** $**$

A mass $M$ moving in the positive $x$-direction collides elastically with a stationary mass $m$. The collision is not necessarily head-on, so the masses may come off at angles, as shown in Fig. 4.31. Let $\theta$ be the angle of $m$'s resulting motion. What should $\theta$ be so that $m$ has the largest possible speed in the $y$-direction? *Hint*: Think about what the collision should look like in the CM frame.

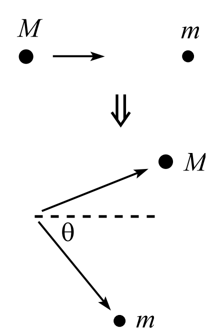


**Figure 4.31**

45. **Maximum deflection** ∗∗∗

A mass $M$ collides with a stationary mass $m$. If $M < m$, then it is possible for $M$ to bounce directly backwards. However, if $M > m$, then there is a maximal angle of deflection of $M$. Show that this maximal angle equals $\sin^{-1}(m/M)$. *Hint:* It is possible to do this problem by working in the lab frame, but you can save yourself a lot of time by considering what happens in the CM frame, and then shifting back to the lab frame.

46. **Balls in a semicircle** ∗∗∗∗

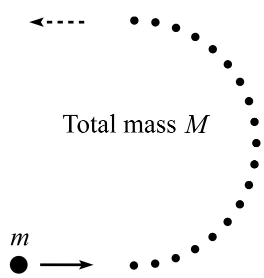

**Figure 4.32**

$N$ identical balls lie equally spaced in a semicircle on a frictionless horizontal table, as shown. The total mass of these balls is $M$. Another ball of mass $m$ approaches the semicircle from the left, with the proper initial conditions so that it bounces (elastically) off all $N$ balls and finally leaves the semicircle, heading directly to the left. See Fig. 4.32.

(a) In the limit $N \to \infty$ (so the mass of each ball in the semicircle, $M/N$, goes to zero), find the minimum value of $M/m$ that allows the incoming ball to come out heading directly to the left. *Hint*: You'll need to do Exercise 45 first.

(b) In the minimum $M/m$ case found in part (a), show that the ratio of $m$'s final speed to initial speed equals $e^{-\pi}$.

47. **Midair collision** ∗∗

A ball is held and then released. At the instant it is released, an identical ball, moving horizontally with speed $v$, collides elastically with it. What is the maximum horizontal distance the latter ball can travel by the time it returns to the height of the collision?

48. **Bouncing between rings** ∗∗

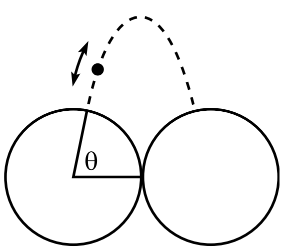

**Figure 4.33**

Two fixed circular rings, in contact with each other, stand in a vertical plane. A ball bounces elastically back and forth between the rings (see Fig. 4.33). Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola. Let this parabola hit the rings at an angle $\theta$ from the horizontal. Show that if you want the magnitude of the change in the horizontal component of the ball's momentum at each bounce to be maximum, then you should pick $\cos\theta = (\sqrt{5}-1)/2$, which just happens to be the inverse of the golden ratio.

49. **Bouncing between surfaces** ∗∗

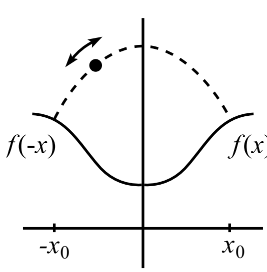

**Figure 4.34**

Consider the following generalization of the previous exercise. A ball bounces back and forth between a surface defined by $f(x)$ and its reflection across the $y$-axis (see Fig. 4.34). Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola, with the contact points located at $\pm x_0$. For what function $f(x)$ is the magnitude of the change in the horizontal component of the ball's momentum at each bounce independent of $x_0$?

50. **Drag force on a sphere** $**$

A sphere of mass $M$ and radius $R$ moves with speed $V$ through a region of space that contains particles of mass $m$ that are at rest. There are $n$ of these particles per unit volume. Assume $m \ll M$, and assume that the particles do not interact with each other. What is the drag force on the sphere?

51. **Block and bouncing ball** $****$

A block with large mass $M$ slides with speed $V_0$ on a frictionless table toward a wall. It collides elastically with a ball with small mass $m$, which is initially at rest at a distance $L$ from the wall. The ball slides towards the wall, bounces elastically, and then proceeds to bounce back and forth between the block and the wall.

(a) How close does the block come to the wall?

(b) How many times does the ball bounce off the block, by the time the block makes its closest approach to the wall?

Assume that $M \gg m$, and give your answers to leading order in $m/M$.

*Section 4.8: Inherently inelastic processes*

52. **Slowing down, speeding up** $*$

A plate of mass $M$ initially moves horizontally at speed $v$ on a frictionless table. A mass $m$ is dropped vertically onto it and soon comes to rest with respect to the plate. How much energy is required to bring the system back up to speed $v$?

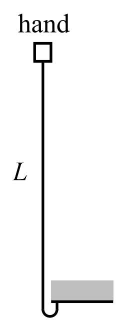


**Figure 4.35**

53. **Falling rope** $**$

A rope with mass $M$ and length $L$ is held in the position shown in Fig. 4.35, with one end attached to a support. Assume that only a negligible length of the rope starts out below the support. The rope is released. Find the force that the support applies to the rope, as a function of time.

54. **Pulling the rope back** $**$

A rope of length $L$ and mass density $\sigma$ kg/m lies outstretched on a frictionless horizontal table. You grab one end and pull it back along itself, in a parallel manner, as shown in Fig. 4.36. If your hand starts from rest and has constant acceleration $a$, what is your force right before the rope is straightened out?

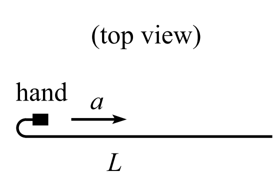


**Figure 4.36**

55. **Pulling the rope** $**$

A rope with mass density $\sigma$ kg/m lies in a heap at the edge of a table. One end of the rope initially sticks out an infinitesimal distance from the heap. You grab this end and accelerate it downward with acceleration $a$. Assume that there is no friction of the rope with itself as it unravels. As a function of time, what force does your hand apply to the rope? Find the value of $a$ that makes your force always equal to zero.

56. **Heap and block** ∗∗

A rope of mass $m$ and length $L$ lies in a heap on the floor, with one end attached to a block of mass $M$. The block is given a sudden kick and instantly acquires a speed $V_0$. Let $x$ be the distance traveled by the block. In terms of $x$, what is the tension in the rope, just to the right of the heap, that is, at the point $P$ shown? See Fig. 4.37. There is no friction in this problem – none with the floor, and none in the rope with itself.

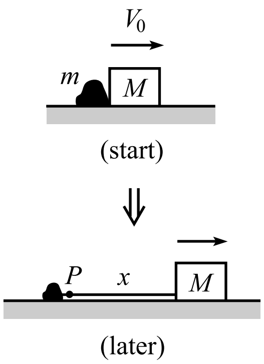


**Figure 4.37**

57. **Downhill dustpan** ∗∗∗

A dustpan slides down a plane inclined at angle $\theta$. Dust is uniformly distributed on the plane, and the dustpan collects the dust in its path. After a long time, what is the acceleration of the dustpan? Assume there is no friction between the dustpan and plane.

58. **Touching the floor** ∗∗∗∗

A rope with mass density $\sigma$ kg/m hangs from a spring with spring-constant $k$. In the equilibrium position, a length $L$ is in the air, and the bottom part of the rope lies in a heap on the floor; see Fig. 4.38. The rope is raised by a very small distance, $b$, and then released. What is the amplitude of the oscillations, as a function of time?

Assume that (1) $L \gg b$, (2) the rope is very thin, so that the size of the heap on the floor is very small compared to $b$, (3) the length of the rope in the initial heap is larger than $b$, so that some of the rope always remains in contact with the floor, and (4) there is no friction of the rope with itself inside the heap.

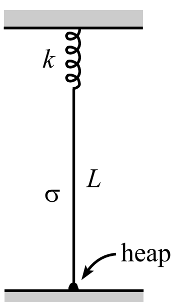


**Figure 4.38**

## 4.10 Problems

*Section 4.1: Conservation of energy in 1-D*

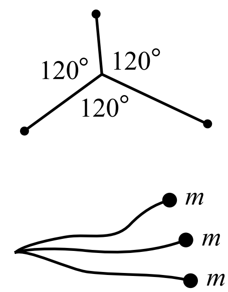


**Figure 4.39**

1. **Minimum length** $*$

   The shortest configuration of string joining three given points is the one shown at the top of Fig. 4.39, where all three angles are 120°.[17] Explain how you could experimentally prove this fact by cutting three holes in a table and making use of three equal masses attached to the ends of strings (the other ends of which are connected), as shown in Fig. 4.39.

2. **Heading to zero** $*$

   A particle moves toward $x = 0$ under the influence of a potential $V(x) = -A|x|^n$, where $A > 0$ and $n > 0$. The particle has barely enough energy to reach $x = 0$. For what values of $n$ will it reach $x = 0$ in a finite time?

3. **Leaving the sphere** $*$

   A small ball rests on top of a fixed frictionless sphere. The ball is given a tiny kick and slides downward. At what point does it lose contact with the sphere?

4. **Pulling the pucks** $**$

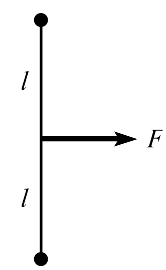


**Figure 4.40**

   (a) A massless string of length $2\ell$ connects two hockey pucks that lie on frictionless ice. A constant horizontal force $F$ is applied to the midpoint of the string, perpendicular to it (see Fig. 4.40). How much kinetic energy is lost when the pucks collide, assuming they stick together?

   (b) The answer you obtained above should be very clean and nice. Find the slick solution (assuming that you solved the problem the "normal" way, above) that makes it transparent why the answer is so nice.

5. **$V(x)$ vs. a hill** $****$

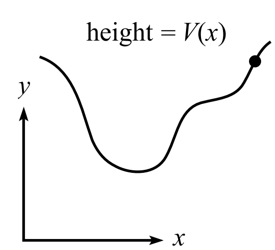


**Figure 4.41**

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $V(x)$ (see Fig. 4.41). Find an expression for the bead's horizontal acceleration, $\ddot{x}$. (It can depend on whatever quantities you need it to depend on.)

   You should find that the result is *not* the same as the $\ddot{x}$ for a particle moving in one dimension in the potential $mgV(x)$, in which case $\ddot{x} = -gV'$. But if you grab hold of the wire, is there any way you can move it so that the bead's $\ddot{x}$ is equal to the $\ddot{x} = -gV'$ result due to the one-dimensional potential $mgV(x)$?

6. **Constant $\dot{y}$** $**$

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. Assume that at position $(x, y) = (0, 0)$,

[17] If the three points form a triangle that has an angle greater than 120°, then the string simply passes through the point where that angle is. We won't worry about this case.

the wire is vertical and the bead passes this point with a given speed $v_0$ downward. What should the shape of the wire be (that is, what is $y$ as a function of $x$) so that the vertical speed remains $v_0$ at all times?

*Section 4.2: Small Oscillations*

7. **Small oscillations** $*$

   A particle moves under the influence of the potential $V(x) = -Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.

8. **Hanging mass**

   The potential for a mass hanging from a spring is $V(y) = ky^2/2 + mgy$, where $y = 0$ corresponds to the position of the spring when nothing is hanging from it. Find the frequency of small oscillations around the equilibrium point.

*Section 4.4: Gravity*

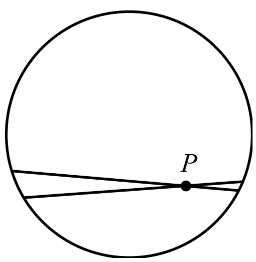


**Figure 4.42**

9. **Zero force inside a sphere** $*$

   Show that the gravitational force inside a spherical shell is zero by showing that the pieces of mass at the ends of the thin cones in Fig. 4.42 give canceling forces at point $P$.

10. **Escape velocity** $*$

    (a) Find the escape velocity (that is, the velocity above which a particle will escape to $r = \infty$) for a particle on a spherical planet of radius $R$ and mass $M$. What is the numerical value for the earth? The moon? The sun?

    (b) Approximately how small must a spherical planet be in order for a human to be able to jump off? Assume a density roughly equal to the earth's.

11. **Through the hole** $**$

    (a) A hole of radius $R$ is cut out from an infinite flat sheet of mass density $\sigma$. Let $L$ be the line that is perpendicular to the sheet and that passes through the center of the hole. What is the force on a mass $m$ that is located on $L$, at a distance $x$ from the center of the hole? *Hint*: Consider the plane to consist of many concentric rings.

    (b) If a particle is released from rest on $L$, very close to the center of the hole, show that it undergoes oscillatory motion, and find the frequency of these oscillations.

    (c) If a particle is released from rest on $L$, at a distance $x$ from the sheet, what is its speed when it passes through the center of the hole? What is your answer in the limit $x \gg R$?

12. **Ratio of potentials** ∗∗

Consider a cube of uniform mass density. Find the ratio of the gravitational potential energy of a mass at a corner to that of a mass at the center. *Hint:* There's a slick way that doesn't involve any messy integrals.

*Section 4.5: Momentum*

13. **Snowball** ∗

A snowball is thrown against a wall. Where does its momentum go? Where does its energy go?

14. **Propelling a car** ∗∗

For some odd reason, you decide to throw baseballs at a car of mass $M$, which is free to move frictionlessly on the ground. You throw the balls at the back of the car at speed $u$, and at a mass rate of $\sigma$ kg/s (assume the rate is continuous, for simplicity). If the car starts at rest, find its speed and position as a function of time, assuming that the balls bounce elastically directly backwards off the back window.

15. **Propelling a car again** ∗∗

Do the previous problem, except now assume that the back window is open, so that the balls collect inside the car.

16. **Leaky bucket** ∗∗

At $t = 0$, a massless bucket contains a mass $M$ of sand. It is connected to a wall by a massless spring with constant tension $T$ (that is, independent of length).[18] See Fig. 4.43. The ground is frictionless, and the initial distance to the wall is $L$. At later times, let $x$ be the distance from the wall, and let $m$ be the mass of sand in the bucket.

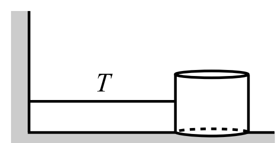


**Figure 4.43**

The bucket is released. On its way to the wall, it leaks sand at a rate $dm/dx = M/L$. In other words, the rate is constant with respect to distance, not time. Note that $dx$ is negative, so $dm$ is also.

(a) What is the kinetic energy of the (sand in the) bucket, as a function of the distance from the wall? What is its maximum value?

(b) What is the magnitude of the momentum of the bucket, as a function of the distance from the wall? What is its maximum value?

17. **Another leaky bucket** ∗∗∗

Consider the setup in Problem 16, but now let the sand leak at a rate proportional to the bucket's acceleration. That is, $dm/dt = b\ddot{x}$. Note that $\ddot{x}$ is negative, so $dm$ is also.

[18] You can construct a constant-tension spring with a regular Hooke's-law spring in the following way. Pick the spring constant to be very small, and stretch the spring a very large distance; have it pass through a hole in the wall, with its other end bolted down a large distance to the left of the wall. Any changes in the bucket's position will then yield a negligible change in the spring's force.

(a) Find the mass as a function of time, $m(t)$.

(b) Find $v(t)$ and $x(t)$ for the times when the bucket contains a nonzero amount of sand. Also find $v(m)$ and $x(m)$. What is the speed right before all the sand leaves the bucket (assuming it hasn't hit the wall yet)?

(c) What is the maximum value of the bucket's kinetic energy, assuming it is achieved before it hits the wall?

(d) What is the maximum value of the magnitude of the bucket's momentum, assuming it is achieved before it hits the wall?

(e) For what value of $b$ does the bucket become empty right when it hits the wall?

*Section 4.7: Collisions*

18. **Right angle in billiards** $*$

A billiard ball collides elastically with an identical stationary one. Use the fact that $mv^2/2$ may be written as $m(\mathbf{v}\cdot\mathbf{v})/2$ to show that the angle between the resulting trajectories is 90°.

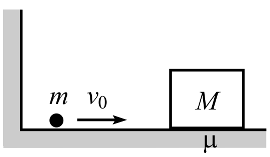


**Figure 4.44**

19. **Bouncing and recoiling** $**$

A ball of mass $m$ and initial speed $v_0$ bounces back and forth between a fixed wall and a block of mass $M$ (with $M \gg m$). See Fig. 4.44. $M$ is initially at rest. Assume that the ball bounces elastically and instantaneously. The coefficient of kinetic friction between the block and the ground is $\mu$. There is no friction between the ball and the ground.

What is the speed of the ball after the $n$th bounce off the block? How far does the block eventually move? How much total time does the block actually spend in motion? Work in the approximation where $M \gg m$, and assume that $\mu$ is large enough so that the block comes to rest by the time the next bounce occurs.

20. **Drag force on a sheet** $**$

A sheet of mass $M$ moves with speed $V$ through a region of space that contains particles of mass $m$ and speed $v$. There are $n$ of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume that the particles do not interact with each other.

(a) If $v \ll V$, what is the drag force per unit area on the sheet?

(b) If $v \gg V$, what is the drag force per unit area on the sheet? Assume, for simplicity, that the component of every particle's velocity in the direction of the sheet's motion is exactly $\pm v/2$.[19]

[19] In reality, the velocities are randomly distributed, but this idealization actually gives the correct answer because the average speed in any direction is $\overline{|v_x|} = v/2$. The result $\overline{v_x^2} = v^2/3$, which may be familiar to you, isn't relevant here.

21. **Drag force on a cylinder** $**$

A cylinder of mass $M$ and radius $R$ moves with speed $V$ through a region of space that contains particles of mass $m$ that are at rest. There are $n$ of these particles per unit volume. The cylinder moves in a direction perpendicular to its axis. Assume $m \ll M$, and assume that the particles do not interact with each other. What is the drag force per unit length on the cylinder?

22. **Basketball and tennis ball** $**$

(a) A tennis ball with a small mass $m_2$ sits on top of a basketball with a large mass $m_1$ (see Fig. 4.45). The bottom of the basketball is a height $h$ above the ground, and the bottom of the tennis ball is a height $h+d$ above the ground. The balls are dropped. To what height does the tennis ball bounce? *Note:* Work in the approximation where $m_1$ is much larger than $m_2$, and assume that the balls bounce elastically. Also assume, for the sake of having a nice clean problem, that the balls are initially separated by a small distance, and that the balls bounce instantaneously.

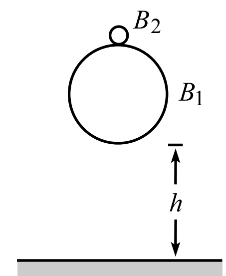


**Figure 4.45**

(b) Now consider $n$ balls, $B_1, \ldots, B_n$, having masses $m_1, m_2, \ldots, m_n$ (with $m_1 \gg m_2 \gg \cdots \gg m_n$), standing in a vertical stack (see Fig. 4.46). The bottom of $B_1$ is a height $h$ above the ground, and the bottom of $B_n$ is a height $h + \ell$ above the ground. The balls are dropped. In terms of $n$, to what height does the top ball bounce? *Note:* Make assumptions and approximations similar to the ones in part (a).

If $h = 1$ meter, what is the minimum number of balls needed for the top one to bounce to a height of at least 1 kilometer? To reach escape velocity? Assume that the balls still bounce elastically (which is a bit absurd here), and ignore wind resistance, etc., and assume that $\ell$ is negligible.

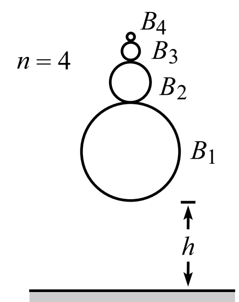


**Figure 4.46**

*Section 4.8: Inherently inelastic processes*

23. **Colliding masses** $*$

A mass $M$, initially moving at speed $v$, collides and sticks to a mass $m$, initially at rest. Assume $M \gg m$, and work in this approximation. What are the final energies of the two masses, and how much energy is lost to heat, in:

(a) The lab frame?

(b) The frame in which $M$ is initially at rest?

24. **Pulling a chain** $**$

A chain of length $L$ and mass density $\sigma$ lies straight on a frictionless horizontal surface. You grab one end and pull it back along itself, in a parallel manner (see Fig. 4.47). Assume that you pull it at constant speed $v$. What force must you apply? What is the total work that you do, by the time the chain is straightened out? How much energy is lost to heat, if any?

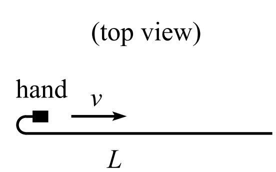


**Figure 4.47**

25. **Pulling a rope** $**$

A rope of mass density $\sigma$ lies in a heap on the floor. You grab an end and pull horizontally with constant force $F$. What is the position of the end of the rope, as a function of time, while it is unravelling? Assume that the rope is greased, so that it has no friction with itself.

26. **Raising the rope** $**$

A rope of length $L$ and mass density $\sigma$ lies in a heap on the floor. You grab one end of the rope and pull upward with a force such that the rope moves at constant speed $v$. What is the total work you do, by the time the rope is completely off the floor? How much energy is lost to heat, if any? Assume that the rope is greased, so that it has no friction with itself.

27. **Falling rope** $***$

(a) A rope of length $L$ lies in a straight line on a frictionless table, except for a very small piece at one end which hangs down through a hole in the table. This piece is released, and the rope slides down through the hole. What is the speed of the rope at the instant it loses contact with the table?

(b) Answer the same question, but now let the rope lie in a heap on a table, except for a very small piece at one end which hangs down through the hole. Assume that the rope is greased, so that it has no friction with itself.

28. **The raindrop** $****$

Assume that a cloud consists of tiny water droplets suspended (uniformly distributed, and at rest) in air, and consider a raindrop falling through them. What is the acceleration of the raindrop? Assume that the raindrop is initially of negligible size and that when it hits a water droplet, the droplet's water gets added to it. Also, assume that the raindrop is spherical at all times.

## 4.11 Solutions

1. **Minimum length**

   Cut three holes in the table at the locations of the three given points. Drop the masses through the holes, and let the system reach its equilibrium position. The equilibrium position is the one with the lowest potential energy of the masses, that is, the one with the most string hanging below the table. In other words, it is the one with the least string lying on the table. This is the desired minimum-length configuration.

   What are the angles at the vertex of the string? The tensions in all three strings are equal to $mg$. The vertex of the string is in equilibrium, so the net force on it must be zero. This implies that each string must bisect the angle formed by the other two. Therefore, the angles between the strings must all be $120°$.

2. **Heading to zero**

   Write $F = ma$ as $mv\,dv/dx = -V'(x)$. Separating variables and integrating gives $mv^2/2 = C - V(x)$, where $C$ is a constant of integration. The given information tells us that $v = 0$ when $x = 0$. Therefore $C = 0$. $C$ is simply the total energy of the particle. Writing $v$ as $dx/dt$ and separating variables again gives

$$\frac{dx}{\sqrt{-V(x)}} = \pm dt\sqrt{\frac{2}{m}}\,. \tag{4.96}$$

   Assume that the particle starts at position $x_0 > 0$. Let $T$ be the time to reach the origin. Integrating the previous equation from $x_0$ to $x = 0$ gives

$$\int_{x_0}^{0} \frac{dx}{x^{n/2}} = -\sqrt{\frac{2A}{m}}\int_0^T dt = -T\sqrt{\frac{2A}{m}}\,. \tag{4.97}$$

   The integral on the left is finite only if $n/2 < 1$. Therefore, the condition that $T$ is finite is

$$n < 2. \tag{4.98}$$

   Remark: The particle will take a finite time to reach the top of a triangle or the curve $-Ax^{3/2}$. But it will take an infinite time to reach the top of a parabola, cubic, etc. A circle looks like a parabola at the top, so $T$ is infinite in that case also. In fact, any nice polynomial function $V(x)$ will require an infinite $T$ to reach a local maximum, because the Taylor series starts at order (at least) two around an extremum. ♣

3. **Leaving the sphere**

   **First Solution:** Let $R$ be the radius of the sphere, and let $\theta$ be the angle of the ball, measured from the top of the sphere. The radial $F = ma$ equation is

$$mg\cos\theta - N = \frac{mv^2}{R}\,, \tag{4.99}$$

   where $N$ is the normal force. The ball loses contact with the sphere when the normal force becomes zero (that is, when the normal component of gravity is not large enough to account for the centripetal acceleration of the ball). Therefore, the ball loses contact when

$$\frac{mv^2}{R} = mg\cos\theta. \tag{4.100}$$

But conservation of energy gives $mv^2/2 = mgR(1-\cos\theta)$. Hence, $v = \sqrt{2gR(1-\cos\theta)}$. Plugging this into eq. (4.100), we see that the ball leaves the sphere when

$$\cos\theta = \frac{2}{3}\,. \tag{4.101}$$

This corresponds to $\theta \approx 48.2°$.

**Second Solution:** Let's assume that the ball always stays in contact with the sphere, and then we'll find the point where the horizontal component of $v$ starts to decrease (which it of course can't do, because the normal force doesn't have a "backwards" component). From above, the horizontal component of $v$ is

$$v_x = v\cos\theta = \sqrt{2gR(1-\cos\theta)}\,\cos\theta. \tag{4.102}$$

Taking the derivative of this, we find that the maximum occurs when $\cos\theta = 2/3$. So this is where $v_x$ would start to decrease if the ball were constrained to remain on the sphere. But since there is no such constraining force available, the ball loses contact when $\cos\theta = 2/3$.

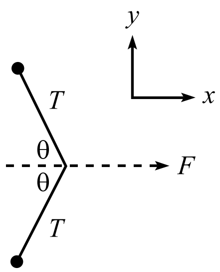


**Figure 4.48**

4. **Pulling the pucks**

   (a) Let $\theta$ be defined as in Fig. 4.48. Then the tension in the string is $T = F/(2\cos\theta)$, because the force on the massless kink in the string must be zero. Consider the "top" puck. The component of the tension in the $y$-direction is $-T\sin\theta = -F\tan\theta/2$. The work done on the puck by this component is therefore

$$\begin{aligned} W_y &= \int_\ell^0 \frac{-F\tan\theta}{2}\,dy \\ &= \int_{\pi/2}^0 \frac{-F\tan\theta}{2}\,d(\ell\sin\theta) \\ &= \int_{\pi/2}^0 \frac{-F\ell\sin\theta}{2}\,d\theta \\ &= \left.\frac{F\ell\cos\theta}{2}\right|_{\pi/2}^0 \\ &= \frac{F\ell}{2}\,. \end{aligned} \tag{4.103}$$

   By the work-energy theorem (or equivalently, by separating variables and integrating $F_y = mv_y\,dv_y/dy$), this work equals $mv_y^2/2$. The kinetic energy lost when the two pucks stick together is twice this quantity ($v_x$ doesn't change during the collision). Therefore,

$$\text{KE}_{\text{loss}} = F\ell. \tag{4.104}$$

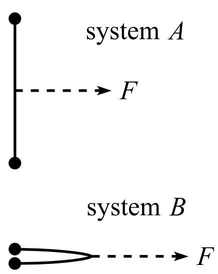


**Figure 4.49**

   (b) Consider two systems, $A$ and $B$ (see Fig. 4.49). $A$ is the original setup, while $B$ starts with $\theta$ already at zero. Let the pucks in both systems start simultaneously at $x = 0$. As the force $F$ is applied, all four pucks will have the same $x(t)$, because the same force in the $x$-direction, namely $F/2$, is being applied to every puck at all times. After the collision, both systems will therefore look exactly the same. Let the collision of the pucks occur at $x = d$. At this point, $F(d + \ell)$ work has been done on system $A$, because the center of the string (where the force is applied) ends up moving a distance $\ell$ more than the masses. However, only $Fd$ work has been done on system $B$. Since both systems have the same kinetic energy after the collision, the extra $F\ell$ work done on system $A$ must be what is lost in the collision.

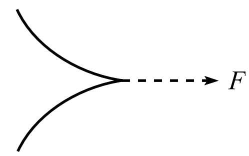


**Figure 4.50**

REMARK: The reasoning in this second solution makes it clear that this $F\ell$ result holds even if we have many masses distributed along the string, or if we have a rope with a continuous mass distribution (so that the rope flops down, as in Fig. 4.50). The only requirement is that the mass be symmetrically distributed around the mid-point. Analyzing this more general setup along the lines of the first solution would be extremely tedious, to say the least. ♣

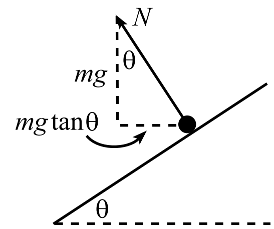


**Figure 4.51**

5. **$V(x)$ vs. a hill**

**First solution:** Consider the normal force, $N$, acting on the bead at a given point. Let $\theta$ be the angle that the tangent to $V(x)$ makes with the horizontal, as shown in Fig. 4.51. The horizontal $F = ma$ equation is

$$-N\sin\theta = m\ddot{x}. \tag{4.105}$$

The vertical $F = ma$ equation is

$$N\cos\theta - mg = m\ddot{y} \qquad \Longrightarrow \qquad N\cos\theta = mg + m\ddot{y}. \tag{4.106}$$

Dividing eq. (4.105) by eq. (4.106) gives

$$-\tan\theta = \frac{\ddot{x}}{g+\ddot{y}}. \tag{4.107}$$

But $\tan\theta = V'(x)$. Therefore,

$$\ddot{x} = -(g+\ddot{y})V'. \tag{4.108}$$

We see that this is not equal to $-gV'$. In fact, there is in general no way to construct a curve with height $y(x)$ that gives the same horizontal motion that a 1-D potential $V(x)$ gives, for all initial conditions. We would need $(g+\ddot{y})y' = V'$, for all $x$. But at a given $x$, the quantities $V'$ and $y'$ are fixed, whereas $\ddot{y}$ depends on the initial conditions. For example, if there is a bend in the wire, then $\ddot{y}$ will be large if $\dot{y}$ is large. And $\dot{y}$ depends (in general) on how far the bead has fallen.

Eq. (4.108) holds the key to constructing a situation that does give the $\ddot{x} = -gV'$ result for a 1-D potential $V(x)$. All we have to do is get rid of the $\ddot{y}$ term. So here's what we do. We grab our $y = V(x)$ wire and then move it up and/or down in precisely the manner that makes the bead stay at the same height with respect to the ground. (Actually, constant vertical speed would be good enough.) This will make the $\ddot{y}$ term vanish, as desired. Note that the vertical movement of the curve doesn't change the slope, $V'$, at a given value of $x$.

REMARK: There is one case where $\ddot{x}$ is (approximately) equal to $-gV'$, even when the wire remains stationary. In the case of small oscillations of the bead near a minimum of $V(x)$, $\ddot{y}$ is small compared to $g$. Hence, eq. (4.108) shows that $\ddot{x}$ is approximately equal to $-gV'$. Therefore, for small oscillations, it is reasonable to model a particle in a 1-D potential $mgV(x)$ as a particle sliding in a valley whose height is given by $y = V(x)$. ♣

**Second solution:** The component of gravity along the wire is what causes the change in speed of the bead. That is,

$$-g\sin\theta = \frac{dv}{dt}\,, \tag{4.109}$$

where $\theta$ is given by

$$\tan\theta = V'(x) \qquad \Longrightarrow \qquad \sin\theta = \frac{V'}{\sqrt{1+V'^2}}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+V'^2}}\,. \tag{4.110}$$

We are, however, not concerned with the rate of change of $v$, but rather with the rate of change of $\dot{x}$. In view of this, let us write $v$ in terms of $\dot{x}$. Since $\dot{x} = v\cos\theta$, we have $v = \dot{x}/\cos\theta = \dot{x}\sqrt{1+V'^2}$. (Dots denote $d/dt$. Primes denote $d/dx$.) Therefore, eq. (4.109) becomes

$$\begin{aligned} \frac{-gV'}{\sqrt{1+V'^2}} &= \frac{d}{dt}\Big(\dot{x}\sqrt{1+V'^2}\Big) \\ &= \ddot{x}\sqrt{1+V'^2} + \frac{\dot{x}V'(dV'/dt)}{\sqrt{1+V'^2}}\,. \end{aligned} \tag{4.111}$$

Hence, $\ddot{x}$ is given by

$$\ddot{x} = \frac{-gV'}{1+V'^2} - \frac{\dot{x}V'(dV'/dt)}{1+V'^2}\,. \tag{4.112}$$

We'll simplify this in a moment, but first a remark.

Remark: A common incorrect solution to this problem is the following. The acceleration along the curve is $g\sin\theta = -g(V'/\sqrt{1+V'^2})$. Calculating the horizontal component of this acceleration brings in a factor of $\cos\theta = 1/\sqrt{1+V'^2}$. Therefore, we might think that

$$\ddot{x} = \frac{-gV'}{1+V'^2} \qquad \text{(incorrect)}. \tag{4.113}$$

But we have missed the second term in eq. (4.112). Where is the mistake? The error is that we forgot to take into account the possible change in the curve's slope. (Eq. (4.113) is true for straight lines.) We addressed only the acceleration due to a change in *speed.* We forgot to consider the acceleration due to a change in the *direction* of motion. (The term we missed is the one with $dV'/dt$.) Intuitively, if we have sharp enough bend in the wire, then $\dot{x}$ can change at an arbitrarily large rate, even if $v$ is roughly constant. In view of this fact, eq. (4.113) is definitely incorrect, because it is bounded (by $g/2$, in fact). ♣

To simplify eq. (4.112), note that $V' \equiv dV/dx = (dV/dt)/(dx/dt) \equiv \dot{V}/\dot{x}$. Therefore,

$$\begin{aligned} \dot{x}V'\frac{dV'}{dt} &= \dot{x}V'\frac{d}{dt}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= \dot{x}V'\left(\frac{\dot{x}\ddot{V} - \dot{V}\ddot{x}}{\dot{x}^2}\right) \\ &= V'\ddot{V} - V'\ddot{x}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= V'\ddot{V} - V'^2\ddot{x}. \end{aligned} \tag{4.114}$$

Substituting this into eq. (4.112), we obtain

$$\ddot{x} = -(g+\ddot{V})V', \tag{4.115}$$

in agreement with eq. (4.108), since $y(x) = V(x)$.

Eq. (4.115) is valid for a curve $V(x)$ that remains fixed. If we grab the wire and start moving it up and down, then the above solution is invalid, because the starting point, eq. (4.109), rests on the assumption that gravity is the only force that does work on the bead. But if we move the wire, then the normal force also does work.

It turns out that for a moving wire, we simply need to replace the $\ddot{V}$ in eq. (4.115) by $\ddot{y}$. This can be seen by looking at things in the (instantaneously inertial) vertically-moving frame in which the wire is at rest. In this new frame, the normal force does no work, so the above solution is valid. And in this new frame, $\ddot{y} = \ddot{V}$. Eq. (4.115) therefore becomes $\ddot{x} = -(g + \ddot{y})V'$. Shifting back to the lab frame (which moves at constant speed with respect to the instantaneous inertial frame of the wire) doesn't change $\ddot{y}$. We thus arrive at eq. (4.108), valid for a stationary or vertically moving wire.

6. **Constant $\dot{y}$**

By conservation of energy, the bead's speed at any time is given by (note that $y$ is negative here)

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 \qquad \Longrightarrow \qquad v = \sqrt{v_0^2 - 2gy}\,. \tag{4.116}$$

The vertical component of the speed is $\dot{y} = v\sin\theta$, where $\tan\theta = y' \equiv dy/dx$ is the slope of the wire. Hence, $\sin\theta = y'/\sqrt{1+y'^2}$. The requirement $\dot{y} = -v_0$, which is equivalent to $v\sin\theta = -v_0$, may therefore be written as

$$\sqrt{v_0^2 - 2gy}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = -v_0. \tag{4.117}$$

Squaring both sides and solving for $y' \equiv dy/dx$ yields $dy/dx = -v_0/\sqrt{-2gy}$. Separating variables and integrating gives

$$\int \sqrt{-2gy}\,dy = -v_0 \int dx \qquad \Longrightarrow \qquad \frac{(-2gy)^{3/2}}{3g} = v_0 x, \tag{4.118}$$

where the constant of integration has been set to zero, because $(x, y) = (0, 0)$ is a point on the curve. Therefore,

$$y = -\frac{(3gv_0x)^{2/3}}{2g}\,. \tag{4.119}$$

7. **Small oscillations**

We will calculate the equilibrium point $x_0$, and then use $\omega = \sqrt{V''(x_0)/m}$. The derivative of $V$ is

$$V'(x) = -Ce^{-ax}x^{n-1}(n - ax). \tag{4.120}$$

Therefore, $V'(x) = 0$ when $x = n/a \equiv x_0$. The second derivative of $V$ is

$$V''(x) = -Ce^{-ax}x^{n-2}\Big((n-1-ax)(n-ax) - ax\Big). \tag{4.121}$$

Plugging in $x_0 = n/a$ simplifies this a bit, and we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{Ce^{-n}n^{n-1}}{ma^{n-2}}}\,. \tag{4.122}$$

8. **Hanging mass**

We will calculate the equilibrium point $y_0$, and then use $\omega = \sqrt{V''(y_0)/m}$. The derivative of $V$ is

$$V'(y) = ky + mg. \tag{4.123}$$

Therefore, $V'(y) = 0$ when $y = -mg/k \equiv y_0$. The second derivative of $V$ is

$$V''(y) = k. \tag{4.124}$$

We therefore have

$$\omega = \sqrt{\frac{V''(y_0)}{m}} = \sqrt{\frac{k}{m}}\,. \tag{4.125}$$

Remark: This is independent of $y_0$, which is what we expect. The only effect of gravity is to change the equilibrium position. If $y_r$ is the position relative to $y_0$ (so that $y \equiv y_0 + y_r$), then the total force as a function of $y_r$ is

$$F(y_r) = -k(y_0 + y_r) - mg = -k\left(-\frac{mg}{k} + y_r\right) - mg = -ky_r, \tag{4.126}$$

so it still looks like a regular spring. (This only works, of course, because the spring force is linear.) Equivalently, we can complete the square and write the given potential as

$$V(y) = \frac{k}{2}\left(y + \frac{mg}{k}\right)^2 - \frac{m^2g^2}{2k}\,. \tag{4.127}$$

The additive constant $-m^2g^2/2k$ is irrelevant in determining the curvature (that is, the second derivative) of the parabola at the minimum, as is the shift in the origin of $y$ by $-mg/k$. We basically have a mass on a spring in zero gravity, in which case the frequency is simply $\sqrt{k/m}$. ♣

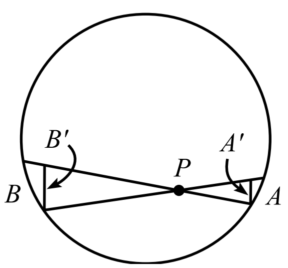

**Figure 4.52**

9. **Zero force inside a sphere**

Let $a$ be the distance from $P$ to piece $A$, and let $b$ be the distance from $P$ to piece $B$ (see Fig. 4.52). Draw the "perpendicular" bases of the cones, and call them $A'$ and $B'$. The ratio of the areas of $A'$ and $B'$ is $a^2/b^2$.

The key point here is that the angle between the planes of $A$ and $A'$ is the same as the angle between $B$ and $B'$; this is true because the chord between $A$ and $B$ meets the circle at equal angles at its ends. So the ratio of the areas of $A$ and $B$ is also $a^2/b^2$. But the gravitational force decreases like $1/r^2$, and this effect exactly cancels the $a^2/b^2$ ratio of the areas. Therefore, the forces at $P$ due to $A$ and $B$ (which can be treated like point masses, because the cones are assumed to be thin) are equal in magnitude (and opposite in direction, of course).

10. **Escape velocity**

(a) The cutoff case is where the particle barely makes it to infinity, that is, where its speed is zero at infinity. Conservation of energy for this situation gives

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{R} = 0 + 0. \tag{4.128}$$

In other words, the initial kinetic energy, $mv_{\text{esc}}^2/2$, must account for the gain in potential energy, $GMm/R$. Therefore,

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}\,. \tag{4.129}$$

In terms of the acceleration, $g = GM/R^2$, at the surface of a planet, we can write $v_{\text{esc}}$ as $v_{\text{esc}} = \sqrt{2gR}$. Using $M = 4\pi\rho R^3/3$, we can also write it as $v_{\text{esc}} = \sqrt{8\pi G R^2 \rho/3}$. So for a given density $\rho$, $v_{\text{esc}}$ grows like $R$.

Using the values of $g$ given in Appendix J, we have:

For the earth, $v_{\text{esc}} = \sqrt{2gR} \approx \sqrt{2(9.8\text{ m/s}^2)(6.4 \cdot 10^6\text{ m})} \approx 11,200\text{ m/s}$.

For the moon, $v_{\text{esc}} = \sqrt{2gR} \approx \sqrt{2(1.6\text{ m/s}^2)(1.7 \cdot 10^6\text{ m})} \approx 2,300\text{ m/s}$.

For the sun, $v_{\text{esc}} = \sqrt{2gR} \approx \sqrt{2(270\text{ m/s}^2)(7.0 \cdot 10^8\text{ m})} \approx 620,000\text{ m/s}$.

REMARK: Another reasonable question to ask is: what is the escape velocity from the sun for an object located where the earth is? (But imagine that the earth isn't there.) The answer is $\sqrt{2GM_S/R_{E,S}}$, where $R_{E,S}$ is the earth-sun distance. Numerically, this is $\sqrt{2(6.67 \cdot 10^{-11})(2 \cdot 10^{30})/(1.5 \cdot 10^{11})} \approx 42,000$ m/s. ♣

(b) To get a rough answer, let's assume that the initial speed of a person's jump on the small planet is the same as it is on the earth. This probably isn't quite true, but it's close enough for the purposes here. A good jump on the earth is about a meter. For this jump, $mv^2/2 = mg(1\,\text{m})$. Therefore, $v = \sqrt{2g(1\,\text{m})} \approx \sqrt{20}\,\text{m/s}$. So we want $\sqrt{20} = \sqrt{8\pi G R^2 \rho/3}$. Using $\rho \approx 5500\,\text{kg/m}^3$, we find $R \approx 2.5\,\text{km}$. On such a planet, you should tread lightly.

11. **Through the hole**

(a) By symmetry, only the component of the gravitational force perpendicular to the plane will survive. A piece of mass $dm$ at radius $r$ on the plane will provide a force equal to $Gm(dm)/(r^2 + x^2)$. To obtain the component perpendicular to the plane, we must multiply this by $x/\sqrt{r^2 + x^2}$. Slicing the plane up into rings with mass $dm = (2\pi r\, dr)\sigma$, we find that the total force is

$$\begin{aligned} F(x) &= -\int_R^\infty \frac{Gm(2\pi r\sigma\, dr)x}{(r^2 + x^2)^{3/2}} \\ &= 2\pi\sigma G m x (r^2 + x^2)^{-1/2} \Big|_{r=R}^{r=\infty} \\ &= -\frac{2\pi\sigma G m x}{\sqrt{R^2 + x^2}} \,. \end{aligned} \tag{4.130}$$

(b) If $x \ll R$, then eq. 4.130 gives

$$F(x) \approx -\frac{2\pi\sigma G m x}{R} \,. \tag{4.131}$$

$F = ma$ then becomes

$$\ddot{x} + \left(\frac{2\pi\sigma G}{R}\right) x = 0. \tag{4.132}$$

The frequency of small oscillations is therefore

$$\omega = \sqrt{\frac{2\pi\sigma G}{R}} \,. \tag{4.133}$$

REMARK: For everyday values of $R$, this is a rather small number because $G$ is so small. Let's determine the rough size. If the sheet has thickness $d$, and if it is made of a material with density $\rho$ (per volume), then $\sigma = \rho d$. Hence, $\omega = \sqrt{2\pi\rho d G/R}$.

In the above analysis, we assumed that the sheet was infinitely thin. In practice, we need $d$ to be much smaller than the amplitude of the motion. But this amplitude must be much smaller than $R$ in order for our approximation to hold. So we conclude that $d \ll R$. To get a rough upper bound on $\omega$, let's pick $d/R = 1/10$. And let's make our sheet out of gold (with $\rho \approx 2 \cdot 10^4\,\text{kg/m}^3$). We then find $\omega \approx 1 \cdot 10^{-3}\,\text{s}^{-1}$, which corresponds to an oscillation about every 100 minutes.

For the analogous system consisting of electrical charges, the frequency is much larger, because the electrical force is so much stronger than the gravitational force. ♣

(c) Integrating the force in eq. 4.130 to obtain the potential energy (relative to the center of the hole) gives

$$\begin{aligned} V(x) &= -\int_0^x F(x)\,dx = \int_0^x \frac{2\pi\sigma Gmx\,dx}{\sqrt{R^2+x^2}} \\ &= 2\pi\sigma Gm\sqrt{R^2+x^2}\Big|_0^x = 2\pi\sigma Gm\big(\sqrt{R^2+x^2}-R\big) \end{aligned} \tag{4.134}$$

By conservation of energy, the speed at the center of the hole is given by $mv^2/2 = V(x)$. Therefore,

$$v = 2\sqrt{\pi\sigma G\big(\sqrt{R^2+x^2}-R\big)}. \tag{4.135}$$

For large $x$ this reduces to $v = 2\sqrt{\pi\sigma Gx}$.

Remark: You can also obtain this last result by noting that for large $x$, the force in eq. (4.130) reduces to $F = -2\pi\sigma Gm$. This is constant, so it's basically just like a gravitational force $F = mg'$, where $g' \equiv 2\pi\sigma G$. But we know that in this familiar case, $v = \sqrt{2g'h} \to \sqrt{2(2\pi\sigma G)x}$, as above. ♣

12. **Ratio of potentials**

Let $\rho$ be the mass density of the cube. Let $V_\ell^{\text{cor}}$ be the potential energy of a mass $m$ at the corner of a cube of side $\ell$, and let $V_\ell^{\text{cen}}$ be the potential energy of a mass $m$ at the center of a cube of side $\ell$. By dimensional analysis,

$$V_\ell^{\text{cor}} \propto \frac{G(\rho\ell^3)m}{\ell} \propto \ell^2. \tag{4.136}$$

Therefore,[20]

$$V_\ell^{\text{cor}} = 4V_{\ell/2}^{\text{cor}}\,. \tag{4.137}$$

But a cube of side $\ell$ can be built from eight cubes of side $\ell/2$. So by superposition, we have

$$V_\ell^{\text{cen}} = 8V_{\ell/2}^{\text{cor}}\,, \tag{4.138}$$

because the center of the larger cube lies at a corner of the eight smaller cubes. Therefore,

$$\frac{V_\ell^{\text{cor}}}{V_\ell^{\text{cen}}} = \frac{4V_{\ell/2}^{\text{cor}}}{8V_{\ell/2}^{\text{cor}}} = \frac{1}{2}\,. \tag{4.139}$$

[20] In other words, imagine expanding a cube of side $\ell/2$ to one of side $\ell$. If we consider corresponding pieces of the two cubes, then the larger piece has $2^3 = 8$ times the mass of the smaller. But corresponding distances are twice as big in the large cube as in the small cube. Therefore, the larger piece contributes $8/2 = 4$ times as much to $V_\ell^{\text{cor}}$ as the smaller piece contributes to $V_{\ell/2}^{\text{cor}}$.

13. **Snowball**

All of the snowball's momentum goes into the earth, which then translates (and rotates) a tiny bit faster (or slower, depending on which way the snowball was thrown).

What about the energy? Let $M$ be the mass of the earth, and let $V$ be the final speed of the earth, with respect to the original rest frame of the earth. Then $m \ll M$ implies $V \approx mv/M$. The kinetic energy of the earth is therefore

$$\frac{1}{2}M\left(\frac{mv}{M}\right)^2 = \frac{1}{2}mv^2\left(\frac{m}{M}\right) \ll \frac{1}{2}mv^2. \tag{4.140}$$

There is also a rotational kinetic-energy term of the same order of magnitude, but that doesn't matter. Wee see that essentially none of the snowball's energy goes into the earth. It therefore must all go into the form of heat, which melts some of the snow. This is a general result for a small object hitting a large object: The large object picks up essentially all of the momentum but essentially none of the energy.

14. **Propelling a car**

Let the speed of the car be $v(t)$. Consider the collision of a ball of mass $dm$ with the car. In the instantaneous rest frame of the car, the speed of the ball is $u - v$. In this frame, the ball reverses velocity when it bounces, so its change in momentum is $-2(u-v)\,dm$. This is also the change in momentum in the lab frame, because the two frames are related by a given speed at any instant. Therefore, in the lab frame the car gains a momentum of $2(u-v)\,dm$ from each ball that hits it. The rate of change in momentum of the car (that is, the force) is thus

$$\frac{dp}{dt} = 2\sigma'(u-v), \tag{4.141}$$

where $\sigma' \equiv dm/dt$ is the rate at which mass hits the car. $\sigma'$ is related to the given $\sigma$ by $\sigma' = \sigma(u-v)/u$, because although you throw the balls at speed $u$, the relative speed of the balls and the car is only $(u - v)$. We therefore have

$$\begin{aligned} M\frac{dv}{dt} &= \frac{2(u-v)^2\sigma}{u} \\ \Longrightarrow \quad \int_0^v \frac{dv}{(u-v)^2} &= \frac{2\sigma}{Mu}\int_0^t dt \\ \Longrightarrow \quad \frac{1}{u-v} - \frac{1}{u} &= \frac{2\sigma t}{Mu} \\ \Longrightarrow \quad v(t) &= \frac{\left(\frac{2\sigma t}{M}\right)u}{1+\frac{2\sigma t}{M}}. \end{aligned} \tag{4.142}$$

Note that $v \to u$ as $t \to \infty$, as it should. Integrating this speed to obtain the position gives

$$x(t) = ut - \frac{Mu}{2\sigma}\ln\left(1 + \frac{2\sigma t}{M}\right). \tag{4.143}$$

We see that even though the speed approaches $u$, the car will eventually be an arbitrarily large distance behind a ball with constant speed $u$ (for example, pretend that your first ball misses the car and continues forward at speed $u$).

15. **Propelling a car again**

We can carry over some of the results from the previous problem. The only change in the calculation of the force on the car is that since the balls don't bounce backwards, we don't pick up the factor of 2 in eq. (4.141). The force on the car is therefore

$$m\frac{dv}{dt} = \frac{(u-v)^2\sigma}{u}\,, \tag{4.144}$$

where $m(t)$ is the mass of the car-plus-contents, as a function of time. The main difference between this problem and the previous one is that this mass $m$ changes because the balls are collecting inside the car. As in the previous problem, the rate at which the balls enter the car is $\sigma' = \sigma(u-v)/u$. Therefore,

$$\frac{dm}{dt} = \frac{(u-v)\sigma}{u}\,. \tag{4.145}$$

We must now solve the two preceding differential equations. Dividing eq. (4.144) by eq. (4.145), and separating variables, gives[21]

$$\int_0^v \frac{dv}{u-v} = \int_M^m \frac{dm}{m} \quad\Longrightarrow\quad -\ln\left(\frac{u-v}{u}\right) = \ln\left(\frac{m}{M}\right) \quad\Longrightarrow\quad m = \frac{Mu}{u-v}\,. \tag{4.147}$$

Note that $m \to \infty$ as $v \to u$, as it should. Substituting this value of $m$ into either eq. (4.144) or eq. (4.145) gives

$$\begin{aligned}
\int_0^v \frac{dv}{(u-v)^3} &= \int_0^t \frac{\sigma\,dt}{Mu^2} \\
\Longrightarrow \quad \frac{1}{2(u-v)^2} - \frac{1}{2u^2} &= \frac{\sigma t}{Mu^2} \\
\Longrightarrow \quad v(t) &= u\left(1 - \frac{1}{\sqrt{1+\frac{2\sigma t}{M}}}\right).
\end{aligned} \tag{4.148}$$

Note that $v \to u$ as $t \to \infty$, as it should. Integrating this speed to obtain the position gives

$$x(t) = ut - \frac{Mu}{\sigma}\sqrt{1+\frac{2\sigma t}{M}}\,. \tag{4.149}$$

Remark: For a given $t$, the $v(t)$ in eq. (4.148) is smaller than the $v(t)$ in eq. (4.142). This makes sense, because the balls have less of an effect on $v(t)$, because now (1) they don't bounce back, and (2) the mass of the car-plus-contents is larger. ♣

16. **Leaky bucket**

(a) **First Solution:** The initial position is $x = L$. The given rate of leaking implies that the mass of the bucket at position $x$ is $m = M(x/L)$. Therefore, $F = ma$

[21] We can also quickly derive this equation by writing down conservation of momentum for the time interval when a mass $dm$ enters the car:

$$dm\,u + mv = (m+dm)(v+dv). \tag{4.146}$$

This yields eq. (4.147). But we still need to use one of eqs. (4.144) and eq. (4.145).

gives $-T = (Mx/L)\ddot{x}$. Writing the acceleration as $v\,dv/dx$, and separating variables and integrating, gives

$$-\frac{TL}{M}\int_L^x \frac{dx}{x} = \int_0^v v\,dv. \qquad \Longrightarrow \qquad -\frac{TL}{M}\ln\left(\frac{x}{L}\right) = \frac{v^2}{2}. \tag{4.150}$$

The kinetic energy at position $x$ is therefore

$$E = \frac{mv^2}{2} = \left(\frac{Mx}{L}\right)\frac{v^2}{2} = -Tx\ln\left(\frac{x}{L}\right). \tag{4.151}$$

In terms of the fraction $z \equiv x/L$, we have $E = -TLz\ln z$. Setting $dE/dz = 0$ to find the maximum gives

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{TL}{e}. \tag{4.152}$$

Note that both $E_{\text{max}}$ and its location are independent of $M$.

Remark: We began this solution by writing down $F = ma$, where $m$ is the mass of the bucket. You may be wondering why we didn't use $F = dp/dt$, where $p$ is the momentum of the bucket. This would certainly give a different result, because $dp/dt = d(mv)/dt = ma + (dm/dt)v$. We used $F = ma$ because at any instant, the mass $m$ is what is being accelerated by the force $F$.

If you want, you can imagine the process occurring in discrete steps: The force pulls on the mass for a short period of time, then a little piece falls off. Then the force pulls again on the new mass, then another little piece falls off. And so on. In this scenario, it is clear that $F = ma$ is the appropriate formula, because it holds for each step in the process.

It is indeed true that $F = dp/dt$, if you let $F$ be *total* force in the problem, and let $p$ be the *total* momentum. The tension $T$ is the only horizontal force in the problem, because we've assumed the ground to be frictionless. However, the total momentum consists of both the sand in the bucket *and* the sand that has leaked out and is sliding along on the ground. If we use $F = dp/dt$, where $p$ is the total momentum, then we obtain

$$-T = \frac{dp_{\text{bucket}}}{dt} + \frac{dp_{\text{leaked}}}{dt} = \left(ma + \frac{dm}{dt}v\right) + \left(-\frac{dm}{dt}\right)v = ma, \tag{4.153}$$

as expected. (Note that $-dm/dt$ is a positive quantity.) See Appendix E for further discussion on the uses of $F = ma$ and $F = dp/dt$. ♣

**Second solution:** Consider a small time interval during which the bucket moves from $x$ to $x + dx$ (where $dx$ is negative). The bucket's kinetic energy changes by $(-T)\,dx$ (this is a positive quantity) due to the work done by the spring, and also changes by a fraction $dx/x$ (this is a negative quantity) due to the leaking. Therefore, $dE = -T\,dx + E\,dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}. \tag{4.154}$$

In solving this differential equation, it is convenient to introduce the variable $y \equiv E/x$. Then $E' = xy' + y$, where a prime denotes differentiation with respect to $x$. Eq. (4.154) then becomes $xy' = -T$, which gives

$$\int_0^{E/x} dy = -T\int_L^x \frac{dx}{x} \qquad \Longrightarrow \qquad E = -Tx\ln\left(\frac{x}{L}\right), \tag{4.155}$$

as in the first solution.

(b) From eq. (4.150), the speed is $v = \sqrt{2TL/M}\,\sqrt{-\ln z}$, where $z \equiv x/L$. Therefore, the magnitude of the momentum is

$$p = mv = (Mz)v = \sqrt{2TLM}\,\sqrt{-z^2 \ln z}. \tag{4.156}$$

Setting $dp/dz = 0$ to find the maximum gives

$$z = \frac{1}{\sqrt{e}} \qquad \Longrightarrow \qquad p_{\text{max}} = \sqrt{\frac{TLM}{e}}\,. \tag{4.157}$$

We see that the location of $p_{\text{max}}$ is independent of $M$, $T$, and $L$, but its value is not.

REMARK: $E_{\text{max}}$ occurs at a later time (that is, closer to the wall) than $p_{\text{max}}$. This is because $v$ matters more in $E = mv^2/2$ than it does in $p = mv$. As far as $E$ is concerned, it is beneficial for the bucket to lose a little more mass if it means being able to pick up a little more speed (up to a certain point). ♣

17. **Another leaky bucket**

(a) $F = ma$ says that $-T = m\ddot{x}$. Combining this with the given $dm/dt = b\ddot{x}$ yields $m\,dm = -bT\,dt$. Integration then gives $m^2/2 = C - bTt$. But $m = M$ when $t = 0$, so we have $C = M^2/2$. Therefore,

$$m(t) = \sqrt{M^2 - 2bTt}\,. \tag{4.158}$$

This holds for $t < M^2/2bT$, provided that the bucket hasn't hit the wall yet.

(b) The given equation $dm/dt = b\ddot{x} = b\,dv/dt$ integrates to $v = m/b + D$. But $v = 0$ when $m = M$, so we have $D = -M/b$. Therefore,

$$v(m) = \frac{m - M}{b} \qquad \Longrightarrow \qquad v(t) = \frac{\sqrt{M^2 - 2bTt}}{b} - \frac{M}{b}\,. \tag{4.159}$$

At the instant right before all the sand leaves the bucket, we have $m = 0$. Therefore, $v = -M/b$.

Integrating $v(t)$ to obtain $x(t)$, we find

$$x(t) = \frac{-(M^2 - 2bTt)^{3/2}}{3b^2T} - \frac{M}{b}t + L + \frac{M^3}{3b^2T}\,, \tag{4.160}$$

where the constant of integration has been chosen to satisfy $x = L$ when $t = 0$. Solving for $t$ in terms of $m$ from eq. (4.158), substituting the result into eq. (4.160), and simplifying, gives

$$x(m) = L - \frac{(M - m)^2(M + 2m)}{6b^2T}\,. \tag{4.161}$$

(c) Using eq. (4.159), the kinetic energy is

$$E = \frac{1}{2}mv^2 = \frac{1}{2b^2}m(m - M)^2. \tag{4.162}$$

Taking the derivative $dE/dm$ to find the maximum, we obtain

$$m = \frac{M}{3} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{2M^3}{27b^2}\,. \tag{4.163}$$

(d) Using eq. (4.159), the momentum is

$$p = mv = \frac{1}{b}m(m-M). \tag{4.164}$$

Taking the derivative to find the maximum magnitude, we obtain

$$m = \frac{M}{2} \qquad \Longrightarrow \qquad |p|_{\text{max}} = \frac{M^2}{4b}\,. \tag{4.165}$$

(e) We want $x = 0$ when $m = 0$. Eq. (4.161) then gives

$$0 = L - \frac{M^3}{6b^2T} \qquad \Longrightarrow \qquad b = \sqrt{\frac{M^3}{6TL}}\,. \tag{4.166}$$

18. **Right angle in billiards**

Let $\mathbf{v}$ be the initial velocity, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the final velocities. Conservation of momentum and energy give

$$\begin{aligned} m\mathbf{v} &= m\mathbf{v}_1 + m\mathbf{v}_2, \\ \frac{1}{2}m(\mathbf{v}\cdot\mathbf{v}) &= \frac{1}{2}m(\mathbf{v}_1\cdot\mathbf{v}_1) + \frac{1}{2}m(\mathbf{v}_2\cdot\mathbf{v}_2). \end{aligned} \tag{4.167}$$

Substituting the $\mathbf{v}$ from the first equation into the second, and using $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1+\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_2$, gives

$$\mathbf{v}_1\cdot\mathbf{v}_2 = 0. \tag{4.168}$$

In other words, the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $90^\circ$. (Or $\mathbf{v}_1 = \mathbf{0}$, which means the incoming mass stops because the collision is head-on. Or $\mathbf{v}_2 = \mathbf{0}$, which means the masses miss each other.)

19. **Bouncing and recoiling**

Let $v_i$ be the speed of the ball after the $i$th bounce, and let $V_i$ be the speed of the block right after the $i$th bounce. Then conservation of momentum gives

$$mv_i = MV_{i+1} - mv_{i+1}. \tag{4.169}$$

But Theorem 4.3 says that $v_i = V_{i+1} + v_{i+1}$. Solving this system of two linear equations gives

$$v_{i+1} = \frac{(M-m)v_i}{M+m} \equiv \frac{(1-\epsilon)v_i}{1+\epsilon} \approx (1-2\epsilon)v_i, \qquad \text{and} \qquad V_{i+1} \approx 2\epsilon v_i, \tag{4.170}$$

where $\epsilon \equiv m/M \ll 1$. This expression for $v_{i+1}$ implies that the speed of the ball after the $n$th bounce is

$$v_n = (1-2\epsilon)^n v_0. \tag{4.171}$$

The total distance traveled by the block can be obtained by looking at the work done by friction. Eventually, the ball has negligible energy, so all of its initial kinetic energy goes into heat from friction. Therefore, $mv_0^2/2 = F_f d = (\mu Mg)d$, which gives

$$d = \frac{mv_0^2}{2\mu Mg}\,. \tag{4.172}$$

To find the total time, we can add up the times, $t_n$, the block moves after each bounce. Since force times time equals the change in momentum, we have $F_f t_n = MV_n$, and so $(\mu Mg)t_n = M(2\epsilon v_{n-1}) = 2M\epsilon(1-2\epsilon)^{n-1}v_0$. Therefore,

$$t = \sum_{n=1}^{\infty} t_n = \frac{2\epsilon v_0}{\mu g}\sum_{n=0}^{\infty}(1-2\epsilon)^n = \frac{2\epsilon v_0}{\mu g}\cdot\frac{1}{1-(1-2\epsilon)} = \frac{v_0}{\mu g}. \tag{4.173}$$

REMARKS: This $t$ is independent of the masses. Note that it is much larger than the result obtained in the case where the ball sticks to the block on the first hit, in which case the answer is $mv_0/(\mu Mg)$. The total time is proportional to the total momentum that the block picks up, and the present answer is larger because the wall keeps transferring positive momentum to the ball, which then transfers it to the block.

The calculation of $d$ above can also be done by adding up the geometric series of the distances moved after each bounce. Note that $d$ is the same as it would be in the case where the ball sticks to the block on the first hit. The total distance is proportional to the total energy that the block picks up, and in both cases the total energy given to the block is $mv_0^2/2$. The wall (which is attached to the very massive earth) transfers essentially no energy to the ball. ♣

20. **Drag force on a sheet**

(a) We will set $v = 0$ here. When the sheet hits a particle, the particle acquires a speed of essentially $2V$. This follows from Theorem 4.3, or by working in the frame of the heavy sheet. The momentum of the particle is then $2mV$. In time $t$, the sheet sweeps through a volume $AVt$, where $A$ is the area of the sheet. Therefore, in time $t$, the sheet hits $AVtn$ particles. The sheet therefore loses momentum at a rate of $dP/dt = (AVn)(2mV)$. But $F = dP/dt$, so the force per unit area is

$$\frac{F}{A} = 2nmV^2 \equiv 2\rho V^2, \tag{4.174}$$

where $\rho$ is the mass density of the particles. We see that the force depends quadratically on $V$.

(b) If $v \gg V$, the particles now hit the sheet on both sides. Note that we can't set $V$ exactly equal to zero here, because we would obtain a result of zero and miss the lowest-order effect. In solving this problem, we need only consider the particles' motions in the direction of the sheet's motion. As stated in the problem, we will assume that all velocities in this direction are equal to $\pm v/2$.

Consider a particle in front of the sheet, moving backward toward the sheet. The relative speed between the particle and the sheet is $v/2 + V$. This relative speed simply reverses direction during the collision, so the change in momentum of this particle is $2m(v/2 + V)$. We have used the fact that the speed of the heavy sheet is essentially unaffected by the collision. The rate at which particles collide with the sheet is $A(v/2 + V)(n/2)$, from the reasoning in part (a). The $n/2$ factor comes from the fact that half of the particles move toward the sheet, and half move away from it.

Now consider a particle in back of the sheet, moving forward toward the sheet. The relative speed between the particle and the sheet is $v/2 - V$. This relative speed simply reverses direction during the collision, so the change in momentum of this particle is $-2m(v/2 - V)$. And the rate at which particles collide with the sheet is $A(v/2 - V)(n/2)$.

Therefore, the force per unit area on the sheet is

$$\begin{aligned} \frac{F}{A} &= \frac{1}{A} \cdot \frac{dP}{dt} \\ &= \Big(\frac{n}{2}(v/2+V)\Big)\Big(2m(v/2+V)\Big) + \Big(\frac{n}{2}(v/2-V)\Big)\Big(-2m(v/2-V)\Big) \\ &= 4nm(v/2)V \\ &\equiv 2\rho v V. \end{aligned} \tag{4.175}$$

We see that the force depends linearly on $V$. The fact that it agrees with the result in part (a) in the case of $v = V$ is coincidence. Neither result is valid when $v = V$.

21. **Drag force on a cylinder**

Consider a particle that makes contact with the cylinder at an angle $\theta$ with respect to the line of motion. In the frame of the heavy cylinder (see Fig. 4.53), the particle comes in with velocity $-V$ and then bounces off with a horizontal velocity component of $V\cos 2\theta$. So in this frame (and therefore also in the lab frame, because the heavy cylinder is essentially unaffected by the collision), the particle increases its horizontal momentum by $mV(1+\cos 2\theta)$.

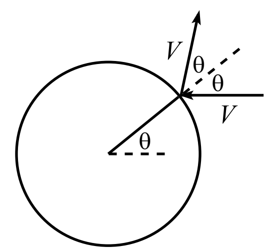


**Figure 4.53**

The area on the cylinder that lies between $\theta$ and $\theta + d\theta$ sweeps out volume at a rate $(R\,d\theta\cos\theta)V\ell$, where $\ell$ is the length of the cylinder. The $\cos\theta$ factor here gives the projection orthogonal to the direction of motion.

The force per unit length on the cylinder (that is, the rate of change of momentum, per unit length) is therefore

$$\begin{aligned} \frac{F}{\ell} &= \int_{-\pi/2}^{\pi/2} \Big(n(R\,d\theta\,\cos\theta)V\Big)\Big(mV(1+\cos 2\theta)\Big) \\ &= 2nmRV^2 \int_{-\pi/2}^{\pi/2} \cos\theta(1-\sin^2\theta)\,d\theta \\ &= 2nmRV^2 \left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Bigg|_{-\pi/2}^{\pi/2} \\ &= \frac{8}{3}nmRV^2 \equiv \frac{8}{3}\rho RV^2. \end{aligned} \tag{4.176}$$

Note that the average force per cross-sectional area, $F/(2R\ell)$, equals $(4/3)\rho V^2$. This is smaller than the result for the sheet in the previous problem, as it should be, because the particles bounce off somewhat sideways in the cylinder case.

22. **Basketball and tennis ball**

(a) Right before the basketball hits the ground, both balls move downward with speed (using $mv^2/2 = mgh$)

$$v = \sqrt{2gh}. \tag{4.177}$$

Right after the basketball bounces off the ground, it moves upward with speed $v$, while the tennis ball still moves downward with speed $v$. The relative speed is therefore $2v$. After the balls bounce off each other, the relative speed is still

$2v$. This follows from Theorem 4.3, or by working in the frame of the heavy basketball. Since the upward speed of the basketball essentially stays equal to $v$, the upward speed of the tennis ball is $2v+v=3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$. But $v^2 = 2gh$, so we have

$$H = d + 9h. \tag{4.178}$$

(b) Right before $B_1$ hits the ground, all of the balls move downward with speed $v = \sqrt{2gh}$.

We will inductively determine the speed of each ball after it bounces off the one below it. If $B_i$ achieves a speed of $v_i$ after bouncing off $B_{i-1}$, then what is the speed of $B_{i+1}$ after it bounces off $B_i$? The relative speed of $B_{i+1}$ and $B_i$ (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. Therefore, since $B_i$ is still moving upward at essentially speed $v_i$, we see that the final upward speed of $B_{i+1}$ equals $(v + v_i) + v_i$. Thus,

$$v_{i+1} = 2v_i + v. \tag{4.179}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), and then $v_3 = 7v$, and then $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{4.180}$$

which is easily seen to satisfy eq. (4.179), with the initial value $v_1 = v$.

From conservation of energy, $B_n$ will bounce to a height of

$$H = \ell + \frac{\big((2^n-1)v\big)^2}{2g} = \ell + (2^n-1)^2 h. \tag{4.181}$$

If $h$ is 1 meter, and we want this height to equal 1000 meters, then (assuming $\ell$ is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

Escape velocity from the earth (which is $v_{\text{esc}} = \sqrt{2gR} \approx 11{,}200\,\text{m/s}$) is reached when

$$v_n \geq v_{\text{esc}} \implies (2^n-1)\sqrt{2gh} \geq \sqrt{2gR} \implies n \geq \ln_2\left(\sqrt{\frac{R}{h}} + 1\right). \tag{4.182}$$

With $R = 6.4 \cdot 10^6\,\text{m}$ and $h = 1\,\text{m}$, we find $n \geq 12$. Of course, the elasticity assumption is absurd in this case, as is the notion that one can find 12 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{12}$.

23. **Colliding masses**

(a) By conservation of momentum, the final speed of the combined masses is $Mv/(M+m) \approx (1-m/M)v$, plus higher-order corrections. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(1-\frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}mv^2, \\ E_M &= \frac{1}{2}M\left(1-\frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}Mv^2 - mv^2. \end{aligned} \tag{4.183}$$

These energies add up to $Mv^2/2 - mv^2/2$, which is $mv^2/2$ less than the initial energy of mass $M$, namely $Mv^2/2$. Therefore, $mv^2/2$ is lost to heat.

(b) In this frame, mass $m$ has initial speed $v$, so its initial energy is $E_i = mv^2/2$. By conservation of momentum, the final speed of the combined masses is $mv/(M + m) \approx (m/M)v$, plus higher-order corrections. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right)^2 E_i \approx 0, \\ E_M &= \frac{1}{2}M\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right) E_i \approx 0. \end{aligned} \tag{4.184}$$

This negligible final energy is $mv^2/2$ less than $E_i$. Therefore, $mv^2/2$ is lost to heat, in agreement with part (a).

24. **Pulling a chain**

Let $x$ be the distance your hand has moved. Then $x/2$ is the length of the moving part of the chain, because the chain gets "doubled up". The momentum of this moving part is therefore $p = (\sigma x/2)\dot{x}$. The force that your hand applies is found from $F = dp/dt$, which gives $F = (\sigma/2)(\dot{x}^2 + x\ddot{x})$. But since $v$ is constant, the $\ddot{x}$ term vanishes. The change in momentum here is due simply to additional mass acquiring speed $v$, and not due to any increase in speed of the part already moving. Hence,

$$F = \frac{\sigma v^2}{2}, \tag{4.185}$$

which is constant. Your hand applies this force over a total distance $2L$, so the total work you do is

$$F(2L) = \sigma L v^2. \tag{4.186}$$

The mass of the chain is $\sigma L$, so its final kinetic energy is $(\sigma L)v^2/2$. This is only half of the work you do. Therefore, an energy of $\sigma L v^2/2$ is lost to heat.

Each atom in the chain goes abruptly from rest to speed $v$, and there is no way to avoid heat loss in such a process. This is clear when viewed in the reference frame of your hand. In this frame, the chain initially moves at speed $v$ and eventually comes to rest, piece by piece. All of its initial kinetic energy, $(\sigma L)v^2/2$, goes into heat.

25. **Pulling a rope**

Let $x$ be the position of the end of the rope. The momentum of the rope is then $p = (\sigma x)\dot{x}$. $F = dp/dt$ gives (using the fact that $F$ is constant) $Ft = p$, so we have $Ft = (\sigma x)\dot{x}$. Separating variables and integrating yields

$$\begin{aligned} \int_0^x \sigma x\, dx &= \int_0^t Ft\, dt \\ \Longrightarrow \qquad \frac{\sigma x^2}{2} &= \frac{Ft^2}{2} \\ \Longrightarrow \qquad x &= t\sqrt{F/\sigma}\,. \end{aligned} \tag{4.187}$$

The position therefore grows linearly with time. In other words, the speed is constant, and it equals $\sqrt{F/\sigma}$.

REMARK: Realistically, when you grab the rope, there is some small initial value of $x$ (call it $\epsilon$). The $dx$ integral above now starts at $\epsilon$ instead of 0, so $x$ takes the form, $x = \sqrt{Ft^2/\sigma + \epsilon^2}$. If $\epsilon$ is very small, the speed very quickly approaches $\sqrt{F/\sigma}$. Even if $\epsilon$ is not small, the position becomes arbitrarily close to $t\sqrt{F/\sigma}$, as $t$ becomes large. The "head-start" of $\epsilon$ will therefore not help you in the long run. ♣

26. **Raising the rope**

Let $y$ be the height of the top of the rope. Let $F(y)$ be the desired force applied by your hand. Consider the moving part of the rope. The net force on this part is $F - (\sigma y)g$, with upward taken to be positive. The momentum is $(\sigma y)\dot{y}$. Equating the net force on the moving part with the rate of change in momentum gives[22]

$$\begin{aligned} F - \sigma y g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{4.188}$$

But $\ddot{y} = 0$, and $\dot{y} = v$. Therefore,

$$F = \sigma y g + \sigma v^2. \tag{4.189}$$

The work that you do is the integral of this force, from $y = 0$ to $y = L$. Since $v$ is constant, we have

$$W = \int_0^L (\sigma y g + \sigma v^2)\, dy = \frac{\sigma L^2 g}{2} + \sigma L v^2. \tag{4.190}$$

The final potential energy of the rope is $(\sigma L)g(L/2)$, because the center of mass is raised by distance $L/2$. This is the first term in eq. (4.190). The final kinetic energy is $(\sigma L)v^2/2$. This accounts for half of the last term. The missing energy, $(\sigma L)v^2/2$, is converted into heat.

27. **Falling rope**

(a) **First Solution:** Let $\sigma$ be the mass density of the rope. From conservation of energy, we know that the rope's final kinetic energy, which is $(\sigma L)v^2/2$, equals the loss in potential energy. This loss equals $(\sigma L)(L/2)g$, because the center of mass falls a distance $L/2$. Therefore,

$$v = \sqrt{gL}\,. \tag{4.191}$$

This is the same as the speed obtained by an object that falls a distance $L/2$. Note that if the initial piece hanging down through the hole is arbitrarily short, then the rope will take an arbitrarily long time to fall down. But the final speed will be still be (arbitrarily close to) $\sqrt{gL}$.

**Second Solution:** Let $x$ be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the entire rope, which is $(\sigma L)\dot{x}$. Therefore, $F = dp/dt$ gives $(\sigma x)g = (\sigma L)\ddot{x}$, which is simply the $F = ma$ equation. Hence, $\ddot{x} = (g/L)x$, and the general solution to this equation is

$$x(t) = Ae^{t\sqrt{g/L}} + Be^{-t\sqrt{g/L}}. \tag{4.192}$$

Note that if $\epsilon$ is the initial value for $x$, then $A = B = \epsilon/2$ satisfies the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$, in which case we may write $x(t) = \epsilon\cosh(t\sqrt{g/L})$. But we won't need this information in what follows.

[22] If you instead wanted to use the entire rope as your system, then eq. (4.188) would still look the same, because the net force is the same (the extra weight of the rope on the floor is cancelled by normal force from the floor), and the momentum is the same (only the moving part has nonzero $p$).

Let $T$ be the time for which $x(T) = L$. If $\epsilon$ is very small, then $T$ will be very large. But for large $t$,[23] we may neglect the negative-exponent term in eq. (4.192). We then have

$$x \approx Ae^{t\sqrt{g/L}} \qquad \Longrightarrow \qquad \dot{x} \approx Ae^{t\sqrt{g/L}}\sqrt{g/L} \approx x\sqrt{g/L} \qquad \text{(for large } t\text{)}. \tag{4.193}$$

When $x = L$, we obtain

$$\dot{x}(T) = L\sqrt{g/L} = \sqrt{gL}\,, \tag{4.194}$$

in agreement with the first solution.

(b) Let $\sigma$ be the mass density of the rope, and let $x$ be the length that hangs down through the hole. The gravitational force on this length, which is $(\sigma x)g$, is responsible for changing the momentum of the rope. This momentum is $(\sigma x)\dot{x}$, because only the hanging part is moving. Therefore, $F = dp/dt$ gives

$$xg = x\ddot{x} + \dot{x}^2. \tag{4.195}$$

Note that $F = ma$ gives the wrong equation, because it neglects the fact that the moving mass, $\sigma x$, is changing. It therefore misses the second term on the right-hand side of eq. (4.195). In short, the momentum of the rope increases because it is speeding up (which gives the $x\ddot{x}$ term) *and* because additional mass is continually being added to the moving part (which gives the $\dot{x}^2$ term, as you can show).

To solve eq. (4.195) for $x(t)$, note that $g$ is the only parameter in the equation. Therefore, the solution for $x(t)$ can involve only $g$'s and $t$'s.[24] By dimensional analysis, $x(t)$ must then be of the form $x(t) = bgt^2$, where $b$ is a numerical constant to be determined. Plugging this expression for $x(t)$ into eq. (4.195) and dividing by $g^2t^2$ gives $b = 2b^2 + 4b^2$. Therefore, $b = 1/6$, and our solution may be written as

$$x(t) = \frac{1}{2}\left(\frac{g}{3}\right)t^2. \tag{4.196}$$

This is the equation for something that accelerates downward with acceleration $g' = g/3$. The time the rope takes to fall a distance $L$ is then given by $L = g't^2/2$, which yields $t = \sqrt{2L/g'}$. The final speed in thus

$$v = g't = \sqrt{2Lg'} = \sqrt{\frac{2gL}{3}}\,. \tag{4.197}$$

This is smaller than the $\sqrt{gL}$ result from part (a). We therefore see that although the total time for the scenario in part (a) is very large, the final speed in that case is in fact larger than that in the present scenario.

REMARKS: Using eq. (4.197), you can show that 1/3 of the available potential energy is lost to heat. This inevitable loss occurs during the abrupt motions that suddenly bring the atoms from zero to non-zero speed when they join the moving part of the

[23] More precisely, for $t \gg \sqrt{L/g}$.

[24] The other dimensionful quantities in the problem, $L$ and $\sigma$, do not appear in eq. (4.195), so they cannot appear in the solution. Also, the initial position and speed (which will in general appear in the solution for $x(t)$, because eq. (4.195) is a second-order differential equation) do not appear in this case, because they are equal to zero.

rope. The use of conservation of energy is therefore *not* a valid way to solve this problem.

You can show that the speed in part (a)'s scenario is smaller than the speed in part (b)'s scenario for $x$ less than $2L/3$, but larger for $x$ greater than $2L/3$.

28. **The raindrop**

Let $\rho$ be the mass density of the raindrop, and let $\lambda$ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

We need three equations to solve for the above three unknowns. The equations we will use are two different expressions for $dM/dt$, and the $F = dp/dt$ expression for the raindrop.

The first expression for $\dot{M}$ is obtained by simply taking the derivative of $M = (4/3)\pi r^3 \rho$, which gives

$$\begin{aligned} \dot{M} &= 4\pi r^2 \dot{r} \rho && (4.198) \\ &= 3M\frac{\dot{r}}{r}\,. && (4.199) \end{aligned}$$

The second expression for $\dot{M}$ is obtained by noting that the change in $M$ is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v \lambda. \quad (4.200)$$

The $F = dp/dt$ equation is found as follows. The gravitational force is $Mg$, and the momentum is $Mv$. Therefore, $F = dp/dt$ gives

$$Mg = \dot{M}v + M\dot{v}. \quad (4.201)$$

We now have three equations involving the three unknowns, $r$, $M$, and $v$.[25]

Our goal is to find $\dot{v}$. We will do this by first finding $\ddot{r}$. Eqs. (4.198) and (4.200) give

$$\begin{aligned} v &= \frac{4\rho}{\lambda}\dot{r} && (4.202) \\ \Longrightarrow \quad \dot{v} &= \frac{4\rho}{\lambda}\ddot{r}. && (4.203) \end{aligned}$$

Plugging eqs. (4.199, 4.202, 4.203) into eq. (4.201) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \quad (4.204)$$

Therefore,

$$\tilde{g}r = 12\dot{r}^2 + 4r\ddot{r}, \quad (4.205)$$

where we have defined $\tilde{g} \equiv g\lambda/\rho$, for convenience. The only parameter in eq. (4.205) is $\tilde{g}$. Therefore, $r(t)$ can depend only on $\tilde{g}$ and $t$. Hence, by dimensional analysis, $r$ must take the form

$$r = A\tilde{g}t^2, \quad (4.206)$$

[25] Note that we *cannot* write down the naive conservation-of-energy equation (which would say that the decrease in the water's potential energy equals the increase in its kinetic energy), because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.

where $A$ is a numerical constant, to be determined. Plugging this expression for $r$ into eq. (4.205) gives

$$\begin{aligned} \tilde{g}(A\tilde{g}t^2) &= 12(2A\tilde{g}t)^2 + 4(A\tilde{g}t^2)(2A\tilde{g}) \\ \Longrightarrow \quad A &= 48A^2 + 8A^2. \end{aligned} \tag{4.207}$$

Therefore, $A = 1/56$, and so $\ddot{r} = 2A\tilde{g} = \tilde{g}/28 = g\lambda/28\rho$. Eq. (4.203) then gives the acceleration of the raindrop as

$$\dot{v} = \frac{g}{7}\,, \tag{4.208}$$

independent of $\rho$ and $\lambda$.

REMARKS: A common invalid solution to this problem is the following, which (incorrectly) uses conservation of energy.

The fact that $v$ is proportional to $\dot{r}$ (shown in eq. (4.202)) means that the volume swept out by the raindrop is a cone. The center of mass of a cone is 1/4 of the way from the base to the apex (as you can show by integrating over horizontal circular slices). Therefore, if $M$ is the mass of the raindrop after it has fallen a height $h$, then an (incorrect) application of conservation of energy gives

$$\frac{1}{2}Mv^2 = Mg\frac{h}{4} \qquad \Longrightarrow \qquad v^2 = \frac{gh}{2}\,. \tag{4.209}$$

Taking the derivative of this (or equivalently, using the general result, $v^2 = 2ad$), we obtain

$$\dot{v} = \frac{g}{4} \qquad \text{(incorrect)}. \tag{4.210}$$

The reason why this solution is invalid is that the collisions between the raindrop and the droplets are completely inelastic. Heat is generated, and the overall kinetic energy of the raindrop is smaller than you would otherwise expect.

Let's calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen. The loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{4.211}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{4.212}$$

where $\Delta E_{\text{int}}$ is the gain in internal thermal energy.

The energy required to heat 1g of water by 1 C° is 1 calorie (= 4.18 joules). Therefore, the energy required to heat 1 kg of water by 1 C° is ≈ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{4.213}$$

where $M$ is measured in kilograms, and $T$ is measured in Celsius. Eqs. (4.212) and (4.213) give the increase in temperature as a function of $h$,

$$4200\, \Delta T = \frac{3}{28}gh. \tag{4.214}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100$ C° is found from eq. (4.214) to be

$$h \approx 400\,\text{km}, \tag{4.215}$$

which is much larger than the height of the atmosphere. We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for $h$ is a few kilometers, which would raise the temperature by only about one degree. This effect, of course, is washed out by many other factors. ♣

# Chapter 5

# The Lagrangian Method



Consider the problem of a mass on the end of a spring. We can solve this, of course, by using $F = ma$ to write down $m\ddot{x} = -kx$. The solutions to this equation are sinusoidal functions, as we well know. We can, however, solve this problem by using another method which doesn't explicitly use $F = ma$. In many (in fact, probably most) physical situations, this new method is far superior to using $F = ma$. You will soon discover this for yourself when you tackle the problems for this chapter.

We will present our new method by first stating its rules (without any justification) and showing that they somehow end up magically giving the correct answer. We will then give the method proper justification.

## 5.1 The Euler-Lagrange equations

Here is the procedure. Form the following seemingly silly combination of the kinetic and potential energies ($T$ and $V$, respectively),

$$\boxed{L \equiv T - V}. \tag{5.1}$$

This is called the *Lagrangian*. Yes, there is a minus sign in the definition (a plus sign would simply give the total energy). In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.2}$$

Now write

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}. \tag{5.3}$$

Don't worry, we'll show you in Section 5.2 where this comes from. This equation is called the *Euler-Lagrange (E-L) equation*. For the problem at hand, we have $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial x = -kx$, so eq. (5.3) gives

$$m\ddot{x} = -kx, \tag{5.4}$$

which is exactly the result obtained by using $F = ma$. An equation such as eq. (5.4), which is derived from the Euler-Lagrange equation, is called an *equation of motion*.[1]

If the problem involves more than one coordinate, as most problems do, we simply have to apply eq. (5.3) to each coordinate. We will obtain as many equations as there are coordinates. Each equation may very well involve many of the coordinates (see the example below, where both equations involve both $x$ and $\theta$).

At this point, you may be thinking, "That was a nice little trick, but we just got lucky in the spring problem. The procedure won't work in a more general situation." Well, let's see. How about if we consider the more general problem of a particle moving in an arbitrary potential, $V(x)$ (we'll just stick to one dimension for now). Then the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \tag{5.5}$$

The Euler-Lagrange equation, eq. (5.3), gives

$$m\ddot{x} = -\frac{dV}{dx}\,. \tag{5.6}$$

But $-dV/dx$ is simply the force on the particle. So we see that eqs. (5.1) and (5.3) together say exactly the same thing that $F = ma$ says, when using a cartesian coordinate in one dimension (but this result is in fact quite general, as we will see in Section 5.4).

Note that shifting the potential by a given constant has no effect on the equation of motion, because eq. (5.3) involves only derivatives of $V$. This, of course, is equivalent to saying that only differences in energy are relevant, and not the actual values, as we well know.

In a three-dimensional problem, where the potential takes the form $V(x, y, z)$, it immediately follows that the three Euler-Lagrange equations (obtained by applying eq. (5.3) to $x$, $y$, and $z$) may be combined into the vector statement,

$$m\ddot{\mathbf{x}} = -\nabla V. \tag{5.7}$$

But $-\nabla V = \mathbf{F}$, so we again arrive at Newton's second law, $\mathbf{F} = m\mathbf{a}$, now in three dimensions.

Let's now do one more example to convince you that there's really something nontrivial going on here.

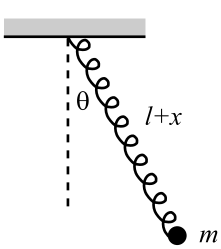

**Figure 5.1**

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**Example (Spring pendulum):** Consider a pendulum made out of a spring with a mass $m$ on the end (see Fig. 5.1). The spring is arranged to lie in a straight line (which we can arrange by, say, wrapping the spring around a rigid massless rod). The

[1] The term "equation of motion" is a little ambiguous. It is understood to refer to the second-order differential equation satisfied by $x$, and *not* the actual equation for $x$ as a function of $t$, namely $x(t) = A\cos(\omega t + \phi)$ in this problem, which is obtained by integrating the equation of motion twice.

equilibrium length of the spring is $\ell$. Let the spring have length $\ell + x(t)$, and let its angle with the vertical be $\theta(t)$. Assuming that the motion takes place in a vertical plane, find the equations of motions for $x$ and $\theta$.

**Solution:** The kinetic energy may be broken up into the radial and tangential parts, so we have

$$T = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big). \tag{5.8}$$

The potential energy comes from both gravity and the spring, so we have

$$V(x, \theta) = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2. \tag{5.9}$$

The Lagrangian therefore equals

$$L \equiv T - V = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big) + mg(\ell + x)\cos\theta - \frac{1}{2}kx^2. \tag{5.10}$$

There are two variables here, $x$ and $\theta$. As mentioned above, the nice thing about the Lagrangian method is that we can simply use eq. (5.3) twice, once with $x$ and once with $\theta$. Hence, the two Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad m\ddot{x} = m(\ell + x)\dot{\theta}^2 + mg\cos\theta - kx, \tag{5.11}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad &\Longrightarrow \qquad \frac{d}{dt}\Big(m(\ell + x)^2\dot{\theta}\Big) = -mg(\ell + x)\sin\theta \\ &\Longrightarrow \qquad m(\ell + x)^2\ddot{\theta} + 2m(\ell + x)\dot{x}\dot{\theta} = -mg(\ell + x)\sin\theta. \\ &\Longrightarrow \qquad m(\ell + x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta. \end{aligned} \tag{5.12}$$

Eq. (5.11) is simply the radial $F = ma$ equation, complete with the centripetal acceleration, $-(\ell + x)\dot{\theta}^2$. The first line of eq. (5.12) is the statement that the torque equals the rate of change of the angular momentum (one of the subjects of Chapter 7).[2]

After writing down the E-L equations, it is always best to double-check them by trying to identify them as $F = ma$ or $\tau = dL/dt$ equations. Sometimes, however, this identification is not obvious. For the times when everything is clear (that is, when you look at the E-L equations and say, "Oh, of course!"), it is usually clear only *after* you've derived them. The Lagrangian method is generally the safer method to use.

The present example should convince you of the great utility of the Lagrangian method. Even if you've never heard of the terms "torque", "centripetal", "centrifugal", or "Coriolis", you can still get the correct equations by simply writing down the kinetic and potential energies, and then taking a few derivatives.

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[2] Alternatively, if you want to work in a rotating frame, then eq. (5.11) is the radial $F = ma$ equation, complete with the centrifugal force, $m(\ell + x)\dot{\theta}^2$. And the third line of eq. (5.12) is the tangential $F = ma$ equation, complete with the Coriolis force, $-2m\dot{x}\dot{\theta}$. But never mind about this now. We'll deal with rotating frames in Chapter 9.

At this point it seems to be personal preference, and all academic, whether you use the Lagrangian method or the $F = ma$ method. The two methods produce the same equations. However, in problems involving more than one variable, it usually turns out to be *much* easier to write down $T$ and $V$, as opposed to writing down all the forces. This is because $T$ and $V$ are nice and simple scalars. The forces, on the other hand, are vectors, and it's easy to get confused if they point in various directions. The Lagrangian method has the advantage that once you've written down $L \equiv T - V$, you don't have to think anymore. All you have to do is blindly take some derivatives.[3]

When jumping from high in a tree,
Just write down del $L$ by del $z$.
Take del $L$ by $z$ dot,
Then $t$-dot what you've got,
And equate the results (but quickly!)

But ease and speed of computation aside, is there any fundamental difference between the two methods? Is there any deep reasoning behind eq. (5.3)? Indeed, there is...

## 5.2 The principle of stationary action

Consider the quantity,

$$S \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt. \tag{5.13}$$

$S$ is called the *action*. It is a number with the dimensions of (Energy) $\times$ (Time). $S$ depends on $L$, and $L$ in turn depends on the function $x(t)$ via eq. (5.1).[4] Given any function $x(t)$, we can produce the number $S$. We'll just deal with one coordinate, $x$, for now.

$S$ is called a *functional*, and is sometimes denoted by $S[x(t)]$. It depends on the entire function $x(t)$, and not on just one input number, as a regular function $f(t)$ does. $S$ can be thought of as a function of an infinite number of values, namely all the $x(t)$ for $t$ ranging from $t_1$ to $t_2$. If you don't like infinities, you can imagine breaking up the time interval into, say, a million pieces, and then replacing the integral by a discrete sum.

Let us now pose the following question: Consider a function $x(t)$, for $t_1 \le t \le t_2$, which has its endpoints fixed (that is, $x(t_1) = x_1$ and $x(t_2) = x_2$, where $x_1$ and $x_2$ are given), but is otherwise arbitrary. What function $x(t)$ yields a stationary value of $S$? A stationary value is a local minimum, maximum, or saddle point.[5]

[3] Of course, you eventually have to *solve* the resulting equations of motion, but you have to do that when using the $F = ma$ method, too.

[4] In some situations, the kinetic and potential energies in $L \equiv T - V$ may explicitly depend on time, so we have included the "$t$" in eq. (5.13).

[5] A saddle point is a point where there are no first-order changes in $S$, and where some of the second-order changes are positive and some are negative (like the middle of a saddle, of course).

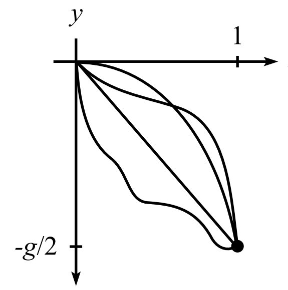


**Figure 5.2**

For example, consider a ball dropped from rest, and consider the function $y(t)$ for $0 \le t \le 1$. Assume that we somehow know that $y(0) = 0$ and $y(1) = -g/2$.[6] A number of possibilities for $y(t)$ are shown in Fig. 5.2, and each of these can (in theory) be plugged into eqs. (5.1) and (5.13) to generate $S$. Which one yields a stationary value of $S$? The following theorem gives us the answer.

**Theorem 5.1** *If the function $x_0(t)$ yields a stationary value (that is, a local minimum, maximum, or saddle point) of $S$, then*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}\,. \tag{5.14}$$

*It is understood that we are considering the class of functions whose endpoints are fixed. That is, $x(t_1) = x_1$ and $x(t_2) = x_2$.*

**Proof:** We will use the fact that if a certain function $x_0(t)$ yields a stationary value of $S$, then any other function very close to $x_0(t)$ (with the same endpoint values) yields essentially the same $S$, up to first order in any deviations. This is actually the definition of a stationary value. The analogy with regular functions is that if $f(b)$ is a stationary value of $f$, then $f(b+\epsilon)$ differs from $f(b)$ only at second order in the small quantity $\epsilon$. This is true because $f'(b) = 0$, so there is no first-order term in the Taylor series around $b$.

Assume that the function $x_0(t)$ yields a stationary value of $S$, and consider the function

$$x_a(t) \equiv x_0(t) + a\beta(t), \tag{5.15}$$

where $\beta(t)$ satisfies $\beta(t_1) = \beta(t_2) = 0$ (to keep the endpoints of the function fixed), but is otherwise arbitrary.

The action $S[x_a(t)]$ is a function of $a$ (the $t$ is integrated out, so $S$ is just a number, and it depends on $a$), and we demand that there be no change in $S$ at first order in $a$. How does $S$ depend on $a$? Using the chain rule, we have

$$\begin{aligned} \frac{d}{da}S[x_a(t)] &= \frac{d}{da}\int_{t_1}^{t_2} L\,dt \\ &= \int_{t_1}^{t_2} \frac{dL}{da}\,dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial a}\right) dt. \end{aligned} \tag{5.16}$$

In other words, $a$ influences $S$ through its effect on $x$, and also through its effect on $\dot{x}$. From eq. (5.15), we have

$$\frac{\partial x_a}{\partial a} = \beta, \qquad \text{and} \qquad \frac{\partial \dot{x}_a}{\partial a} = \dot{\beta}, \tag{5.17}$$

[6]This follows from $y = -gt^2/2$, but pretend that we don't know this formula.

so eq. (5.16) becomes[7]

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta} \right) dt. \tag{5.18}$$

Now comes the one sneaky part of the proof. You will see this trick many times in your physics career. We will integrate the second term by parts. Using

$$\int \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\, dt = \frac{\partial L}{\partial \dot{x}_a}\beta - \int \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} \right) \beta\, dt, \tag{5.19}$$

eq. (5.18) becomes

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} \right) \beta\, dt + \frac{\partial L}{\partial \dot{x}_a}\beta \bigg|_{t_1}^{t_2}. \tag{5.20}$$

But $\beta(t_1) = \beta(t_2) = 0$, so the last term (the "boundary term") vanishes. We now use the fact that $(d/da)S[x_a(t)]$ must be zero for *any* function $\beta(t)$, because we are assuming that $x_0(t)$ yields a stationary value. The only way this can be true is if the quantity in parentheses above (evaluated at $a = 0$) is identically equal to zero, that is,

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}_0} \right) = \frac{\partial L}{\partial x_0}. \qquad \blacksquare \tag{5.21}$$

The E-L equation, eq. (5.3), therefore doesn't just come out of the blue. It is a consequence of requiring that the action be at a stationary value. We may therefore replace $F = ma$ by the following principle.

- **The Principle of Stationary-Action:**

  *The path of a particle is the one that yields a stationary value of the action.*

This principle is equivalent to $F = ma$ because the above theorem shows that if (and only if, as you can show by working backwards) we have a stationary value of $S$, then the E-L equations hold. And the E-L equations are equivalent to $F = ma$ (as we showed for Cartesian coordinates in Section 5.1 and which we'll prove for any coordinate system in Section 5.4). Therefore, "stationary-action" is equivalent to $F = ma$.

If we have a multidimensional problem, where the lagrangian is a function of the variables $x_1(t), x_2(t), \ldots$, then the above principle of stationary action is still all we need. With more than one variable, we can now vary the path by varying each coordinate (or combinations thereof). The variation of each coordinate produces an E-L equation which, as we caw in the cartesian case, is equivalent to an $F = ma$ equation.

Given a classical mechanics problem, we can solve it with $F = ma$, or we can solve it with the E-L equations, which derive from the principle of stationary action

[7]Note that nowhere do we assume that $x_a$ and $\dot{x}_a$ are independent variables. The partial derivatives in eq. (5.17) are very much related, in that one is the derivative of the other. The use of the chain rule in eq. (5.16) is still perfectly valid.

(often called the principle of "minimal action", but see the third remark below). Either method will get the job done. But as mentioned at the end of Section 5.1, it is often easier to use the latter, because it avoids the use of force, and it's easy to get confused if you have forces pointing in all sorts of complicated directions.

It just stood there and did nothing, of course,
A harmless and still wooden horse.
But the minimal action
Was just a distraction;
The plan involved no use of force.

Let's now return to the example of a ball dropped from rest, mentioned above. The Lagrangian is $L = T - V = m\dot{y}^2/2 - mgy$, so eq. (5.21) gives $\ddot{y} = -g$, which is simply the $F = ma$ equation, as expected. The solution is $y(t) = -gt^2/2 + v_0 t + y_0$, as we well know. But the initial conditions tell us that $v_0 = y_0 = 0$, so our solution is $y(t) = -gt^2/2$. You are encouraged to verify explicitly that this $y(t)$ yields an action that is stationary with respect to variations of the form, say, $y(t) = -gt^2/2 + \epsilon t(t-1)$, which also satisfies the endpoint conditions (this is the task of Exercise 3). There are, of course, an infinite number of other ways to vary $y(t)$, but this specific result should help convince you of the general result of Theorem 5.1.

Note that the stationarity implied by the Euler-Lagrange equation, eq. (5.21), is a *local* statement. It gives information only about nearby paths. It says nothing about the *global* nature of how the action depends on all possible paths. If we find that a solution to eq. (5.21) happens to produce a local minimum, there is no reason to conclude that it is a global minimum, although in many cases it turns out to be.

Remarks:

1. Theorem 5.1 is based on the assumption that the ending time, $t_2$, of the motion is given. But how do we know this final time? Well, we don't. In the example of a ball thrown upward, the total time to rise and fall back to your hand can be anything, depending on the ball's initial speed. This initial speed will show up as an integration constant when solving the E-L Equations. The motion has to end sometime, and the principle of stationary action says that for whatever time this happens to be, the physical path has a stationary action.

2. Theorem 5.1 shows that we can explain the E-L equations by the principle of stationary action. This, however, simply shifts the burden of proof. We are now left with the task of justifying why we should want the action to have a stationary value. The good news is that there is a very solid reason for this. The bad news is that the reason involves quantum mechanics, so we won't be able to discuss it properly here. Suffice it to say that a particle actually takes all possible paths in going from one place to another, and each path is associated with the complex number $e^{iS/\hbar}$ (where $\hbar = 1.05 \cdot 10^{-34}$ Js is *Planck's constant*). These complex numbers have absolute value 1 and are called "phases". It turns out that the phases from all possible paths must be added up to give the "amplitude" of going from one point to another. The absolute value of the amplitude must then be squared to obtain the probability.[8]

[8]This is one of those remarks that is completely useless, because it is incomprehensible to those who haven't seen the topic before, and trivial to those who have. My apologies. But this and the

The basic point, then, is that at a non-stationary value of $S$, the phases from different paths differ (greatly, because $\hbar$ is so small) from one another, which effectively leads to the addition of many random vectors in the complex plane. These end up cancelling each other, yielding a sum of essentially zero. There is therefore no contribution to the overall amplitude from non-stationary values of $S$. Hence, we do not observe the paths associated with these $S$'s. At a stationary value of $S$, however, all the phases take on essentially the same value, thereby adding constructively instead of destructively. There is therefore a non-zero probability for the particle to take a path that yields a stationary value of $S$. So this is the path we observe.

3. But again, the preceding remark simply shifts the burden of proof one step further. We must now justify why these phases $e^{iS/\hbar}$ should exist, and why the Lagrangian that appears in $S$ should equal $T - V$. But here's where we're going to stop.

4. The principle of stationary action is sometimes referred to as the principle of "least" action, but this is misleading. True, it is often the case that the stationary value turns out to be a minimum value, but it need not be, as we can see in the following example.

   Consider a harmonic oscillator. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.22}$$

   Let $x_0(t)$ be a function which yields a stationary value of the action. Then we know that $x_0(t)$ satisfies the E-L equation, $m\ddot{x}_0 = -kx_0$.

   Consider a slight variation on this path, $x_0(t) + \xi(t)$, where $\xi(t)$ satisfies $\xi(t_1) = \xi(t_2) = 0$. With this new function, the action becomes

$$S_\xi = \int_{t_1}^{t_2} \left( \frac{m}{2}\left(\dot{x}_0^2 + 2\dot{x}_0\dot{\xi} + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + 2x_0\xi + \xi^2\right)\right) dt. \tag{5.23}$$

   The two cross-terms add up to zero, because after integrating the $\dot{x}_0\dot{\xi}$ term by parts, their sum is

$$m\dot{x}_0\xi\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (m\ddot{x}_0 + kx_0)\xi\, dt. \tag{5.24}$$

   The first term is zero, due to the boundary conditions on $\xi(t)$. The second term is zero, due to the E-L equation. We've basically just reproduced the proof of Theorem 5.1 for the special case of the harmonic oscillator here.

   The terms involving only $x_0$ give the stationary value of the action (call it $S_0$). To determine whether $S_0$ is a minimum, maximum, or saddle point, we must look at the difference,

$$\Delta S \equiv S_\xi - S_0 = \frac{1}{2}\int_{t_1}^{t_2} (m\dot{\xi}^2 - k\xi^2)\, dt. \tag{5.25}$$

   It is always possible to find a function $\xi$ that makes $\Delta S$ positive. Simply choose $\xi$ to be small, but make it wiggle very fast, so that $\dot{\xi}$ is large. Therefore, it is *never* the case that $S_0$ is a maximum. Note that this reasoning works for any potential, not just a harmonic oscillator, as long as it is a function of only position (that is, it contains no derivatives, as we always assume).

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following remarks are by no means necessary for an understanding of the material in this chapter. If you're interested in reading more about these quantum mechanics issues, you should take a look at Richard Feynman's book, *QED*. Feynman was, after all, the one who thought of this idea.

You might be tempted to use the same line of reasoning to say that it is also always possible to find a function $\xi$ that makes $\Delta S$ negative, by making $\xi$ large and $\dot{\xi}$ small. If this were true, then we could put everything together and conclude that all stationary points are saddle points, for a harmonic oscillator. However, it is *not* always possible to make $\xi$ large enough and $\dot{\xi}$ small enough so that $\Delta S$ is negative, due to the boundary conditions $\xi(t_1) = \xi(t_2) = 0$. If $\xi$ changes from zero to a large value and then back to zero, then $\dot{\xi}$ may also have to be large, if the time interval is short enough. Problem 6 deals quantitatively with this issue. For now, let's just say that in some cases $S_0$ is a minimum, in some cases it is a saddle point, and it is never a maximum. "Least action" is therefore a misnomer.

5. It is sometimes said that nature has a "purpose", in that it seeks to take the path that produces the minimum action. In view of the second remark above, this is incorrect. In fact, nature does exactly the opposite. It takes *every* path, treating them all on equal footing. We simply end up seeing the path with a stationary action, due to the way the quantum mechanical phases add.

   It would be a harsh requirement, indeed, to demand that nature make a "global" decision (that is, to compare paths that are separated by large distances), and to choose the one with the smallest action. Instead, we see that everything takes place on a "local" scale. Nearby phases simply add, and everything works out automatically.

   When an archer shoots an arrow through the air, the aim is made possible by all the other arrows taking all the other nearby paths, each with essentially the same action. Likewise, when you walk down the street with a certain destination in mind, you're not alone.

   When walking, I know that my aim
   Is caused by the ghosts with my name.
   And although I don't see
   Where they walk next to me,
   I know they're all there, just the same.

6. Consider a function, $f(x)$, of one variable (for ease of terminology). Let $f(b)$ be a local minimum of $f$. There are two basic properties of this minimum. The first is that $f(b)$ is smaller than all nearby values. The second is that the slope of $f$ is zero at $b$. From the above remarks, we see that (as far as the action $S$ is concerned) the first property is completely irrelevant, and the second one is the whole point. In other words, saddle points (and maxima, although we showed above that these never exist for $S$) are just as good as minima, as far as the constructive addition of the $e^{iS/\hbar}$ phases is concerned.

7. Given that classical mechanics is an approximate theory, while quantum mechanics is the (more) correct one, it is quite silly to justify the principle of stationary action by demonstrating its equivalence with $F = ma$, as we did above. We should be doing it the other way around. However, because our intuition is based on $F = ma$, we'll assume that it's easier to start with $F = ma$ as the given fact, rather than calling upon the latent quantum-mechanical intuition hidden deep within all of us. Maybe someday...

   At any rate, in more advanced theories dealing with fundamental issues concerning the tiny building blocks of matter (where the action is of the same order of magnitude as $\hbar$), the approximate $F = ma$ theory is invalid, and you *have* to use the Lagrangian method. ♣

## 5.3 Forces of constraint

One nice thing about the Lagrangian method is that we are free to impose any given constraints at the beginning of the problem, thereby immediately reducing the number of variables. This is always done (perhaps without thinking) whenever a particle is constrained to move on a wire or surface, etc. Often we are not concerned with the exact nature of the forces doing the constraining, but only with the resulting motion, given that the constraints hold. By imposing the constraints at the outset, we can find the motion, but we can't say anything about the constraining forces.

If we want to determine the constraining forces, we must take a different approach. The main idea of the strategy, as we will show below, is that we must not impose the constraints too soon. This, of course, leaves us with a larger number of variables to deal with, so the calculations are more cumbersome. But the benefit is that we are able to find the constraining forces.

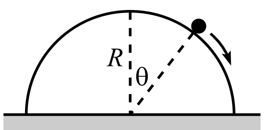

**Figure 5.3**

Consider the problem of a particle sliding off a fixed frictionless hemisphere of radius $R$ (see Fig. 5.3). Let's say that we are concerned only with finding the equation of motion for $\theta$, and not the constraining force. Then we can write everything in terms of $\theta$, because we know that the radial distance, $r$, is constrained to be $R$. The kinetic energy is $mR^2\dot{\theta}^2/2$, and the potential energy (relative to the bottom of the hemisphere) is $mgr\cos\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta, \tag{5.26}$$

and the equation of motion, via eq. (5.3), is

$$\ddot{\theta} = (g/R)\sin\theta, \tag{5.27}$$

which is simply the tangential $F = ma$ statement.

Now let's say that we want to find the constraining normal force that the hemisphere applies to the particle. To do this, let's solve the problem in a different way and write things in terms of both $r$ and $\theta$. Also (and here's the critical step), let's be really picky and say that $r$ isn't *exactly* constrained to be $R$, because in the real world the particle actually sinks into the hemisphere a little bit. This may seem a bit silly, but it's really the whole point. The particle pushes and sinks inward a tiny distance until the hemisphere gets squashed enough to push back with the appropriate force to keep the particle from sinking in any more. (Just consider the hemisphere to be made of lots of little springs with very large spring constants.) The particle is therefore subject to a (very) steep potential due to the hemisphere. The constraining potential, $V(r)$, looks something like the plot in Fig. 5.4. The *true* Lagrangian for the system is thus

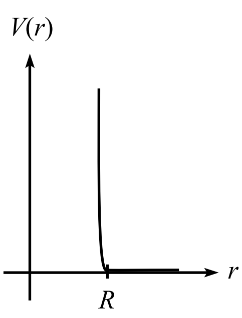

**Figure 5.4**

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta - V(r). \tag{5.28}$$

(The $\dot{r}^2$ term in the kinetic energy will turn out to be insignificant.) The equations of motion obtained from varying $\theta$ and $r$ are therefore

$$\begin{aligned} mr^2\ddot{\theta} &= mgr\sin\theta, \\ m\ddot{r} &= mr\dot{\theta}^2 - mg\cos\theta - V'(r). \end{aligned} \tag{5.29}$$

Having written down the equations of motion, we will *now* apply the constraint condition that $r = R$. This condition implies $\dot{r} = \ddot{r} = 0$. (Of course, $r$ isn't *really* equal to $R$, but any differences are inconsequential from this point onward.) The first of eqs. (5.29) then simply gives eq. (5.27), while the second yields

$$-\frac{dV}{dr}\bigg|_{r=R} = mg\cos\theta - mR\dot{\theta}^2. \tag{5.30}$$

But $F \equiv -dV/dr$ is the constraint force applied in the $r$ direction, which is precisely the force we are looking for. The normal force of constraint is therefore

$$F(\theta, \dot{\theta}) = mg\cos\theta - mR\dot{\theta}^2, \tag{5.31}$$

which is simply the radial $F = ma$ statement. Note that this result is valid only if $F(\theta, \dot{\theta}) > 0$. If the normal force becomes zero, then this means that the particle has left the sphere, in which case $r$ is no longer equal to $R$ (except at the instant right when it leaves).

REMARKS:

1. What if we instead had (unwisely) chosen Cartesian coordinates, $x$ and $y$, instead of polar coordinates, $r$ and $\theta$? Since the distance from the particle to the surface of the sphere is $\eta \equiv \sqrt{x^2 + y^2} - R$, we obtain a true Lagrangian equal to

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(\eta). \tag{5.32}$$

The equations of motion are (using the chain rule)

$$m\ddot{x} = -\frac{dV}{d\eta}\frac{\partial\eta}{\partial x}, \qquad \text{and} \qquad m\ddot{y} = -mg - \frac{dV}{d\eta}\frac{\partial\eta}{\partial y}. \tag{5.33}$$

We now apply the constraint condition $\eta = 0$. Since $-dV/d\eta$ equals the constraint force $F$, you can show that the equations we end up with (namely, the two E-L equations and the constraint equation) are

$$m\ddot{x} = F\frac{x}{R}, \qquad m\ddot{y} = -mg + F\frac{y}{R}, \qquad \text{and} \qquad \sqrt{x^2 + y^2} - R = 0. \tag{5.34}$$

These three equations are sufficient to determine the three unknowns $\ddot{x}$, $\ddot{y}$, and $F$ as functions of the quantities $x$, $\dot{x}$, $y$, and $\dot{y}$ (see Exercise 9, which should convince you that polar coordinates are the way to go).

2. You can see from eqs. (5.29) and (5.34) that the E-L equations end up taking the form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + F\frac{\partial\eta}{\partial q_i}, \tag{5.35}$$

for each coordinate, $q_i$. Here $\eta$ is the constraint equation of the form $\eta = 0$. In our hemisphere problem, we have $\eta = r - R$ in polar coordinates, and $\eta = \sqrt{x^2 + y^2} - R$ in cartesian coordinates. The E-L equations, combined with the $\eta = 0$ condition, give us exactly the number of equations ($N + 1$ of them, where $N$ is the number of coordinates) needed to determine all of the $N + 1$ unknowns (the $\ddot{q}_i$ and $F$), in terms of the $q_i$ and $\dot{q}_i$.

3. When trying to determine the forces of constraint, you can simply start with eqs. (5.35), without bothering to write down $V(\eta)$. But you must be careful to make sure that $\eta$ does indeed represent the distance the particle is from where it should be. In polar coordinates, if someone gives you the constraint condition as $7(r-R)=0$, and if you use the left-hand side of this as the $\eta$ in eq. (5.35), then you will get the wrong constraint force, off by a factor of 7. Likewise, in cartesian coordinates, writing the constraint as $y-\sqrt{R^2-x^2}=0$ would give you the wrong force.

   The best way to avoid this problem is, of course, to pick one of your variables as the distance the particle is from where it should be (up to an additive constant, as in the case of the $r$ in eq. (5.28)). ♣

## 5.4 Change of coordinates

When $L$ is written in terms of cartesian coordinates $x$,$y$,$z$, we showed in Section 5.1 that the Euler-Lagrange equations are equivalent to Newton's $\mathbf{F} = m\mathbf{a}$ equations; see eq. (5.7). But what about the case where we use polar, spherical, or some other coordinates? The equivalence of the E-L equations and $\mathbf{F} = m\mathbf{a}$ is not so obvious. As far as trusting the E-L equations for such coordinates goes, you can achieve peace-of-mind in two ways. You can accept the principle of stationary action as something so beautiful and profound that it simply has to work for any choice of coordinates. Or, you can take the more mundane road and show through a change of coordinates that if the E-L equations hold for one set of coordinates,[9] then they also hold for any other coordinates (of a certain form, described below). In this section, we will demonstrate the validity of the E-L equations through the explicit change of coordinates.[10]

Consider the set of coordinates,

$$x_i : \quad (x_1, x_2, \dots, x_N). \tag{5.36}$$

For example, $x_1$,$x_2$,$x_3$ could be the cartesian $x$,$y$,$z$ coordinates of one particle, and $x_4$,$x_5$,$x_6$ could be the $r$,$\theta$,$\phi$ polar coordinates of a second particle, and so on. Assume that the E-L equations hold for these variables, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \qquad (1 \le i \le N). \tag{5.37}$$

We know that there is at least one set of variables for which this is true, namely the cartesian coordinates. Consider a new set of variables which are functions of the $x_i$ and $t$,

$$q_i = q_i(x_1, x_2, \dots, x_N; t). \tag{5.38}$$

We will restrict ourselves to the case where the $q_i$ do not depend on the $\dot{x}_i$. (This is quite reasonable. If the coordinates depended on the velocities, then we wouldn't be able to label points in space with definite coordinates. We'd have to worry about

[9] We know that they *do* hold for cartesian coordinates, because we showed in this case that the E-L equations are equivalent to $F = ma$, and we are assuming $F = ma$ to be true.

[10] This calculation is straightforward but a bit messy, so you may want to skip this section and just settle for the "beautiful and profound" reasoning.

how the particles were behaving when they were at the points. These would be strange coordinates indeed.) Note that we can, in theory, invert eqs. (5.38) and express the $x_i$ as functions of the $q_i$ and $t$,

$$x_i = x_i(q_1, q_2, \ldots, q_N; t). \tag{5.39}$$

**Claim 5.2** *If eq. (5.37) is true for the $x_i$ coordinates, and if the $x_i$ and $q_i$ are related by eqs. (5.39), then eq. (5.37) is also true for the $q_i$ coordinates. That is,*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}\,, \qquad (1 \le m \le N). \tag{5.40}$$

**Proof:** We have

$$\frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_m}\,. \tag{5.41}$$

(Note that if the $x_i$ depended on the $\dot{q}_i$, then we would have the additional term, $\sum(\partial L/\partial x_i)(\partial x_i/\partial \dot{q}_m)$, but we have excluded such dependence.) Let's rewrite the $\partial \dot{x}_i/\partial \dot{q}_m$ term. From eq. (5.39), we have

$$\dot{x}_i = \sum_{m=1}^{N} \frac{\partial x_i}{\partial q_m}\dot{q}_m + \frac{\partial x_i}{\partial t}\,. \tag{5.42}$$

Therefore,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m}\,. \tag{5.43}$$

Substituting this into eq. (5.41) and taking the time derivative of both sides gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right). \tag{5.44}$$

In the second term here, it is legal to switch the order of the total derivative, $d/dt$, and the partial derivative, $\partial/\partial q_m$.

Remark: In case you have your doubts, let's prove that this switching is legal.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right) &= \sum_{k=1}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_m}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial q_m}\right) \\ &= \frac{\partial}{\partial q_m}\left(\sum_{k=1}^{N} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right) \\ &= \frac{\partial \dot{x}_i}{\partial q_m}\,, \end{aligned} \tag{5.45}$$

as was to be shown. ♣

In the first term on the right-hand side of eq. (5.44), we can use the given information in eq. (5.37) and rewrite the $(d/dt)(\partial L/\partial \dot{x}_i)$ term. Eq. (5.44) then becomes

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) &= \sum_{i=1}^{N} \frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_m} \\ &= \frac{\partial L}{\partial q_m}, \end{aligned} \tag{5.46}$$

as we wanted to show. ■

We have therefore demonstrated that if the Euler-Lagrange equations are true for one set of coordinates, $x_i$ (and they *are* true for cartesian coordinates), then they are also true for any other set of coordinates, $q_i$, satisfying eq. (5.38). For those of you who look at the principle of stationary action with distrust (thinking that it might be a coordinate-dependent statement), this proof should put you at ease. The Euler-Lagrange equations are valid in any coordinates.

Note that the above proof did not in any way use the precise form of the Lagrangian. If $L$ were equal to $T+V$, or $7T+\pi V^2/T$, or any other arbitrary function, our result would still be true: If eqs. (5.37) are true for one set of coordinates, then they are also true for any other coordinates $q_i$ satisfying eqs. (5.38). The point is that the only $L$ for which the hypothesis is true at all (that is, for which eq. (5.37) holds) is $L \equiv T - V$ (or any constant multiple of this).

REMARK: On one hand, it is quite amazing how little we assumed in proving the above claim. *Any* new coordinates of the very general form (5.38) satisfy the E-L equations, as long as the original coordinates do. If the E-L equations had, say, a factor of 5 on the right-hand side of eq. (5.37), then they would *not* hold in arbitrary coordinates. To see this, just follow the proof through with the factor of 5.

On the other hand, the claim is quite believable, if you make an analogy with a function instead of a functional. Consider the function $f(z) = z^2$. This has a minimum at $z = 0$, consistent with the fact that $df/dz = 0$ at $z = 0$. But let's now write $f$ in terms of the variable $y$ defined by, say, $z = y^4$. Then $f(y) = y^8$, and $f$ has a minimum at $y = 0$, consistent with the fact that $df/dy$ equals zero at $y = 0$. So $f' = 0$ holds in both coordinates at the corresponding points $y = z = 0$. This is the (simplified) analog of the E-L equations holding in both coordinates. In both cases, the derivative equation describes where the stationary value occurs.

This change-of-variables result may be stated in a more geometrical (and friendly) way. If you plot a function and then stretch the horizontal axis in an arbitrary manner (which is what happens when you change coordinates), then a stationary value (that is, one where the slope is zero) will still be a stationary value after the stretching. A picture is worth a dozen equations, it appears.

As an example of an equation that does *not* hold for all coordinates, consider the preceding example, but with $f' = 1$ instead of $f' = 0$. In terms of $z$, $f' = 1$ when $z = 1/2$. And in terms of $y$, $f' = 1$ when $y = (1/8)^{1/7}$. But the points $z = 1/2$ and $y = (1/8)^{1/7}$ are not the same point. In other words, $f' = 1$ is not a coordinate-independent statement. Most equations, of course, are coordinate dependent. The special thing about $f' = 0$ is that a stationary point is a stationary point no matter how you look at it.[11] ♣

[11] There is, however, one exception. A stationary point in one coordinate system might be located at a kink in another coordinate system, so that $f'$ is not defined there. For example, if we had

## 5.5 Conservation Laws

### 5.5.1 Cyclic coordinates

Consider the case where the Lagrangian does not depend on a certain coordinate $q_k$. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0. \tag{5.47}$$

Therefore

$$\frac{\partial L}{\partial \dot{q}_k} = C, \tag{5.48}$$

where $C$ is a constant, independent of time. In this case, we say that $q_k$ is a *cyclic* coordinate, and that $\partial L/\partial \dot{q}_k$ is a *conserved* quantity (meaning that it doesn't change with time).

If cartesian coordinates are used, then $\partial L/\partial \dot{x}_k$ is simply the momentum, $m\dot{x}_k$, because $\dot{x}_k$ appears in only the $m\dot{x}_k^2/2$ term (we exclude cases where $V$ depends on $\dot{x}_k$). We therefore call $\partial L/\partial \dot{q}_k$ the *generalized momentum* corresponding to the coordinate $q_k$. And in cases where $\partial L/\partial \dot{q}_k$ does not change with time, we call it a *conserved momentum*. Note that a generalized momentum need not have the units of linear momentum, as the angular-momentum examples below show.

---

**Example 1: Linear momentum**

Consider a ball thrown through the air. In the full three dimensions, the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz. \tag{5.49}$$

There is no $x$ or $y$ dependence here, so both $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial \dot{y} = m\dot{y}$ are constant, as we well know. The fancy way of saying this is that conservation of $p_x \equiv m\dot{x}$ arises from spatial translation invariance in the $x$-direction. The fact that the Lagrangian doesn't depend on $x$ means that it doesn't matter if you throw the ball in one spot, or in another spot a mile down the road. The setup is independent of the $x$ value. This independence leads to conservation of $p_x$.

**Example 2: Angular and linear momentum in cylindrical coordinates**

Consider a potential that depends only on the distance to the $z$-axis. In cylindrical coordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - V(r). \tag{5.50}$$

There is no $z$ dependence here, so $\partial L/\partial \dot{z} = m\dot{z}$ is constant. Also, there is no $\theta$ dependence, so $\partial L/\partial \dot{\theta} = mr^2\dot{\theta}$ is constant. Since $r\dot{\theta}$ is the speed in the tangential direction around the $z$-axis, we see that our conserved quantity, $mr(r\dot{\theta})$, is the angular momentum around the $z$-axis. (We actually haven't defined angular momentum yet; we'll talk about it at great length in Chapters 6-8.) In the same manner as in the

instead defined $y$ by $z = y^{1/4}$, then $f(y) = y^{1/2}$, which has an undefined slope at $y = 0$. Basically, we've stretched (or shrunk) the horizontal axis by a factor of infinity at the origin, and this is a process that can indeed change a zero slope into an undefined one. But let's not worry about this.

preceding example, conservation of angular momentum around the $z$-axis arises from rotation invariance around the $z$-axis.

**Example 3: Angular momentum in spherical coordinates**

In spherical coordinates, consider a potential that depends only on $r$ and $\theta$. (Our convention for spherical coordinates will be that $\theta$ is the angle down from the north pole, and $\phi$ is the angle around the equator.) The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r,\theta). \tag{5.51}$$

There is no $\phi$ dependence here, so $\partial L/\partial\dot{\phi} = mr^2\sin^2\theta\dot{\phi}$ is constant. Since $r\sin\theta$ is the distance from the $z$-axis, and since $r\sin\theta\dot{\phi}$ is the speed in the tangential direction around the $z$-axis, we see that our conserved quantity, $m(r\sin\theta)(r\sin\theta\dot{\phi})$, is the angular momentum around the $z$-axis.

### 5.5.2 Energy conservation

We will now derive another conservation law, namely conservation of energy. The conservation of momentum or angular momentum above arose when the Lagrangian was independent of $x$, $y$, $z$, $\theta$, or $\phi$. Conservation of energy arises when the Lagrangian is independent of time. This conservation law is different from those in the above momenta examples, because $t$ is not a coordinate which the stationary-action principle can be applied to. You can imagine varying the coordinates $x$, $\theta$, etc., which are functions of $t$. But it makes no sense to vary $t$. Therefore, we're going to have to prove this conservation law in a different way.

Consider the quantity

$$E = \left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - L. \tag{5.52}$$

$E$ will (usually) turn out to be the energy. We'll show this below. The motivation for this expression for $E$ comes from the theory of Legendre transforms, but we won't get into that here. Let's just accept the definition in eq. (5.52), and now we'll prove a nice little theorem about it.

**Claim 5.3** *If $L$ has no explicit time dependence (that is, if $\partial L/\partial t = 0$), then $E$ is conserved (that is, $dE/dt = 0$), assuming the motion obeys the E-L equations (which it does).*

Note that there is one partial derivative and one total derivative in this statement.

**Proof:** $L$ is a function of the $q_i$, the $\dot{q}_i$, and possibly $t$. Making copious use of the chain rule, we have

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - \frac{dL}{dt}\\ &= \sum_{i=1}^{N}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right) - \left(\sum_{i=1}^{N}\left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right) + \frac{\partial L}{\partial t}\right).\end{aligned} \tag{5.53}$$

There are five terms here. The second cancels with the fourth. And the first (after using the E-L equation, eq. (5.3), to rewrite it) cancels with the third. We therefore arrive at the simple result,

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,. \tag{5.54}$$

In the event that $\partial L/\partial t = 0$ (that is, there are no $t$'s sitting on the paper when you write down $L$), which is invariably the case in the situations we consider (because we won't consider potentials that depend on time), we have $dE/dt = 0$. ■

Not too many things are constant with respect to time, and the quantity $E$ has units of energy, so it's a good bet that it is the energy. Let's show this in cartesian coordinates (however, see the remark below). The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z), \tag{5.55}$$

so eq. (5.52) gives

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z), \tag{5.56}$$

which is, of course, the total energy. The effect of the operations in eq. (5.52) in most cases is to simply switch the sign in front of the potential.

Of course, taking the kinetic energy $T$ and subtracting the potential energy $V$ to obtain $L$, and then using eq. (5.52) to produce $E = T + V$, seems like a rather convoluted way of arriving at $T + V$. But the point of all this is that we used the E-L equations to *prove* that $E$ is conserved. Although we know very well from the $F = ma$ methods in Chapter 4 that the sum $T + V$ is conserved, it's not fair to assume that it is conserved in our new Lagrangian formalism. We have to show that this *follows* from the E-L equations.

As with the translation and rotation invariance we observed in the examples in Section 5.5.1, we now see that energy conservation arises from time translation invariance. If the Lagrangian has no explicit $t$ dependence, then the setup looks the same today as it did yesterday. This fact leads to conservation of energy.

Remark: The quantity $E$ in eq. (5.52) gives the energy of the system only if the entire system is represented by the Lagrangian. That is, the Lagrangian must represent a closed system with no external forces. If the system is not closed, then Claim 5.3 (or more generally, eq. (5.54)) is still perfectly valid for the $E$ defined in eq. (5.52), but this $E$ may simply not be the energy of the system. Problem 8 is a good example of such a situation.

Another example is projectile motion in the $x$-$y$ plane. The normal thing to do is to say that the particle moves under the influence of the potential $V(y) = mgy$. The Lagrangian for this closed system is $L = m(\dot{x}^2 + \dot{y})^2/2 - mgy$, and so eq. (5.52) gives $E = m(\dot{x}^2 + \dot{y})^2/2 + mgy$, which is indeed the energy of the particle. However, another way to do this problem is to consider the particle to be subject to an *external* gravitational force, which gives an acceleration of $-g$ in the $y$ direction. If we assume that the mass starts at rest, then $\dot{y} = -gt$. The Lagrangian is therefore $L = m\dot{x}^2/2 + m(gt)^2/2$, and so eq. (5.52) gives $E = m\dot{x}^2/2 - m(gt)^2/2$. This is not the energy.

At any rate, most of the systems we will deal with are closed, so you can usually ignore this remark and assume that the $E$ in eq. (5.52) gives the energy. ♣

## 5.6 Noether's Theorem

We now present one of the most beautiful and useful theorems in physics. It deals with two fundamental concepts, namely *symmetry* and *conserved quantities*. The theorem (due to Emmy Noether) may be stated as follows.

**Theorem 5.4** (**Noether's Theorem**) *For each symmetry of the Lagrangian, there is a conserved quantity.*

By "symmetry", we mean that if the coordinates are changed by some small quantities, then the Lagrangian has no first-order change in these quantities. By "conserved quantity", we mean a quantity that does not change with time. The result in Section 5.5.1 for cyclic coordinates is a special case of this theorem.

**Proof:** Let the Lagrangian be invariant (to first order in the small number $\epsilon$) under the change of coordinates,

$$q_i \longrightarrow q_i + \epsilon K_i(q). \tag{5.57}$$

Each $K_i(q)$ may be a function of all the $q_i$, which we collectively denote by the shorthand, $q$.

REMARK: As an example of what these $K_i$'s might look like, consider the Lagrangian (which we just pulled out of a hat), $L = (m/2)(5\dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{y}^2) + C(2x - y)$. This is invariant under the transformation $x \to x + \epsilon$ and $y \to y + 2\epsilon$, as you can easily check. (It is actually invariant to all orders in $\epsilon$, and not just first order. But this isn't necessary for the theorem to hold.) Therefore, $K_x = 1$ and $K_y = 2$. In the problems we'll be doing, the $K_i$'s can generally be determined by simply looking at the potential term.

Of course, someone else might come along with $K_x = 3$ and $K_y = 6$, which is also a symmetry. And indeed, any factor can be taken out of $\epsilon$ and put into the $K_i$'s without changing the quantity $\epsilon K_i(q)$ in eq. (5.57). Any such modification will simply bring an overall constant factor (and hence not change the property of being conserved) into the conserved quantity in eq. (5.60) below. It is therefore irrelevant. ♣

The fact that the Lagrangian does not change at first order in $\epsilon$ means that

$$\begin{aligned} 0 = \frac{dL}{d\epsilon} &= \sum_i \left( \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon} \right) \\ &= \sum_i \left( \frac{\partial L}{\partial q_i} K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i \right). \end{aligned} \tag{5.58}$$

Using the E-L equation, eq. (5.3), we may rewrite this as

$$\begin{aligned} 0 &= \sum_i \left( \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i \right) \\ &= \frac{d}{dt}\left( \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i \right). \end{aligned} \tag{5.59}$$

Therefore, the quantity

$$P(q, \dot{q}) \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i(q) \tag{5.60}$$

does not change with time. It is given the generic name of *conserved momentum*. But it need not have the units of linear momentum. ■

As Noether most keenly observed
(And for which much acclaim is deserved),
We can easily see,
That for each symmetry,
A quantity must be conserved.

---

**Example 1:** Consider the Lagrangian in the above remark, $L = (m/2)(5\dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{y}^2) + C(2x - y)$. We saw that $K_x = 1$ and $K_y = 2$. The conserved momentum is therefore

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(5\dot{x} - \dot{y})(1) + m(-\dot{x} + 2\dot{y})(2) = m(3\dot{x} + 3\dot{y}). \tag{5.61}$$

The overall factor of $3m$ doesn't matter, of course.

**Example 2:** Consider a thrown ball. We have $L = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$. This is invariant under translations in $x$, that is, $x \to x + \epsilon$; and also under translations in $y$, that is, $y \to y + \epsilon$. (Both $x$ and $y$ are cyclic coordinates.) Note that we only need invariance to first order in $\epsilon$ for Noether's theorem to hold, but this $L$ is invariant to all orders.

We therefore have two symmetries in our Lagrangian. The first has $K_x = 1$, $K_y = 0$, and $K_z = 0$. The second has $K_x = 0$, $K_y = 1$, and $K_z = 0$. Of course, the nonzero $K_i$'s here may be chosen to be any constant, but we may as well pick them to be 1. The two conserved momenta are

$$\begin{aligned} P_1(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y + \frac{\partial L}{\partial \dot{z}} K_z = m\dot{x}, \\ P_2(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y + \frac{\partial L}{\partial \dot{z}} K_z = m\dot{y}. \end{aligned} \tag{5.62}$$

These are simply the $x$- and $y$-components of the linear momentum, as we saw in Example 1 in Section 5.5.1.

Note that any combination of these momenta, say $3P_1 + 8P_2$, is also conserved. (In other words, $x \to x + 3\epsilon$, $y \to y + 8\epsilon$, $z \to z$ is a symmetry of the Lagrangian.) But the above $P_1$ and $P_2$ are the simplest conserved momenta to choose as a "basis" for the infinite number of conserved momenta (which is how many you have, if there are two or more independent symmetries).

**Example 3:** Consider a mass on a spring, with zero equilibrium length, in the $x$-$y$ plane. The Lagrangian, $L = (m/2)(\dot{x}^2 + \dot{y}^2) - (k/2)(x^2 + y^2)$, is invariant under the change of coordinates, $x \to x + \epsilon y$, $y \to y - \epsilon x$, to first order in $\epsilon$ (as you can check). So we have $K_x = y$ and $K_y = -x$. The conserved momentum is therefore

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(\dot{x}y - \dot{y}x). \tag{5.63}$$

This is simply the (negative of the) $z$-component of the angular momentum. The angular momentum is conserved here because the potential, $V(x, y) = x^2 + y^2 = r^2$, depends only on the distance from the origin; we'll discuss such potentials in Chapter 6.

In contrast with the first two examples above, the $x \to x + \epsilon y$, $y \to y - \epsilon x$ transformation isn't so obvious here. How did we get this? Well, unfortunately there doesn't seem to be any fail-proof method of determining the $K_i$'s in general, so sometimes you just have to guess around, as was the case here. But in many problems, the $K_i$'s are simple constants which are easy to see.

---

REMARKS:

1. As we saw above, in some cases the $K_i$'s are functions of the coordinates, and in some cases they are not.

2. The cyclic-coordinate result in eq. (5.48) is a special case of Noether's theorem, for the following reason. If $L$ doesn't depend on a certain coordinate $q_k$, then $q_k \longrightarrow q_k + \epsilon$ is certainly a symmetry. Hence $K_k = 1$ (with all the other $K_i$'s equal to zero), and eq. (5.60) reduces to eq. (5.48).

3. We use the word "symmetry" to describe the situation where the transformation in eq. (5.57) produces no first-order change in the Lagrangian. This is an appropriate choice of word, because the Lagrangian describes the system, and if the system essentially doesn't change when the coordinates are changed, then we say that the system is symmetric. For example, if we have a setup that doesn't depend on $\theta$, then we say that the setup is symmetric under rotations. Rotate the system however you want, and it looks the same. The two most common applications of Noether's theorem are the conservation of angular momentum, which arises from symmetry under rotations; and conservation of linear momentum, which arises from symmetry under translations.

4. In simple systems, as in Example 2 above, it is clear why the resulting $P$ is conserved. But in more complicated systems, as in Example 1 above (which has an $L$ of the type that arises in Atwood's machine problems; see Exercise 11 and Problem 9), the resulting $P$ might not have an obvious interpretation. But at least you know that it is conserved, and this will invariably help in solving a problem.

5. Although conserved quantities are extremely useful in studying a physical situation, it should be stressed that there is no more information contained in the them than there is in the E-L equations. Conserved quantities are simply the result of integrating the E-L equations. For example, if you write down the E-L equations for Example 1 above, and then add the "$x$" equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) to twice the "$y$" equation (which is $-m\ddot{x} + 2m\ddot{y} = -C$), then you find $3m(\ddot{x} + \ddot{y}) = 0$. In other words, $3m(\dot{x} + \dot{y})$ is constant, as we found from Noether's theorem.

   Of course, you might have to do some guesswork to find the proper combination of the E-L equations that gives a zero on the right-hand side. But you'd have to do some guesswork anyway, to find the symmetry for Noether's theorem. At any rate, a conserved quantity is useful because it is an integrated form of the E-L equations. It puts you one step closer to solving the problem, compared to where you would be if you started with the second-order E-L equations.

6. Does every system have a conserved momentum? Certainly not. The one-dimensional problem of a falling ball ($m\ddot{z} = -mg$) doesn't have one. And if you write down an arbitrary potential in 3-D, odds are that there won't be one. In a sense, things have to contrive nicely for there to be a conserved momentum. In some problems, you can just look at the physical system and see what the symmetry is, but in others (for example, in the Atwood's-machine problems for this chapter), the symmetry is not at all obvious.

7. By "conserved quantity", we mean a quantity that depends on (at most) the coordinates and their first derivatives (that is, not on their second derivatives). If we do not make this restriction, then it is trivial to construct quantities that do not vary with time. For example, in Example 1 above, the "$x$" E-L equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) tells us that $5m\ddot{x} - m\ddot{y}$ has its time derivative equal to zero. Note that an equivalent way of excluding these trivial cases is to say that the value of a conserved quantity depends on initial conditions (that is, velocities and positions). The quantity $5m\ddot{x} - m\ddot{y}$ does not satisfy this criterion, because its value is always constrained to be $2C$. ♣

## 5.7 Small oscillations

In many physical systems, a particle undergoes small oscillations around an equilibrium point. In Section 4.2, we showed that the frequency of these small oscillations is

$$\omega = \sqrt{\frac{V''(x_0)}{m}}\,, \tag{5.64}$$

where $V(x)$ is the potential energy, and $x_0$ is the equilibrium point.

However, this result holds only for *one-dimensional* motion (we will see below why this is true). In more complicated systems, such as the one described below, it is necessary to use another procedure to obtain the frequency $\omega$. This procedure is a fail-proof one, applicable in all situations. It is, however, a bit more involved than simply writing down eq. (5.64). So in one-dimensional problems, eq. (5.64) is still what you want to use.

We'll demonstrate our fail-proof method through the following problem.

---

**Problem:** A mass $m$ is free to move on a frictionless table and is connected by a string, which passes through a hole in the table, to a mass $M$ which hangs below (see Fig. 5.5). Assume that $M$ moves in a vertical line only, and assume that the string always remains taut.

(a) Find the equations of motion for the variables $r$ and $\theta$ shown in the figure.

(b) Under what condition does $m$ undergo circular motion?

(c) What is the frequency of small oscillations (in the variable $r$) about this circular motion?

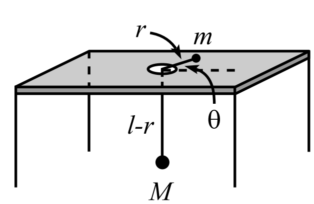


**Figure 5.5**

**Solution:**

(a) Let the string have length $\ell$ (this length won't matter). Then the Lagrangian (we'll call it "$\mathcal{L}$" here, to save "$L$" for the angular momentum, which arises below) is

$$\mathcal{L} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mg(\ell - r). \tag{5.65}$$

For the purposes of the potential energy, we've taken the table to be at height zero, but any other value could be chosen, of course. The equations of motion obtained from varying $\theta$ and $r$ are

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0, \\ (M+m)\ddot{r} &= mr\dot{\theta}^2 - Mg. \end{aligned} \tag{5.66}$$

The first equation says that angular momentum is conserved (much more about this in Chapter 6). The second equation says that the $Mg$ gravitational force accounts for the acceleration of the two masses along the direction of the string, plus the centripetal acceleration of $m$.

(b) The first of eqs. (5.66) says that $mr^2\dot{\theta} = L$, where $L$ is some constant (the angular momentum) which depends on the initial conditions. Plugging $\dot{\theta} = L/mr^2$ into the second of eqs. (5.66) gives

$$(M+m)\ddot{r} = \frac{L^2}{mr^3} - Mg. \tag{5.67}$$

Circular motion occurs when $\dot{r} = \ddot{r} = 0$. Therefore, the radius of the circular orbit is given by

$$r_0^3 = \frac{L^2}{Mmg}\,. \tag{5.68}$$

REMARK: Note that since $L = mr^2\dot{\theta}$, eq. (5.68) is equivalent to

$$mr_0\dot{\theta}^2 = Mg, \tag{5.69}$$

which can be obtained by simply letting $\ddot{r} = 0$ in the second of eqs. (5.66). In other words, the gravitational force on $M$ exactly accounts for the centripetal acceleration of $m$ if the motion is circular. Given $r_0$, eq. (5.69) determines what $\dot{\theta}$ must be (in order to have circular motion), and vice versa. ♣

(c) To find the frequency of small oscillations about the circular motion, we need to look at what happens to $r$ if we perturb it slightly from its equilibrium value, $r_0$. Our fail-proof procedure is the following.

Let $r(t) \equiv r_0 + \delta(t)$, where $\delta(t)$ is very small (more precisely, $\delta(t) \ll r_0$), and expand eq. (5.67) to first order in $\delta(t)$. Using

$$\frac{1}{r^3} \equiv \frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.70}$$

we obtain

$$(M+m)\ddot{\delta} \approx \frac{L^2}{mr_0^3}\left(1 - \frac{3\delta}{r_0}\right) - Mg. \tag{5.71}$$

The terms not involving $\delta$ on the right-hand side cancel, by the definition of $r_0$ given in eq. (5.68). This cancellation will always occur in such a problem at this stage, due to the definition of the equilibrium point. We are therefore left with

$$\ddot{\delta} + \left(\frac{3L^2}{(M+m)mr_0^4}\right)\delta \approx 0. \tag{5.72}$$

This is a good old simple-harmonic-oscillator equation in the variable $\delta$. Therefore, the frequency of small oscillations about a circle of radius $r_0$ is

$$\omega \approx \sqrt{\frac{3L^2}{(M+m)mr_0^4}} = \sqrt{\frac{3M}{M+m}}\sqrt{\frac{g}{r_0}}\,, \tag{5.73}$$

where we have used eq. (5.68) to eliminate $L$ in the second expression.

To sum up, the above frequency is the frequency of small oscillations in the variable $r$. In other words, if you plot $r$ as a function of time (and ignore what $\theta$ is doing), then you will get a nice sinusoidal graph whose frequency is given by eq. (5.73), provided that that amplitude is small. Note that this frequency need not have anything to do with the other relevant frequency in this problem, namely the frequency of the circular motion, which is $\sqrt{Mg/mr_0}$, from eq. (5.69).

Remarks: Let's look at some limits. For a given $r_0$, if $m \gg M$, then $\omega \approx \sqrt{3Mg/mr_0} \approx 0$. This makes sense, because everything will be moving very slowly. Note that this frequency is equal to $\sqrt{3}$ times the frequency of circular motion, $\sqrt{Mg/mr_0}$, which isn't at all obvious.

For a given $r_0$, if $m \ll M$, then $\omega \approx \sqrt{3g/r_0}$, which isn't so obvious, either.

Note that the frequency of small oscillations is equal to the frequency of circular motion if $M = 2m$ (once again, not obvious). This condition is independent of $r_0$. ♣

---

The above procedure for finding the frequency of small oscillations may be summed up in three steps: (1) Find the equations of motion, (2) Find the equilibrium point, and (3) Let $x(t) \equiv x_0 + \delta(t)$, where $x_0$ is the equilibrium point of the relevant variable, and expand one of the equations of motion (or a combination of them) to first order in $\delta$, to obtain a simple-harmonic-oscillator equation for $\delta$.

Remark: Note that if you simply used the potential energy in the above problem (which is $Mgr$, up to a constant) in eq. (5.64), then you would obtain a frequency of zero, which is incorrect. You *can* use eq. (5.64) to find the frequency, if you instead use the "effective potential" for this problem, namely $L^2/(2mr^2) + Mgr$, and if you use the total mass, $M + m$, as the mass in eq. (5.64), as you can check. The reason why this works will become clear in Chapter 6 when we introduce the effective potential.

In many problems, however, it is not obvious what "modified potential" should be used, or what mass should be used in eq. (5.64), so it is generally much safer to take a deep breath and go through an expansion similar to the one in part (c) above. ♣

The one-dimensional result in eq. (5.64) is, of course, simply a special case of our above expansion procedure. We can repeat the derivation of Section 4.2 in the present language. In one dimension, we have $m\ddot{x} = -V'(x)$. Let $x_0$ be the equilibrium point (so that $V'(x_0) = 0$), and let $x(t) \equiv x_0 + \delta(t)$. Expanding $m\ddot{x} = -V'(x)$ to first order in $\delta$, we have $m\ddot{\delta} = -V'(x_0) - V''(x_0)\delta - \cdots$. Hence, $m\ddot{\delta} \approx -V''(x_0)\delta$, as desired.

## 5.8 Other applications

The formalism developed in Section 5.2 works for *any* function $L(x, \dot{x}, t)$. If our goal is to find the stationary points of $S \equiv \int L$, then eq. (5.14) holds, no matter what $L$ is. There is no need for $L$ to be equal to $T - V$, or indeed, to have anything to do with physics. And $t$ need not have anything to do with time. All that is required is that the quantity $x$ depend on the parameter $t$, and that $L$ depend on only $x$, $\dot{x}$, and $t$ (and not, for example, on $\ddot{x}$; see Exercise 6). The formalism is very general and powerful, as the following example demonstrates.

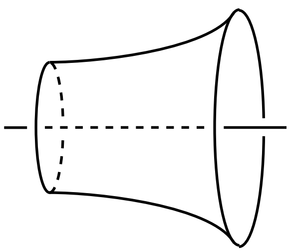

**Figure 5.6**

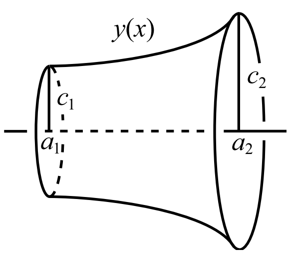


**Figure 5.7**

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**Example (Minimal surface of revolution):** A surface of revolution has two given rings as its boundary; see Fig. 5.6. What should the shape of the surface be so that it has the minimum possible area? We'll present two solutions. A third is left for Problem 23.

**First solution:** Let the surface be generated by rotating the curve $y = y(x)$ around the $x$-axis. The boundary conditions are $y(a_1) = c_1$ and $y(a_2) = c_2$; see Fig. 5.7. Slicing the surface up into vertical rings, we see that the area is given by

$$A = \int_{a_1}^{a_2} 2\pi y \sqrt{1 + y'^2}\, dx. \tag{5.74}$$

The goal is to find the function $y(x)$ that minimizes this integral. We therefore have exactly the same situation as in Section 5.2, except that $x$ is now the parameter (instead of $t$), and $y$ in now the function (instead of $x$). Our "Lagrangian" is thus $L \propto y\sqrt{1 + y'^2}$. To minimize the integral $A$, we "simply" have to apply the E-L equation to this Lagrangian. This calculation, however, gets a bit tedious, so we've relegated it to Lemma 5.5 at the end of this section. For now we'll just use the result in eq. (5.83) which gives (with $f(y) = y$ here)

$$1 + y'^2 = By^2. \tag{5.75}$$

At this point we can cleverly guess (motivated by the fact that $1 + \sinh^2 z = \cosh^2 z$) that the solution is

$$y(x) = \frac{1}{b}\cosh b(x + d), \tag{5.76}$$

where $b = \sqrt{B}$, and $d$ is a constant of integration. Or, we can separate variables to obtain (again with $b = \sqrt{B}$)

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}\,, \tag{5.77}$$

and then use the fact that the integral of $1/\sqrt{z^2 - 1}$ is $\cosh^{-1} z$, to obtain the same result.

The answer to our problem, therefore, is that $y(x)$ takes the form of eq. (5.76), with $b$ and $d$ determined by the boundary conditions,

$$c_1 = \frac{1}{b}\cosh b(a_1 + d), \qquad \text{and} \qquad c_2 = \frac{1}{b}\cosh b(a_2 + d). \tag{5.78}$$

In the symmetrical case where $c_1 = c_2$, we know that the minimum occurs in the middle, so we may choose $d = 0$ and $a_1 = -a_2$.

REMARK: Solutions for $b$ and $d$ exist only for certain ranges of the $a$'s and $c$'s. Basically, if $a_2 - a_1$ is too large, then there is no solution. In this case, the minimal "surface" turns out to be the two given circles, attached by a line (which isn't a nice two-dimensional surface). If you perform an experiment with soap bubbles (which want to minimize their area), and if you pull the rings too far apart, then the surface will break and disappear as it tries to form the two circles. Problem 24 deals with this issue. ♣

**Second solution:** Consider the curve that we rotate around the $x$-axis to be described now by the function $x(y)$. That is, let $x$ be a function of $y$. The area is then given by

$$A = \int_{a_1}^{a_2} 2\pi y\sqrt{1+x'^2}\,dy, \tag{5.79}$$

where $x' \equiv dx/dy$. Note that the function $x(y)$ may be double-valued, so it may not really be a function. But it looks like a function locally, and all of our formalism deals with local variations.

Our "Lagrangian" is now $L \propto y\sqrt{1+x'^2}$, and the E-L equation is

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) = 0. \tag{5.80}$$

The nice thing about this solution is the "0" on the right-hand side, which arises from the fact that $L$ does not depend on $x$ (that is, $x$ is a cyclic coordinate). Therefore, $yx'/\sqrt{1+x'^2}$ is constant. If we define this constant to be $1/b$, then we may solve for $x'$ and then separate variables to obtain

$$dx = \frac{dy}{\sqrt{(by)^2-1}}\,, \tag{5.81}$$

in agreement with eq. (5.77). The solution proceeds as above.

---

Numerous other "extremum" problems are solvable with these general techniques. A few are presented in the problems for this chapter.

Let us now prove the following lemma, which we invoked in the first solution above. This lemma is very useful, because it is common to encounter problems where the quantity to be extremized depends on the arclength, $\sqrt{1+y'^2}$, and takes the form $\int f(y)\sqrt{1+y'^2}\,dx$.

We will give two proofs. The first proof uses the Euler-Lagrange equation. The calculation gets a bit messy, so it's a good idea to work through it once and for all, and then just invoke the result whenever needed. The derivation isn't something you'd want to repeat too often. The second proof makes use of a conserved quantity. And in contrast with the first proof, this method is exceedingly clean and simple. It actually *is* something you'd want to repeat quite often. But we'll still do it once and for all.

**Lemma 5.5** *Let $f(y)$ be a given function of $y$. Then the function $y(x)$ that extremizes the integral,*

$$\int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\,dx, \tag{5.82}$$

*satisfies the differential equation,*

$$1+y'^2 = Bf(y)^2, \tag{5.83}$$

*where $B$ is a constant of integration.*[12]

**First Proof:** Our goal is to find the function $y(x)$ that extremizes the integral in eq. (5.82). We therefore have exactly the same situation as in Section 5.2, except with $x$ in place of $t$, and $y$ in place of $x$. Our "Lagrangian" is thus $L = f(y)\sqrt{1+y'^2}$, and the Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(f\cdot y'\cdot\frac{1}{\sqrt{1+y'^2}}\right) = f'\sqrt{1+y'^2}, \tag{5.84}$$

where $f' \equiv df/dy$. We must now perform some straightforward (albeit tedious) differentiations. Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, we obtain

$$\frac{f'y'^2}{\sqrt{1+y'^2}} + \frac{fy''}{\sqrt{1+y'^2}} - \frac{fy'^2y''}{(1+y'^2)^{3/2}} = f'\sqrt{1+y'^2}. \tag{5.85}$$

Multiplying through by $(1+y'^2)^{3/2}$ and simplifying gives

$$fy'' = f'(1+y'^2). \tag{5.86}$$

We have completed the first step of the proof, namely producing the Euler-Lagrange differential equation. We must now integrate it. Eq. (5.86) happens to be integrable for arbitrary functions $f(y)$. If we multiply through by $y'$ and rearrange, we obtain

$$\frac{y'y''}{1+y'^2} = \frac{f'y'}{f}. \tag{5.87}$$

Taking the $dx$ integral of both sides gives $(1/2)\ln(1+y'^2) = \ln(f) + C$, where $C$ is a constant of integration. Exponentiation then gives (with $B \equiv e^{2C}$)

$$1+y'^2 = Bf(y)^2, \tag{5.88}$$

as we wanted to show. In an actual problem, we would solve this equation for $y'$, and then separate variables and integrate. But we would need to be given a specific function $f(y)$ to be able do this.

**Second Proof:** Note that our "Lagrangian", $L = f(y)\sqrt{1+y'^2}$, is independent of $x$. Therefore, in analogy with the conserved energy given in eq. (5.52), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{-f(y)}{\sqrt{1+y'^2}} \tag{5.89}$$

is independent of $x$. Call it $1/\sqrt{B}$. Then we have easily reproduced eq. (5.88). ■

IMPORTANT REMARK: As demonstrated by the brevity of this second proof, it is highly advantageous to make use of a conserved quantity (for example, the $E$ here, which arose from independence of $x$) whenever you can. ♣

[12] The constant $B$, and also one other constant of integration (arising when eq. (5.83) is integrated to solve for $y$), is determined by the boundary conditions on $y(x)$.

## 5.9 Exercises

*Section 5.1: The Euler-Lagrange equations*

1. **Three falling sticks** ∗∗∗

   Three massless sticks of length $2r$, each with a mass $m$ fixed at its middle, are hinged at their ends, as shown in Fig. 5.8. The bottom end of the lower stick is hinged at the ground. They are held such that the lower two sticks are vertical, and the upper one is tilted at a small angle $\epsilon$ with respect to the vertical. They are then released. At this instant, what are the angular accelerations of the three sticks? Work in the approximation where $\epsilon$ is very small. (You may want to look at Problem 3 first.)

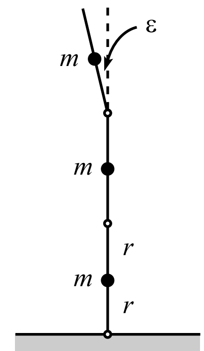



**Figure 5.8**

2. **Coffee cup and mass** ∗∗∗

   A coffee cup of mass $M$ is connected to a mass $m$ by a string. The coffee cup hangs over a pulley (of negligible size), and the mass $m$ is held horizontally, as shown in Fig. 5.9. The mass $m$ is released. Find the equations of motion for $r$ (the length of string between $m$ and the pulley) and $\theta$ (the angle that the string to $m$ makes with the horizontal). Assume that $m$ somehow doesn't run into the string holding the cup up.

   The coffee cup will of course initially fall, but it turns out that it will reach a lowest point and then rise back up. Write a program (see Appendix D) that numerically determines the ratio of the $r$ at this point to the $r$ at the start, for a given value of $m/M$. (To check your program, a value of $m/M = 1/10$ yields a ratio of about 0.208.)

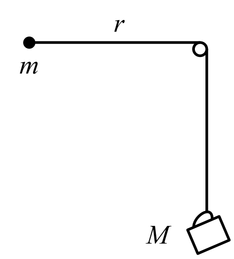



**Figure 5.9**

*Section 5.2: The principle of stationary action*

3. **Dropped ball** ∗

   Consider that action, from $t = 0$ to $t = 1$, of a ball dropped from rest. From the E-L equation (or from $F = ma$), we know that $y(t) = -gt^2/2$ yields a stationary value of the action. Show explicitly that the function $y(t) = -gt^2/2 + \epsilon t(t-1)$ yields an action that has no first-order dependence on $\epsilon$.

4. **Second-order change** ∗

   Let $x_a(t) \equiv x_0(t) + a\beta(t)$. Eq. (5.16) gives the first derivative of the action with respect to $a$. Show that the second derivative is

$$\frac{d^2}{da^2}S[x_a(t)] = \int_{t_1}^{t_2}\left(\frac{\partial^2 L}{\partial x^2}\beta^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\beta\dot{\beta} + \frac{\partial^2 L}{\partial \dot{x}^2}\dot{\beta}^2\right) dt. \tag{5.90}$$

5. **Explicit minimization** ∗

   A ball is thrown upward. Let $y(t)$ be the height as a function of time, and assume $y(0) = 0$ and $y(T) = 0$. Guess a solution for $y$ of the form $y(t) = a_0 + a_1 t + a_2 t^2$, and explicitly calculate the action between $t = 0$ and $t = T$. Show that the action is minimized when $a_2 = -g/2$. (This gets slightly messy.)

6. **$\ddot{x}$ dependence** $**$

   Let there be $\ddot{x}$ dependence (in addition to $x$,$\dot{x}$,$t$ dependence) in the Lagrangian in Theorem 5.1. There will then be the additional term $(\partial L/\partial \ddot{x}_a)\ddot{\beta}$ in eq. (5.18). It is tempting to integrate this term by parts twice, and then arrive at a modified form of eq. (5.21):

$$\frac{\partial L}{\partial x_0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}_0}\right) = 0. \tag{5.91}$$

   Is this a valid result? If not, where is the error in its derivation?

*Section 5.3: Forces of constraint*

7. **Constraint on a circle**

   A bead slides with speed $v$ around a horizontal loop of radius $R$. What force does the loop apply to the bead? (Ignore gravity.)

8. **Constraint on a curve** $**$

   Let the horizontal plane be the $x$-$y$ plane. A bead slides with speed $v$ along a curve described by the function $y = f(x)$. What force does the curve apply to the bead? (Ignore gravity.)

9. **Cartesian coordinates** $**$

   In eq. (5.34), take two derivatives of the $\sqrt{x^2+y^2} - R = 0$ equation to obtain

$$R^2(x\ddot{x} + y\ddot{y}) - (x\dot{y} - y\dot{x})^2 = 0, \tag{5.92}$$

   and then combine this with the other two equations to solve for $F$. Show that your result agrees with eq. (5.31).

*Section 5.5: Conservation Laws*

10. **Bead on stick, using $F = ma$** $*$

    After doing Problem 8, show again that the quantity $E$ is conserved, but now use $F = ma$. Do this is two ways:

    (a) Use the first of eqs. (2.52). *Hint*: multiply through by $\dot{r}$.

    (b) Use the second of eqs. (2.52) to calculate the work done on the bead.

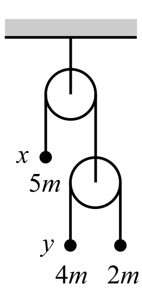


**Figure 5.10**

11. **Atwood's machine** $**$

    Consider the Atwood's machine shown in Fig. 5.10. The masses are $5m$, $4m$, and $2m$. Let $x$ and $y$ be the heights of the left two masses, relative to their initial positions. Use Noether's Theorem to find the conserved momentum. (The solution to Problem 9 gives some other methods, too.)

## 5.10 Problems

*Section 5.1: The Euler-Lagrange equations*

1. **Moving plane** $**$

   A block of mass $m$ is held motionless on a frictionless plane of mass $M$ and angle of inclination $\theta$ (see Fig. 5.11). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane? (This problem already showed up as Problem 2.2. If you haven't already done so, try solving it using $F = ma$. You will then have a greater appreciation for the Lagrangian method.)

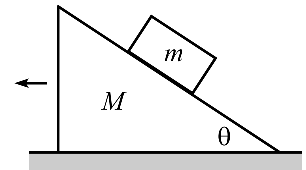



**Figure 5.11**

2. **Two masses, one swinging** $***$

   Two equal masses, $m$, connected by a string, hang over two pulleys (of negligible size), as shown in Fig. 5.12. The left one moves in a vertical line, but the right one is free to swing back and forth (in the plane of the masses and pulleys). Find the equations of motion for $r$ and $\theta$, as shown.

   Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude $\epsilon$ (with $\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?

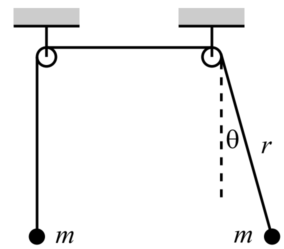



**Figure 5.12**

3. **Two falling sticks** $**$

   Two massless sticks of length $2r$, each with a mass $m$ fixed at its middle, are hinged at an end. One stands on top of the other, as shown in Fig. 5.13. The bottom end of the lower stick is hinged at the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle $\epsilon$ with respect to the vertical. They are then released. At this instant, what are the angular accelerations of the two sticks? Work in the approximation where $\epsilon$ is very small.

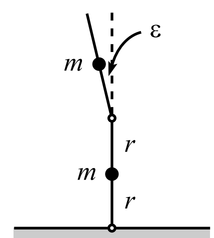



**Figure 5.13**

4. **Pendulum with an oscillating support** $**$

   A pendulum consists of a mass $m$ and a massless stick of length $\ell$. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$ (see Fig. 5.14). Calculate the angle of the pendulum as a function of time.

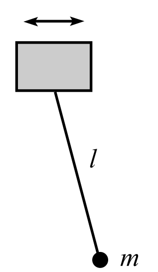



**Figure 5.14**

5. **Inverted pendulum** $****$

   A pendulum consists of a mass $m$ at the end of a massless stick of length $\ell$. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$, where $A \ll \ell$. See Fig. 5.15. It turns out that if $\omega$ is large enough, and if the pendulum is initially nearly upside-down, then it will surprisingly *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position.

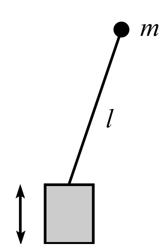



**Figure 5.15**

Find the equation of motion for the angle of the pendulum (measured relative to its upside-down position). And explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.

*Section 5.2: The principle of stationary action*

6. **Minimum or saddle** $**$

   (a) In eq. (5.25), let $t_1 = 0$ and $t_2 = T$, for convenience. And let $\xi(t)$ be an easy-to-deal-with "triangular" function, of the form

$$\xi(t) = \begin{cases} \epsilon t/T, & 0 \le t \le T/2, \\ \epsilon(1 - t/T), & T/2 \le t \le T. \end{cases} \tag{5.93}$$

   Under what conditions is the harmonic-oscillator $\Delta S$ in eq. (5.25) negative?

   (b) Answer the same question, but now with $\xi(t) = \epsilon \sin(\pi t/T)$.

*Section 5.3: Forces of constraint*

7. **Normal force from a plane** $**$

   A mass $m$ slides down a frictionless plane that is inclined at angle $\theta$. Show, using the method in Section 5.3, that the normal force from the plane is the familiar $mg\cos\theta$.

*Section 5.5: Conservation Laws*

8. **Bead on a stick** $*$

   A stick is pivoted at the origin and is arranged to swing around in a horizontal plane at constant angular speed $\omega$. A bead of mass $m$ slides frictionlessly along the stick. Let $r$ be the radial position of the bead. Find the conserved quantity $E$ given in eq. (5.52). Explain why this quantity is *not* the energy of the bead.

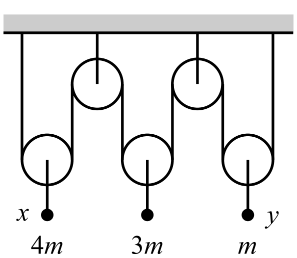



**Figure 5.16**

*Section 5.6: Noether's Theorem*

9. **Atwood's machine** $**$

   Consider the Atwood's machine shown in Fig. 5.16. The masses are $4m$, $3m$, and $m$. Let $x$ and $y$ be the heights of the left and right masses, relative to their initial positions. Find the conserved momentum.

*Section 5.7: Small oscillations*

10. **Hoop and pulley** $**$

   A mass $M$ is attached to a massless hoop (of radius $R$) which lies in a vertical plane. The hoop is free to rotate about its fixed center. $M$ is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass $m$ hangs on the other end of the string (see Fig. 5.17). Find the equation of motion for the angle of rotation of the hoop. What is the frequency of small oscillations? Assume that $m$ moves only vertically, and assume $M > m$.

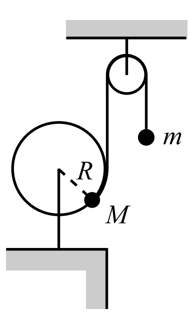



**Figure 5.17**

11. **Bead on a rotating hoop** $**$

A bead is free to slide along a frictionless hoop of radius $R$. The hoop rotates with constant angular speed $\omega$ around a vertical diameter (see Fig. 5.18). Find the equation of motion for the position of the bead. What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium?

There is one value of $\omega$ that is rather special. What is it, and why is it special?

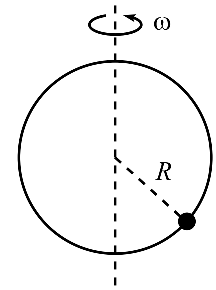



**Figure 5.18**

12. **Another bead on a rotating hoop** $**$

A bead is free to slide along a frictionless hoop of radius $r$. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius $R$, with constant angular speed $\omega$, about a given point (see Fig. 5.19). Find the equation of motion for the position of the bead. Also, find the frequency of small oscillations about the equilibrium point.

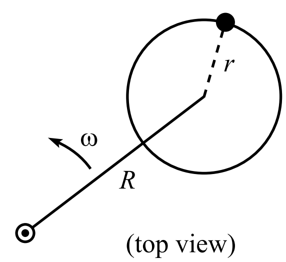



**Figure 5.19**

13. **Rotating curve** $***$

The curve $y(x) = b(x/a)^\lambda$ is rotated around the $y$-axis with constant frequency $\omega$ (see Fig. 5.20). A bead moves frictionlessly along the curve. Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist? (This problem gets a little messy.)

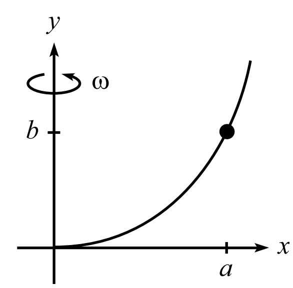



**Figure 5.20**

14. **Mass on a wheel** $**$

A mass $m$ is fixed to a given point on the edge of a wheel of radius $R$. The wheel is massless, except for a mass $M$ located at its center (see Fig. 5.21). The wheel rolls without slipping on a horizontal table. Find the equation of motion for the angle through which the wheel rolls. For the case where the wheel undergoes small oscillations, find the frequency.

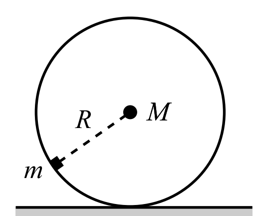



**Figure 5.21**

15. **Double pendulum** $****$

Consider a double pendulum made of two masses, $m_1$ and $m_2$, and two rods of lengths $\ell_1$ and $\ell_2$ (see Fig. 5.22). Find the equations of motion.

For small oscillations, find the normal modes and their frequencies for the special case $\ell_1 = \ell_2$ (and consider the cases $m_1 = m_2$, $m_1 \gg m_2$, and $m_1 \ll m_2$). Do the same for the special case $m_1 = m_2$ (and consider the cases $\ell_1 = \ell_2$, $\ell_1 \gg \ell_2$, and $\ell_1 \ll \ell_2$).

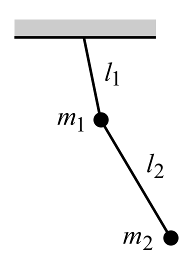



**Figure 5.22**

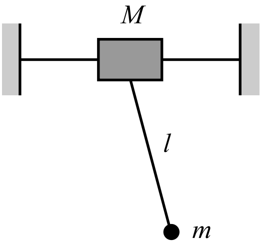



**Figure 5.23**

16. **Pendulum with a free support** $**$

    A mass $M$ is free to slide along a frictionless rail. A pendulum of length $\ell$ and mass $m$ hangs from $M$ (see Fig. 5.23). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

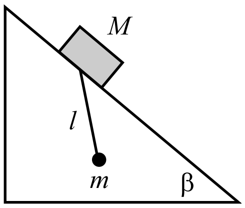



**Figure 5.24**

17. **Pendulum support on an inclined plane** $**$

    A mass $M$ is free to slide down a frictionless plane inclined at angle $\beta$. A pendulum of length $\ell$ and mass $m$ hangs from $M$ (see Fig. 5.24). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

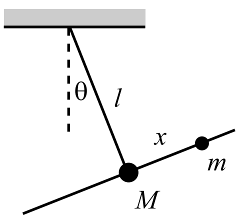



**Figure 5.25**

18. **Tilting plane** $***$

    A mass $M$ is fixed at the right-angled vertex where a massless rod of length $\ell$ is connected to a very long massless rod (see Fig. 5.25). A mass $m$ is free to move frictionlessly along the long rod. The rod of length $\ell$ is hinged at a support, and the whole system is free to rotate, in the plane of the rods, about the support.

    Let $\theta$ be the angle of rotation of the system, and let $x$ be the distance between $m$ and $M$. Find the equations of motion. Find the normal modes when $\theta$ and $x$ are both very small.

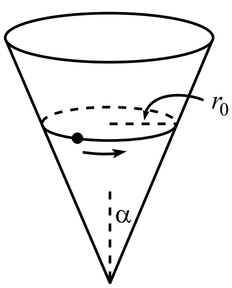



**Figure 5.26**

19. **Motion in a cone** $***$

    A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle at the tip is $\alpha$ (see Fig. 5.26). Let $r(t)$ be the distance from the particle to the axis, and let $\theta(t)$ be the angle around the cone. Find the equations of motion.

    If the particle moves in a circle of radius $r_0$, what is the frequency, $\omega$, of this motion? If the particle is then perturbed slightly from this circular motion, what is the frequency, $\Omega$, of the oscillations about the radius $r_0$? Under what conditions does $\Omega = \omega$?

*Section 5.8: Other applications*

20. **Shortest distance in a plane**

    In the spirit of Section 5.8, show that the shortest path between two points in a plane is a straight line.

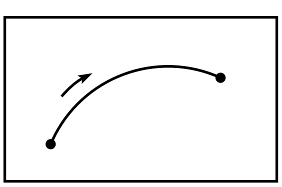


**Figure 5.27**

21. **Index of refraction** $**$

    Assume that the speed of light in a given slab of material is proportional to the height above the base of the slab.[13] Show that light moves in circular arcs in this material; see Fig. 5.27. You may assume that light takes the path of least time between two points (Fermat's principle of least time).

[13] In other words, the index of refraction of the material, $n$, as a function of the height, $y$, is given by $n(y) = y_0/y$, where $y_0$ is some length that is larger than the height of the slab.

22. **The brachistochrone** ∗∗∗

A bead is released from rest at the origin and slides down a frictionless wire that connects the origin to a given point, as shown in Fig. 5.28. You wish to shape the wire so that the bead reaches the endpoint in the shortest possible time. Let the desired curve be described by the function $y(x)$, with downward taken to be positive. Show that $y(x)$ satisfies

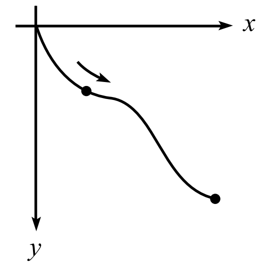



**Figure 5.28**

$$1 + y'^2 = \frac{C}{y}\,. \tag{5.94}$$

where $C$ is a constant. Show that $x$ and $y$ may be written as

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta). \tag{5.95}$$

This is the parametrization of a *cycloid*, which is the path taken by a point on the edge of a rolling wheel.

23. **Minimal surface** ∗∗

Derive the shape of the minimal surface discussed in Section 5.8, by demanding that a cross-sectional "ring" (that is, the region between the planes $x = x_1$ and $x = x_2$) is in equilibrium; see Fig. 5.29. *Hint:* The tension must be constant throughout the surface.

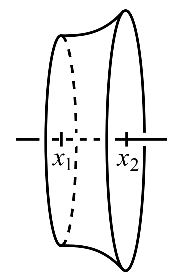



**Figure 5.29**

24. **Existence of a minimal surface** ∗∗

Consider the minimal surface from Section 5.8, and look at the special case where the two rings have the same radius (see Fig. 5.30). Let $2\ell$ be the distance between the rings. What is the largest value of $\ell/r$ for which a minimal surface exists? You will have to solve something numerically here.

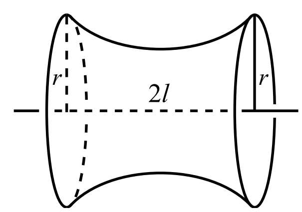



**Figure 5.30**

## 5.11 Solutions

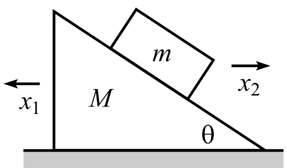


**Figure 5.31**

1. **Moving plane**

   Let $x_1$ be the horizontal coordinate of the plane (with positive $x_1$ to the left), and let $x_2$ be the horizontal coordinate of the block (with positive $x_2$ to the right); see Fig. 5.31. The relative horizontal distance between the plane and the block is $x_1+x_2$, so the height fallen by the block is $(x_1+x_2)\tan\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\Big(\dot{x}_2^2 + (\dot{x}_1+\dot{x}_2)^2\tan^2\theta\Big) + mg(x_1+x_2)\tan\theta. \qquad (5.96)$$

   The equations of motion obtained from varying $x_1$ and $x_2$ are

$$\begin{aligned} M\ddot{x}_1 + m(\ddot{x}_1+\ddot{x}_2)\tan^2\theta &= mg\tan\theta, \\ m\ddot{x}_2 + m(\ddot{x}_1+\ddot{x}_2)\tan^2\theta &= mg\tan\theta. \end{aligned} \qquad (5.97)$$

   Note that the difference of these two equations immediately yields conservation of momentum, $M\ddot{x}_1 - m\ddot{x}_2 = 0 \Longrightarrow (d/dt)(M\dot{x}_1 - m\dot{x}_2) = 0$. Eqs. (5.97) are two linear equations in the two unknowns, $\ddot{x}_1$ and $\ddot{x}_2$, so we can solve for $\ddot{x}_1$. After a little simplification, we arrive at

$$\ddot{x}_1 = \frac{mg\sin\theta\cos\theta}{M+m\sin^2\theta}. \qquad (5.98)$$

   For some limiting cases, see the remark in the solution to Problem 2.2.

2. **Two masses, one swinging**

   With $r$ and $\theta$ being the distance from the swinging mass to the pulley, and the angle of the swinging mass, respectively, the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr + mgr\cos\theta. \qquad (5.99)$$

   The last two terms are the (negatives of the) potentials of each mass, relative to where they would be if the right mass were located at the right pulley. The equations of motion obtained from varying $r$ and $\theta$ are

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - g(1-\cos\theta), \\ \frac{d}{dt}(r^2\dot{\theta}) &= -gr\sin\theta. \end{aligned} \qquad (5.100)$$

   The first equation deals with the forces and accelerations along the direction of the string. The second equation equates the torque from gravity with the change in angular momentum of the right mass.

   If we do a (coarse) small-angle approximation and keep only terms up to first order in $\theta$, we find that at $t=0$ (using the initial condition, $\dot{r}=0$), eqs. (5.100) become

$$\begin{aligned} \ddot{r} &= 0, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \qquad (5.101)$$

   These say that the left mass stays still, and the right mass behaves just like a pendulum.

   If we want to find the leading term in the initial acceleration of the left mass (that is, the leading term in $\ddot{r}$), we need to be a little less coarse in our approximation. So let's keep terms in eq. (5.100) up to second order in $\theta$. We then have at $t = 0$ (using the initial condition, $\dot{r} = 0$)

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - \frac{1}{2}g\theta^2, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \tag{5.102}$$

The second equation still says that the right mass undergoes harmonic motion. We are told that the amplitude is $\epsilon$, so we have

$$\theta(t) = \epsilon\cos(\omega t + \phi), \tag{5.103}$$

where $\omega = \sqrt{g/r}$. Plugging this into the first equation gives

$$2\ddot{r} = \epsilon^2 g\left(\sin^2(\omega t + \phi) - \frac{1}{2}\cos^2(\omega t + \phi)\right). \tag{5.104}$$

If we average this over a few periods, both $\sin^2\alpha$ and $\cos^2\alpha$ average to $1/2$, so we find

$$\ddot{r}_{\text{avg}} = \frac{\epsilon^2 g}{8}. \tag{5.105}$$

This is a small second-order effect. It is positive, so the left mass slowly begins to climb.

3. **Two falling sticks**

Let $\theta_1(t)$ and $\theta_2(t)$ be defined as in Fig. 5.32. Then the position of the bottom mass in cartesian coordinates is $(r\sin\theta_1, r\cos\theta_1)$, and the position of the top mass is $(2r\sin\theta_1 - r\sin\theta_2, 2r\cos\theta_1 + r\cos\theta_2)$. So the potential energy of the system is

$$V(\theta_1, \theta_2) = mgr(3\cos\theta_1 + \cos\theta_2). \tag{5.106}$$

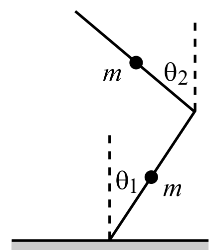



**Figure 5.32**

The kinetic energy is somewhat more complicated. The kinetic energy of the bottom mass is simply $mr^2\dot{\theta}_1^2/2$. Taking the derivative of the top mass's position given above, we find that the kinetic energy of the top mass is

$$\frac{1}{2}mr^2\Big((2\cos\theta_1\dot{\theta}_1 - \cos\theta_2\dot{\theta}_2)^2 + (-2\sin\theta_1\dot{\theta}_1 - \sin\theta_2\dot{\theta}_2)^2\Big). \tag{5.107}$$

We can simplify this, using the small-angle approximations. The terms involving $\sin\theta$ will be fourth order in the small $\theta$'s, so we may neglect them. Also, we may approximate $\cos\theta$ by 1, because this entails dropping only terms of at least fourth order. So the top mass's kinetic energy becomes $(1/2)mr^2(2\dot{\theta}_1 - \dot{\theta}_2)^2$. In retrospect, it would have been easier to obtain the kinetic energies of the masses by first applying the small-angle approximations to the positions, and then taking the derivatives to obtain the velocities. This strategy will show that both masses move essentially horizontally (initially). You will probably want to use this strategy when solving Exercise 1.

Using the small-angle approximation $\cos\theta \approx 1 - \theta^2/2$ to rewrite the potential energy in eq. (5.106), we have

$$L \approx \frac{1}{2}mr^2\left(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) - mgr\left(4 - \frac{3}{2}\theta_1^2 - \frac{1}{2}\theta_2^2\right). \tag{5.108}$$

The equations of motion obtained from varying $\theta_1$ and $\theta_2$ are, respectively,

$$\begin{aligned} 5\ddot{\theta}_1 - 2\ddot{\theta}_2 &= \frac{3g}{r}\theta_1 \\ -2\ddot{\theta}_1 + \ddot{\theta}_2 &= \frac{g}{r}\theta_2. \end{aligned} \tag{5.109}$$

At the instant the sticks are released, we have $\theta_1 = 0$ and $\theta_2 = \epsilon$. Solving eqs. (5.109) for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ gives

$$\ddot{\theta}_1 = \frac{2g\epsilon}{r}, \qquad \text{and} \qquad \ddot{\theta}_2 = \frac{5g\epsilon}{r}. \tag{5.110}$$

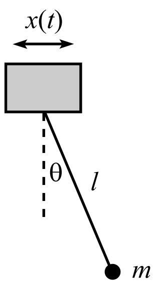



**Figure 5.33**

4. **Pendulum with an oscillating support**

Let $\theta$ be defined as in Fig. 5.33. With $x(t) = A\cos(\omega t)$, the position of the mass $m$ is given by

$$(X, Y)_m = (x + \ell\sin\theta, -\ell\cos\theta). \tag{5.111}$$

The square of the speed is

$$V_m^2 = \ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta. \tag{5.112}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.113}$$

The equation of motion obtained from varying $\theta$ is

$$\begin{aligned} \frac{d}{dt}(m\ell^2\dot{\theta} + m\ell\dot{x}\cos\theta) &= -m\ell\dot{x}\dot{\theta}\sin\theta - mg\ell\sin\theta \\ \Longrightarrow \qquad \ell\ddot{\theta} + \ddot{x}\cos\theta &= -g\sin\theta. \end{aligned} \tag{5.114}$$

Plugging in the explicit form of $x(t)$, we have

$$\ell\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0. \tag{5.115}$$

This makes sense. Someone in the frame of the support (which has horizontal acceleration $\ddot{x} = -A\omega^2\cos(\omega t)$) may as well be living in a world where the acceleration from gravity has a component $g$ downward and a component $A\omega^2\cos(\omega t)$ to the right. Eq. (5.122) is simply the $F = ma$ equation in the tangential direction in this accelerating world.

A small-angle approximation in eq. (5.115) gives

$$\ddot{\theta} + \omega_0^2\theta = a\omega^2\cos(\omega t), \tag{5.116}$$

where $\omega_0 \equiv \sqrt{g/\ell}$ and $a \equiv A/\ell$. This equation is simply that of a driven oscillator, which we solved in Chapter 3. The solution is

$$\theta(t) = \frac{a\omega^2}{\omega_0^2 - \omega^2}\cos(\omega t) + C\cos(\omega_0 t + \phi), \tag{5.117}$$

where $C$ and $\phi$ are determined by the initial conditions.

If $\omega$ happens to equal $\omega_0$, then the amplitude becomes large. Eq. (5.117) would seem to suggest that the amplitude actually goes to infinity in this case. But as soon as the amplitude becomes large, our small-angle approximation breaks down, and eqs. (5.116) and (5.117) are no longer valid.

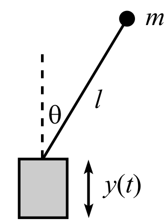



**Figure 5.34**

5. **Inverted pendulum**

Let $\theta$ be defined as in Fig. 5.34. With $y(t) = A\cos(\omega t)$, the position of the mass $m$ is given by

$$(X, Y) = (\ell\sin\theta, y + \ell\cos\theta). \tag{5.118}$$

Taking the derivatives of these coordinates, we see that the square of the speed is

$$V^2 = \dot{X}^2 + \dot{Y}^2 = \ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta. \tag{5.119}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta). \tag{5.120}$$

The equation of motion for $\theta$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \qquad \Longrightarrow \qquad \ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta. \tag{5.121}$$

Plugging in the explicit form of $y(t)$, we have

$$\ell\ddot{\theta} + \sin\theta\Big(A\omega^2\cos(\omega t) - g\Big) = 0. \tag{5.122}$$

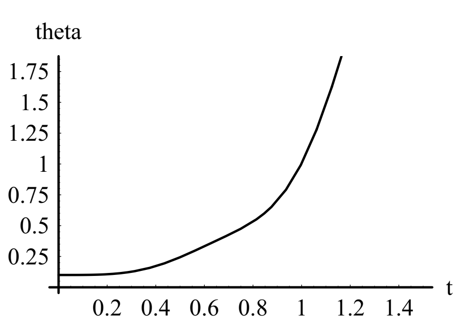
In retrospect, this makes sense. Someone in the reference frame of the support, which has acceleration $\ddot{y} = -A\omega^2\cos(\omega t)$, may as well be living in a world where the acceleration from gravity is $g - A\omega^2\cos(\omega t)$ downward. Eq. (5.122) is simply the $F = ma$ equation in the tangential direction in this accelerated frame.

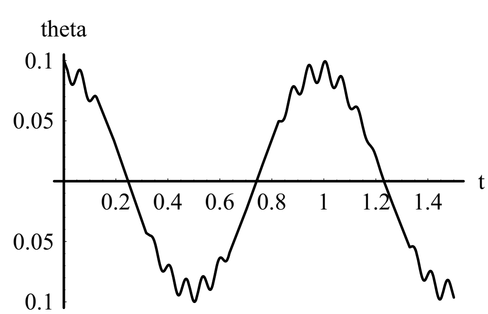
Assuming $\theta$ is small, we may set $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big) = 0, \tag{5.123}$$

where $\omega_0 \equiv \sqrt{g/\ell}$, and $a \equiv A/\ell$. Eq. (5.123) cannot be solved exactly, but we can still get a good idea of how $\theta$ depends on time. We can do this both numerically and (approximately) analytically.

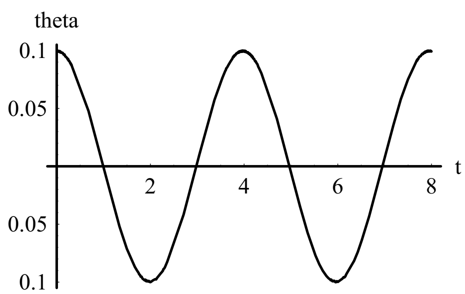
The figures below show how $\theta$ depends on time for parameters with values $\ell = 1\,\text{m}$, $A = 0.1\,\text{m}$, and $g = 10\,\text{m/s}^2$ (so $a = 0.1$, and $\omega_0^2 = 10\,\text{s}^{-2}$). In the first plot, $\omega = 10\,\text{s}^{-1}$. And in the second plot, $\omega = 100\,\text{s}^{-1}$. The stick falls over in first case, but undergoes oscillatory motion in the second case. Apparently, if $\omega$ is large enough the stick will not fall over.

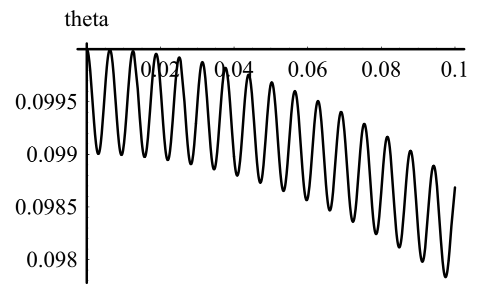


Let's now explain this phenomenon analytically. At first glance, it's rather surprising that the stick stays up. It seems like the average (over a few periods of the $\omega$ oscillations) of the tangential acceleration in eq. (5.123), namely $-\theta(a\omega^2\cos(\omega t) - \omega_0^2)$, equals the positive quantity $\theta\omega_0^2$, because the $\cos(\omega t)$ term averages to zero (or so it appears). So you might think that there is a net force making $\theta$ increase, causing the stick fall over.

The fallacy in this reasoning is that the average of the $-a\omega^2\theta\cos(\omega t)$ term is *not* zero, because $\theta$ undergoes tiny oscillations with frequency $\omega$, as seen below. Both of these plots have $a = 0.005$, $\omega_0^2 = 10\,\text{s}^{-2}$, and $\omega = 1000\,\text{s}^{-1}$ (we'll work with small $a$ and large $\omega$ from now on; more on this below). The second plot is a zoomed-in version of the first one near $t = 0$.




The important point here is that the tiny oscillations in $\theta$ shown in the second plot are correlated with $\cos(\omega t)$. It turns out that the $\theta$ value at the $t$ where $\cos(\omega t) = 1$ is larger than the $\theta$ value at the $t$ where $\cos(\omega t) = -1$. So there is a net negative contribution to the $-a\omega^2\theta\cos(\omega t)$ part of the acceleration. And it may indeed be large enough to keep the pendulum up, as we will now show.

To get a handle on the $-a\omega^2\theta\cos(\omega t)$ term, let's work in the approximation where $\omega$ is large and $a \equiv A/\ell$ is small. More precisely, we will assume $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons we will explain below. Look at one of the little oscillations in the second of the above plots. These oscillations have frequency $\omega$, because they are due simply to the support moving up and down. When the support moves up, $\theta$ increases; and when the support moves down, $\theta$ decreases. Since the average position of the pendulum doesn't change much over one of these small periods, we can look for an approximate solution to eq. (5.123) of the form

$$\theta(t) \approx C + b\cos(\omega t), \tag{5.124}$$

where $b \ll C$. $C$ will change over time, but on the scale of $1/\omega$ it is essentially constant, if $a \equiv A/\ell$ is small enough.

Plugging this guess for $\theta$ into eq. (5.123), and using $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, we find that $-b\omega^2\cos(\omega t) + Ca\omega^2\cos(\omega t) = 0$, to leading order.[14] So we must have $b = aC$.

[14] The reasons for the $a \ll 1$ and $a\omega^2 \gg \omega_0^2$ qualifications are the following. If $a\omega^2 \gg \omega_0^2$, then the $a\omega^2\cos(\omega t)$ term dominates the $\omega_0^2$ term in eq. (5.123). The one exception to this is when $\cos(\omega t) \approx 0$, but this occurs for a negligibly small amount of time if $a\omega^2 \gg \omega_0^2$. If $a \ll 1$, then we can legally ignore the $\ddot{C}$ term when eq. (5.124) is substituted into eq. (5.123). We will find below, in eq. (5.126), that our assumptions lead to $\ddot{C}$ being roughly proportional to $a^2\omega^2$. Since the other terms in eq. (5.123) are proportional to $a\omega^2$, we need $a \ll 1$ in order for the $\ddot{C}$ term to be negligible. In short, $a \ll 1$ is the condition under which $C$ varies slowly on the time scale of $1/\omega$.

Our approximate solution for $\theta$ is therefore

$$\theta \approx C\Big(1 + a\cos(\omega t)\Big). \tag{5.125}$$

Let's now determine how $C$ gradually changes with time. From eq. (5.123), the average acceleration of $\theta$, over a period $T = 2\pi/\omega$, is

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&\approx \overline{-C\Big(1 + a\cos(\omega t)\Big)\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= -C\Big(a^2\omega^2\overline{\cos^2(\omega t)} - \omega_0^2\Big) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&\equiv -C\Omega^2, \qquad (5.126)
\end{aligned}$$

where

$$\Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}\,. \tag{5.127}$$

But if we take two derivatives of eq. (5.124), we see that $\overline{\ddot{\theta}}$ simply equals $\ddot{C}$. Equating this value of $\overline{\ddot{\theta}}$ with the one in eq. (5.126) gives

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0. \tag{5.128}$$

This equation describes nice simple-harmonic motion. Therefore, $C$ oscillates sinusoidally with the frequency $\Omega$ given in eq. (5.127). This is the overall back and forth motion seen in the first of the above plots. Note that we must have $a\omega > \sqrt{2}\omega_0$ if this frequency is to be real so that the pendulum stays up. Since we have assumed $a \ll 1$, we see that $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 \gg \omega_0^2$, which is consistent with our initial assumption above.

If $a\omega \gg \omega_0$, then eq. (5.127) gives $\Omega \approx a\omega/\sqrt{2}$. Such is the case if we change the setup and simply have the pendulum lie flat on a horizontal table where the acceleration from gravity is zero. In this limit where $g$ is irrelevant, dimensional analysis implies that the frequency of the $C$ oscillations must be a multiple of $\omega$, because $\omega$ is the only quantity in the problem with units of frequency. It just so happens that the multiple is $a/\sqrt{2}$.

6. **Minimum or saddle**

   (a) For the given $\xi(t)$, the integrand in eq. (5.25) is symmetric around the midpoint, so we obtain

$$\begin{aligned}
\Delta S &= \int_0^{T/2} \left(m\Big(\frac{\epsilon}{T}\Big)^2 - k\Big(\frac{\epsilon t}{T}\Big)^2\right) dt. \\
&= \frac{m\epsilon^2}{2T} - \frac{k\epsilon^2 T}{24}\,. \qquad (5.129)
\end{aligned}$$

   This is negative if $T > \sqrt{12m/k} \equiv 2\sqrt{3}/\omega$. Since the period of the oscillation is $\tau \equiv 2\pi/\omega$, we see that $T$ must be greater than $(\sqrt{3}/\pi)\tau$ in order for $\Delta S$ to be negative (provided that we are using our triangular function for $\xi$).

(b) With $\xi(t) = \epsilon \sin(\pi t/T)$, the integrand in eq. (5.25) becomes

$$\begin{aligned} \Delta S &= \frac{1}{2}\int_0^T \left( m\left(\frac{\epsilon\pi}{T}\cos(\pi t/T)\right)^2 - k\Big(\epsilon\sin(\pi t/T)\Big)^2\right) dt. \\ &= \frac{m\epsilon^2\pi^2}{4T} - \frac{k\epsilon^2 T}{4}, \end{aligned} \tag{5.130}$$

where we have used the fact that the average value of $\sin^2\theta$ and $\cos^2\theta$ over half of a period is 1/2 (or you can just do the integrals). This result for $\Delta S$ is negative if $T > \pi\sqrt{m/k} \equiv \pi/\omega = \tau/2$, where $\tau$ is the period.

REMARK: It turns out that the $\xi(t) \propto \sin(\pi t/T)$ function gives the best chance of making $\Delta S$ negative. You can show this by invoking a theorem from Fourier analysis that says that any function satisfying $\xi(0) = \xi(T) = 0$ can be written as the sum $\xi(t) = \sum_1^\infty c_n \sin(n\pi t/T)$, where the $c_n$ are numerical coefficients. When this sum is plugged into eq. (5.25), you can show that all the cross terms (terms involving two different values of $n$) integrate to zero. Using the fact that the average value of $\sin^2\theta$ and $\cos^2\theta$ is 1/2, the rest of the integral yields

$$\Delta S = \frac{1}{4}\sum_1^\infty c_n^2 \left(\frac{m\pi^2 n^2}{T} - kT\right). \tag{5.131}$$

In order to obtain the smallest value of $T$ that can make this sum negative, we want only the $n = 1$ term to exist. We then have $\xi(t) = c_1 \sin(\pi t/T)$, and eq. (5.131) reduces to eq. (5.130), as it should.

As mentioned in Remark 4 in Section 5.2, it is always possible to make $\Delta S$ positive by picking a $\xi(t)$ function that is small but wiggles very fast. Therefore, we see that for a harmonic oscillator, if $T > \tau/2$, then the stationary value of $S$ is a saddle point (some $\xi$'s make $\Delta S$ positive, and some make it negative), but if $T < \tau/2$, then the stationary value of $S$ is a minimum (all $\xi$'s make $\Delta S$ positive). In the latter case, the point is that $T$ is small enough so that there is no way for $\xi$ to get large, without making $\dot\xi$ large also. ♣

7. **Normal force from a plane**

**First Solution:** The most convenient coordinates in this problem are $w$ and $z$, where $w$ is the distance upward along the plane, and $z$ is the distance perpendicularly away from it. The Lagrangian is then

$$\frac{1}{2}m(\dot w^2 + \dot z^2) - mg(w\sin\theta + z\cos\theta) - V(z), \tag{5.132}$$

where $V(z)$ is the (very steep) constraining potential. The two equations of motion are

$$\begin{aligned} m\ddot w &= -mg\sin\theta, \\ m\ddot z &= -mg\cos\theta - \frac{dV}{dz}. \end{aligned} \tag{5.133}$$

At this point we invoke the constraint $z = 0$. So $\ddot z = 0$, and the second equation gives

$$F_c \equiv -V'(0) = mg\cos\theta, \tag{5.134}$$

as desired. We also obtain the usual result, $\ddot w = -g\sin\theta$.

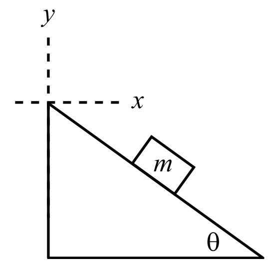



**Figure 5.35**

**Second Solution:** We can also solve this problem by using the horizontal and vertical components, $x$ and $y$. We'll choose $(x, y) = (0, 0)$ to be at the top of the plane; see Fig. 5.35. The (very steep) constraining potential is $V(z)$, where $z \equiv x\sin\theta + y\cos\theta$ is the distance from the mass to the plane (as you can verify). The Lagrangian is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(z) \tag{5.135}$$

Keeping in mind that $z \equiv x\sin\theta + y\cos\theta$, the two equations of motion are (using the chain rule)

$$\begin{aligned} m\ddot{x} &= -\frac{dV}{dz}\frac{\partial z}{\partial x} = -V'(z)\sin\theta, \\ m\ddot{y} &= -mg - \frac{dV}{dz}\frac{\partial z}{\partial y} = -mg - V'(z)\cos\theta. \end{aligned} \tag{5.136}$$

At this point we invoke the constraint condition $x = -y\cot\theta$ (that is, $z = 0$). This condition, along with the two E-L equations, allows us to solve for the three unknowns, $\ddot{x}$, $\ddot{y}$, and $V'(0)$. Using $\ddot{x} = -\ddot{y}\cot\theta$ in eqs. (5.136), we find

$$\ddot{x} = g\cos\theta\sin\theta, \qquad \ddot{y} = -g\sin^2\theta, \qquad F_c \equiv -V'(0) = mg\cos\theta. \tag{5.137}$$

The first two results here are simply the horizontal and vertical components of the acceleration along the plane.

8. **Bead on a stick**

There is no potential energy here, so the Lagrangian simply consists of the kinetic energy, which comes from the radial and tangential motions:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2. \tag{5.138}$$

Eq. (5.52) therefore gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2. \tag{5.139}$$

Claim 5.3 says that this quantity is conserved, because $\partial L/\partial t = 0$. But it is *not* the energy of the bead, due to the minus sign in the second term.

The point here is that in order to keep the stick rotating at a constant angular speed, there must be an external force acting it. This force will cause work to be done on the bead, thereby changing its kinetic energy. The kinetic energy, $T$, is therefore *not* conserved. From the above equations, we see that $E = T - mr^2\omega^2$ is the quantity that is constant in time.

See Exercise 10 for some $F = ma$ ways to show that the quantity $E$ is conserved.

9. **Atwood's machine**

**First solution:** If the left mass goes up by $x$ and the right mass goes up by $y$, then conservation of string says that the middle mass must go down by $x + y$. Therefore, the Lagrangian of the system is

$$\begin{aligned} L &= \frac{1}{2}(4m)\dot{x}^2 + \frac{1}{2}(3m)(-\dot{x} - \dot{y})^2 + \frac{1}{2}m\dot{y}^2 - \Big((4m)gx + (3m)g(-x - y) + mgy\Big) \\ &= \frac{7}{2}m\dot{x}^2 + 3m\dot{x}\dot{y} + 2m\dot{y}^2 - mg(x - 2y). \end{aligned} \tag{5.140}$$

This is invariant under the transformation $x \to x + 2\epsilon$ and $y \to y + \epsilon$. Hence, we can use Noether's theorem, with $K_x = 2$ and $K_y = 1$. The conserved momentum is then

$$P = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(7\dot{x} + 3\dot{y})(2) + m(3\dot{x} + 4\dot{y})(1) = m(17\dot{x} + 10\dot{y}). \quad (5.141)$$

This $P$ is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $-(10/17)\dot{y}$.

**Second solution:** The Euler-Lagrange equations are, from eq. (5.140),

$$\begin{aligned} 7m\ddot{x} + 3m\ddot{y} &= -mg, \\ 3m\ddot{x} + 4m\ddot{y} &= 2mg. \end{aligned} \quad (5.142)$$

Adding the second equation to twice the first gives

$$17m\ddot{x} + 10m\ddot{y} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(17m\dot{x} + 10m\dot{y}\Big) = 0. \quad (5.143)$$

**Third solution:** We can also solve this problem using $F = ma$. Since the tension, $T$, is the same throughout the rope, we see that the three $F = dP/dt$ equations are

$$2T - 4mg = \frac{dP_{4m}}{dt}, \qquad 2T - 3mg = \frac{dP_{3m}}{dt}, \qquad 2T - mg = \frac{dP_m}{dt}. \quad (5.144)$$

The three forces depend on only two parameters, so there will be some combination of them that adds up to zero. If we set $a(2T-4mg) + b(2T-3mg) + c(2T-mg) = 0$, then we have $a + b + c = 0$ and $4a + 3b + c = 0$, which is satisfied by $a = 2$, $b = -3$, and $c = 1$. Therefore,

$$\begin{aligned} 0 &= \frac{d}{dt}(2P_{4m} - 3P_{3m} + P_m) \\ &= \frac{d}{dt}\Big(2(4m)\dot{x} - 3(3m)(-\dot{x} - \dot{y}) + m\dot{y}\Big) \\ &= \frac{d}{dt}(17m\dot{x} + 10m\dot{y}). \end{aligned} \quad (5.145)$$

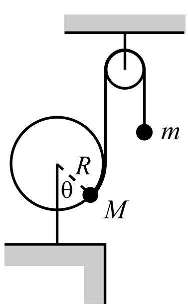



**Figure 5.36**

10. **Hoop and pulley**

Let the radius to $M$ make an angle $\theta$ with the vertical (see Fig. 5.36). Then the coordinates of $M$ are $R(\sin\theta, -\cos\theta)$. The height of $m$, relative to its position when $M$ is at the bottom of the hoop, is $y = -R\theta$. The Lagrangian is therefore (and yes, we've chosen a different $y = 0$ point for each mass, but such a definition only changes the potential by a constant amount, which is irrelevant)

$$L = \frac{1}{2}(M + m)R^2\dot{\theta}^2 + MgR\cos\theta + mgR\theta. \quad (5.146)$$

The equation of motion is then

$$(M + m)R\ddot{\theta} = g(m - M\sin\theta). \quad (5.147)$$

This is, of course, just $F = ma$ along the direction of the string (because $Mg\sin\theta$ is the tangential component of the gravitational force on $M$).

Equilibrium occurs when $\dot\theta = \ddot\theta = 0$. From eq. (5.147), we see that this happens at $\sin\theta_0 = m/M$. Letting $\theta \equiv \theta_0 + \delta$, and expanding eq. (5.147) to first order in $\delta$, gives

$$\ddot\delta + \left(\frac{Mg\cos\theta_0}{(M+m)R}\right)\delta = 0. \tag{5.148}$$

The frequency of small oscillations is therefore

$$\omega = \sqrt{\frac{M\cos\theta_0}{M+m}}\sqrt{\frac{g}{R}} = \left(\frac{M-m}{M+m}\right)^{1/4}\sqrt{\frac{g}{R}}\,, \tag{5.149}$$

where we have used $\cos\theta_0 = \sqrt{1-\sin\theta_0^2}$.

Remarks: If $M \gg m$, then $\theta_0 \approx 0$, and $\omega \approx \sqrt{g/R}$. This makes sense, because $m$ can be ignored, and $M$ essentially oscillates about the bottom of the hoop, just like a pendulum of length $R$.

If $M$ is only slightly greater than $m$, then $\theta_0 \approx \pi/2$, and $\omega \approx 0$. This also makes sense, because if $\theta \approx \pi/2$, the restoring force $g(m - M\sin\theta)$ does not change much as $\theta$ changes (the derivative of $\sin\theta$ is zero at $\theta = \pi/2$), so it's as if we have a pendulum in a weak gravitational field.

We can actually derive the frequency in eq. (5.149) without doing any calculations. Look at $M$ at the equilibrium position. The tangential forces on it cancel, and the radially inward force from the hoop must be $Mg\cos\theta_0$ to balance the radial outward component of the gravitational force. Therefore, for all the mass $M$ knows, it is sitting at the bottom of a hoop of radius $R$ in a world where gravity has strength $g' = g\cos\theta_0$. The general formula for the frequency of a pendulum (as you can quickly show) is $\omega = \sqrt{F'/M'R}$, where $F'$ is the gravitational force (which is $Mg'$ here), and $M'$ is the total mass being accelerated (which is $M + m$ here). This gives the $\omega$ in eq. (5.149). ♣

11. **Bead on a rotating hoop**

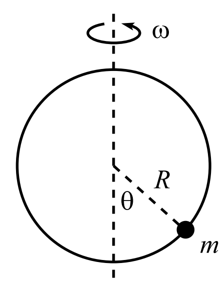


**Figure 5.37**

Let $\theta$ be the angle that the radius to the bead makes with the vertical (see Fig. 5.37). Breaking the velocity up into the component along the hoop plus the component perpendicular to the hoop, we find

$$L = \frac{1}{2}m(\omega^2R^2\sin^2\theta + R^2\dot\theta^2) + mgR\cos\theta. \tag{5.150}$$

The equation of motion is then

$$R\ddot\theta = \sin\theta(\omega^2R\cos\theta - g). \tag{5.151}$$

The $F = ma$ interpretation of this is that the component of gravity pulling downward along the hoop accounts for the acceleration along the hoop plus the component of the centripetal acceleration along the hoop.

Equilibrium occurs when $\dot\theta = \ddot\theta = 0$. The right-hand side of eq. (5.151) equals zero when either $\sin\theta = 0$ (that is, $\theta = 0$ or $\theta = \pi$) or $\cos\theta = g/(\omega^2R)$. Since $\cos\theta$ must be less than or equal to 1, this second condition is possible only if $\omega^2 \ge g/R$. So we have two cases:

- If $\omega^2 < g/R$, then $\theta = 0$ and $\theta = \pi$ are the only equilibrium points.

  The $\theta = \pi$ case is unstable. This is fairly intuitive, but it can also be seen mathematically by letting $\theta \equiv \pi + \delta$, where $\delta$ is small. Eq. (5.151) then becomes

$$\ddot\delta - \delta(\omega^2 + g/R) = 0. \tag{5.152}$$

The coefficient of $\delta$ is negative, so this does not admit oscillatory solutions. The $\theta = 0$ case turns out to be stable. For small $\theta$, eq. (5.151) becomes

$$\ddot{\theta} + \theta(g/R - \omega^2) = 0. \tag{5.153}$$

The coefficient of $\theta$ is positive, so we have sinusoidal solutions. The frequency of small oscillations is $\sqrt{g/R - \omega^2}$. This goes to zero as $\omega \to \sqrt{g/R}$.

- If $\omega^2 \geq g/R$, then $\theta = 0$, $\theta = \pi$, and $\cos\theta_0 \equiv g/(\omega^2 R)$ are all equilibrium points. The $\theta = \pi$ case is again unstable, by looking at eq. (5.152). And the $\theta = 0$ case is also unstable, because the coefficient of $\theta$ in eq. (5.153) is now negative (or zero, if $\omega^2 = g/R$).

  Therefore, $\cos\theta_0 \equiv g/(\omega^2 R)$ is the only stable equilibrium. To find the frequency of small oscillations, let $\theta \equiv \theta_0 + \delta$ in eq. (5.151), and expand to first order in $\delta$. Using $\cos\theta_0 \equiv g/(\omega^2 R)$, we find

$$\ddot{\delta} + \omega^2 \sin^2\theta_0 \delta = 0. \tag{5.154}$$

  The frequency of small oscillations is therefore $\omega \sin\theta_0 = \sqrt{\omega^2 - g^2/R^2\omega^2}$.

  REMARK: This frequency goes to zero as $\omega \to \sqrt{g/R}$. And it approximately equals $\omega$ as $\omega \to \infty$. This second limit can be viewed in the following way. For very large $\omega$, gravity is not very important, and the bead essentially feels a centripetal force of $m\omega^2 R$ as it moves near $\theta = \pi/2$. So for all the bead knows, it is a pendulum of length $R$ in a world where "gravity" pulls sideways with a force $m\omega^2 R \equiv mg'$. The frequency of such a pendulum is $\sqrt{g'/R} = \sqrt{\omega^2 R/R} = \omega$. ♣

The frequency $\omega = \sqrt{g/R}$ is the critical frequency above which there is a stable equilibrium at $\theta \neq 0$, that is, above which the mass will want to move away from the bottom of the hoop.

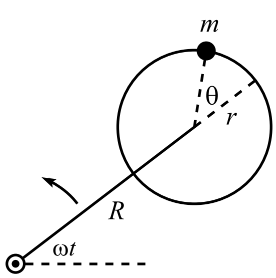



**Figure 5.38**

12. **Another bead on a rotating hoop**

Let the angles $\omega t$ and $\theta$ be defined as in Fig. 5.38. Then the cartesian coordinates for the bead are

$$(x, y) = \Big(R\cos\omega t + r\cos(\omega t + \theta),\ R\sin\omega t + r\sin(\omega t + \theta)\Big). \tag{5.155}$$

The velocity is then

$$(x, y) = \Big(-\omega R\sin\omega t - r(\omega + \dot{\theta})\sin(\omega t + \theta),\ \omega R\cos\omega t + r(\omega + \dot{\theta})\cos(\omega t + \theta)\Big). \tag{5.156}$$

The square of the speed is therefore

$$\begin{aligned} v^2 &= R^2\omega^2 + r^2(\omega + \dot{\theta})^2 \\ &\quad + 2Rr\omega(\omega + \dot{\theta})\Big(\sin\omega t\sin(\omega t + \theta) + \cos\omega t\cos(\omega t + \theta)\Big) \\ &= R^2\omega^2 + r^2(\omega + \dot{\theta})^2 + 2Rr\omega(\omega + \dot{\theta})\cos\theta \end{aligned} \tag{5.157}$$

There is no potential energy, so the Lagrangian is simply $L = mv^2/2$. The equation of motion is then, as you can show,

$$r\ddot{\theta} + R\omega^2\sin\theta = 0. \tag{5.158}$$

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$, and so eq. (5.158) tells us that the equilibrium is located at $\theta = 0$, which makes intuitive sense. (Another solution is $\theta = \pi$, but that's an unstable equilibrium.) A small-angle approximation in eq. (5.158) gives $\ddot{\theta} + (R/r)\omega^2\theta = 0$, so the frequency of small oscillations is $\Omega = \omega\sqrt{R/r}$.

REMARKS: If $R \ll r$, then $\Omega \approx 0$. This makes sense, because the frictionless hoop is essentially not moving. If $R = r$, then $\Omega = \omega$. If $R \gg r$, then $\Omega$ is very large. In this case, we can double-check the $\Omega = \omega\sqrt{R/r}$ result in the following way. In the accelerating frame of the hoop, the bead feels a centrifugal force (discussed in Chapter 9) of $m(R+r)\omega^2$. For all the bead knows, it is in a gravitational field with strength $g' \equiv (R+r)\omega^2$. So the bead (which acts like a pendulum of length $r$), oscillates with a frequency equal to

$$\sqrt{\frac{g'}{r}} = \sqrt{\frac{(R+r)\omega^2}{r}} \approx \omega\sqrt{\frac{R}{r}}\,, \tag{5.159}$$

for $R \gg r$.

Note that if we try to use this "effective gravity" argument as a double check for smaller values of $R$, we get the wrong answer. For example, if $R = r$, we obtain an oscillation frequency of $\omega\sqrt{2R/r}$, instead of the correct value $\omega\sqrt{R/r}$. This is because in reality the centrifugal force fans out near the equilibrium point, while our "effective gravity" argument assumes that the field lines are parallel (and so it gives a frequency that is too large). ♣

13. **Rotating curve**

The speed along the curve is $\dot{x}\sqrt{1+y'^2}$, and the speed perpendicular to the curve is $\omega x$. So the Lagrangian is

$$L = \frac{1}{2}m\Big(\omega^2 x^2 + \dot{x}^2(1+y'^2)\Big) - mgy, \tag{5.160}$$

where $y(x) = b(x/a)^\lambda$. The equation of motion is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \ddot{x}(1+y'^2) + \dot{x}^2 y' y'' = \omega^2 x - g y'. \tag{5.161}$$

Equilibrium occurs when $\dot{x} = \ddot{x} = 0$, so eq. (5.161) implies that the equilibrium value of $x$ satisfies

$$x_0 = \frac{g y'(x_0)}{\omega^2}\,. \tag{5.162}$$

The $F = ma$ explanation for this is that the component of gravity along the curve accounts for the component of the centripetal acceleration along the curve. Using $y(x) = b(x/a)^\lambda$, eq. (5.162) yields

$$x_0 = a\left(\frac{a^2\omega^2}{\lambda g b}\right)^{1/(\lambda-2)}. \tag{5.163}$$

As $\lambda \to \infty$, we see that $x_0$ goes to $a$. This makes sense, because the curve essentially equals zero up to $a$, and then it rises very steeply. You can check numerous other limits.

Letting $x \equiv x_0 + \delta$ in eq. (5.161), and expanding to first order in $\delta$, gives

$$\ddot{\delta}\Big(1 + y'(x_0)^2\Big) = \delta\Big(\omega^2 - g y''(x_0)\Big). \tag{5.164}$$

The frequency of small oscillations is therefore

$$\Omega^2 = \frac{g y''(x_0) - \omega^2}{1 + y'(x_0)^2}\,. \tag{5.165}$$

Using the explicit form of $y$, along with eq. (5.163), we find

$$\Omega^2 = \frac{(\lambda - 2)\omega^2}{1 + \frac{a^2\omega^4}{g^2}\left(\frac{a^2\omega^2}{\lambda g b}\right)^{2/(\lambda-2)}}. \tag{5.166}$$

We see that $\lambda$ must be greater than 2 in order for there to be oscillatory motion around the equilibrium point. For $\lambda < 2$, the equilibrium point is unstable, that is, to the left the force is inward, and to the right the force is outward.

For the case $\lambda = 2$, the equilibrium condition, eq. (5.162), gives $x_0 = (2gb/a^2\omega^2)x_0$. For this to be true for some $x_0$, we must have $\omega^2 = 2gb/a^2$. But if this holds, then eq. (5.162) is true for all $x$. So in the special case of $\lambda = 2$, the bead will happily sit anywhere on the curve if $\omega^2 = 2gb/a^2$. (In the rotating frame of the curve, the tangential components of the centrifugal and gravitational forces exactly cancel at all points.) If $\omega^2 \neq 2gb/a^2$, then the particle feels a force either always inward or always outward.

REMARKS: For $\omega \to 0$, eqs. (5.163) and (5.166) give $x_0 \to 0$ and $\Omega \to 0$. And for $\omega \to \infty$, they give $x_0 \to \infty$ and $\Omega \to 0$. In both cases $\Omega \to 0$, because in both case the equilibrium position is at a place where the curve is very flat (horizontally or vertically, respectively), so the restoring force is very small.

For $\lambda \to \infty$, we have $x_0 \to a$ and $\Omega \to \infty$. The frequency is large here because the equilibrium position at $a$ is where the curve has a sharp corner, so the restoring force changes quickly with position. Or, you can think of it as a pendulum with a very small length (if you approximate the "corner" by a tiny circle). ♣

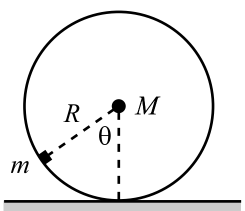


**Figure 5.39**

14. **Mass on a wheel**

Let the angle $\theta$ be defined as in Fig. 5.39, with the convention that $\theta$ is positive if $M$ is to the right of $m$. Then the position of $m$ in cartesian coordinates, relative to the point where $m$ would be in contact with the ground, is

$$(x, y)_m = R(\theta - \sin\theta, 1 - \cos\theta). \tag{5.167}$$

We have used the non-slipping condition to say that the present contact point is a distance $R\theta$ to the right of where $m$ would be in contact with the ground. Differentiating eq. (5.167), we find that the square of $m$'s speed is $v_m^2 = 2R^2\dot\theta^2(1 - \cos\theta)$.

The position of $M$ is $(x, y)_M = R(\theta, 1)$, so the square of its speed is $v_M^2 = R^2\dot\theta^2$. The Lagrangian is therefore

$$L = \frac{1}{2}MR^2\dot\theta^2 + mR^2\dot\theta^2(1 - \cos\theta) + mgR\cos\theta, \tag{5.168}$$

where we have measured both potential energies relative to the height of $M$. The equation of motion is

$$MR^2\ddot\theta + 2mR^2\ddot\theta(1 - \cos\theta) + mR^2\dot\theta^2\sin\theta + mgR\sin\theta = 0. \tag{5.169}$$

In the case of small oscillations, we may use $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$. The second and third terms above are third order in $\theta$ and may be neglected, so we find

$$\ddot\theta + \left(\frac{mg}{MR}\right)\theta = 0. \tag{5.170}$$

The frequency of small oscillations is therefore

$$\omega = \sqrt{\frac{m}{M}}\sqrt{\frac{g}{R}}. \tag{5.171}$$

REMARKS: If $M \gg m$, then $\omega \to 0$. This makes sense.

If $m \gg M$, then $\omega \to \infty$. This also makes sense, because the huge $mg$ force makes the situation similar to one where the wheel is bolted to the floor, in which case the wheel vibrates with a high frequency.

Eq. (5.171) can actually be derived in a much quicker way, using torque (which will be discussed in Chapter 7). For small oscillations, the gravitational force on $m$ produces a torque of $-mgR\theta$ around the contact point on the ground. For small $\theta$, $m$ has essentially no moment of inertia around the contact point, so the total moment of inertia is simply $MR^2$. Therefore, $\tau = I\alpha$ gives $-mgR\theta = MR^2\ddot{\theta}$, from which the result follows. ♣

15. **Double pendulum**

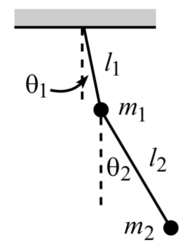


**Figure 5.40**

Relative to the pivot point, the cartesian coordinates of $m_1$ and $m_2$ are, respectively (see Fig. 5.40),

$$\begin{aligned}(x,y)_1 &= (\ell_1 \sin\theta_1, -\ell_1\cos\theta_1),\\ (x,y)_2 &= (\ell_1\sin\theta_1 + \ell_2\sin\theta_2, -\ell_1\cos\theta_1 - \ell_2\cos\theta_2).\end{aligned} \tag{5.172}$$

Taking the derivative to find the velocities, and then squaring, gives

$$\begin{aligned}v_1^2 &= \ell_1^2\dot{\theta}_1^2,\\ v_2^2 &= \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2).\end{aligned} \tag{5.173}$$

The Lagrangian is therefore

$$\begin{aligned}L &= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\Big)\\ &\quad + m_1 g\ell_1\cos\theta_1 + m_2 g(\ell_1\cos\theta_1 + \ell_2\cos\theta_2).\end{aligned} \tag{5.174}$$

The equations of motion obtained from varying $\theta_1$ and $\theta_2$ are

$$\begin{aligned}0 &= (m_1+m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)\\ &\quad +(m_1+m_2)g\ell_1\sin\theta_1,\\ 0 &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)\\ &\quad + m_2 g\ell_2\sin\theta_2.\end{aligned} \tag{5.175}$$

This is a bit of a mess, but it simplifies greatly if we consider small oscillations. Using the small-angle approximations and keeping only the leading-order terms, we obtain

$$\begin{aligned}0 &= (m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1,\\ 0 &= \ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1 + g\theta_2.\end{aligned} \tag{5.176}$$

Consider now the special case, $\ell_1 = \ell_2 \equiv \ell$. We can find the frequencies of the normal modes by using the determinant method, discussed in Section 3.5. You can show that the result is

$$\omega_\pm = \sqrt{\frac{m_1+m_2 \pm \sqrt{m_1m_2+m_2^2}}{m_1}}\sqrt{\frac{g}{\ell}}. \tag{5.177}$$

The normal modes are found to be, after some simplification,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp\sqrt{m_2} \\ \sqrt{m_1+m_2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.178}$$

Some special cases are:

- $m_1 = m_2$: The frequencies are

$$\omega_{\pm} = \sqrt{2 \pm \sqrt{2}}\sqrt{\frac{g}{\ell}}\,. \tag{5.179}$$

  The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.180}$$

- $m_1 \gg m_2$: With $m_2/m_1 \equiv \epsilon$, the frequencies are (to leading nontrivial order in $\epsilon$)

$$\omega_{\pm} = (1 \pm \sqrt{\epsilon}/2)\sqrt{\frac{g}{\ell}}\,. \tag{5.181}$$

  The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp\sqrt{\epsilon} \\ 1 \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.182}$$

  In both modes, the upper (heavy) mass essentially stands still, and the lower (light) mass oscillates like a pendulum of length $\ell$.

- $m_1 \ll m_2$: With $m_1/m_2 \equiv \epsilon$, the frequencies are (to leading order in $\epsilon$)

$$\omega_{+} = \sqrt{\frac{2g}{\epsilon\ell}}\,, \qquad \omega_{-} = \sqrt{\frac{g}{2\ell}}\,. \tag{5.183}$$

  The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp 1 \\ 1 \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.184}$$

  In the first mode, the lower (heavy) mass essentially stands still, and the upper (light) mass vibrates back and forth at a high frequency (because there is a very large tension in the rods). In the second mode, the rods form a straight line, and the system is essentially a pendulum of length $2\ell$.

Consider now the special case, $m_1 = m_2$. Using the determinant method, you can show that the frequencies of the normal modes are

$$\omega_{\pm} = \sqrt{g}\sqrt{\frac{\ell_1 + \ell_2 \pm \sqrt{\ell_1^2 + \ell_2^2}}{\ell_1 \ell_2}}\,. \tag{5.185}$$

The normal modes are found to be, after some simplification,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \ell_2 \\ \ell_2 - \ell_1 \mp \sqrt{\ell_1^2 + \ell_2^2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.186}$$

Some special cases are:

- $\ell_1 = \ell_2$: We already considered this case above. You show that eqs. (5.185) and (5.186) agree with eqs. (5.179) and (5.180), respectively.
- $\ell_1 \gg \ell_2$: With $\ell_2/\ell_1 \equiv \epsilon$, the frequencies are (to leading order in $\epsilon$)

$$\omega_+ = \sqrt{\frac{2g}{\ell_2}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_1}}. \tag{5.187}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} -\epsilon \\ 2 \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.188}$$

In the first mode, the masses essentially move equal distances in opposite directions, at a very high frequency (because $\ell_2$ is so small). In the second mode, the rods form a straight line, and the masses move just like a mass of $2m$. The system is essentially a pendulum of length $\ell$.

- $\ell_1 \ll \ell_2$: With $\ell_1/\ell_2 \equiv \epsilon$, the frequencies are (to leading order in $\epsilon$)

$$\omega_+ = \sqrt{\frac{2g}{\ell_1}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_2}}. \tag{5.189}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.190}$$

In the first mode, the bottom mass essentially stands still, and the top mass oscillates at a very high frequency (because $\ell_1$ is so small). The factor of 2 in the frequency arises because the top mass essentially lives in a world where the acceleration from gravity is $g' = 2g$ (because of the extra $mg$ force downward from the lower mass). In the second mode, the system is essentially a pendulum of length $\ell_2$. The factor of 2 in the angles is what is needed to make the tangential force on the top mass roughly equal to zero (because otherwise it would oscillate at a high frequency, since $\ell_1$ is so small).

16. **Pendulum with a free support**

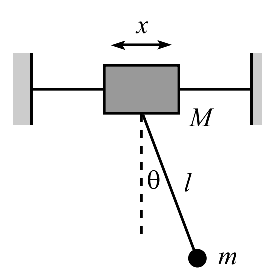



**Figure 5.41**

Let $x$ be the coordinate of $M$, and let $\theta$ be the angle of the pendulum (see Fig. 5.41). Then the position of the mass $m$ in cartesian coordinates is $(x + \ell \sin\theta, -\ell\cos\theta)$. Taking the derivative to find the velocity, and then squaring to find the speed, gives $v_m^2 = \dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.191}$$

The equations of motion from obtained varying $x$ and $\theta$ are

$$\begin{aligned} (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= 0, \\ \ell\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta &= 0. \end{aligned} \tag{5.192}$$

If $\theta$ is small, we can use the small angle approximations, $\cos\theta \approx 1-\theta^2/2$ and $\sin\theta \approx \theta$. Keeping only the terms that are first-order in $\theta$, we obtain

$$\begin{aligned}(M+m)\ddot{x} + m\ell\ddot{\theta} &= 0,\\ \ddot{x} + \ell\ddot{\theta} + g\theta &= 0.\end{aligned} \tag{5.193}$$

The first equation expresses momentum conservation. Integrating it twice gives

$$x = -\left(\frac{m\ell}{M+m}\right)\theta + At + B. \tag{5.194}$$

The second equation is $F = ma$ in the tangential direction. Eliminating $\ddot{x}$ from eqs. (5.193) gives

$$\ddot{\theta} + \left(\frac{M+m}{M}\right)\frac{g}{\ell}\,\theta = 0. \tag{5.195}$$

The solution to this equation is $\theta(t) = C\cos(\omega t + \phi)$, where

$$\omega = \sqrt{1+\frac{m}{M}}\sqrt{\frac{g}{\ell}}\,. \tag{5.196}$$

The general solutions for $\theta$ and $x$ are therefore

$$\theta(t) = C\cos(\omega t + \phi), \qquad x(t) = -\frac{Cm\ell}{M+m}\cos(\omega t + \phi) + At + B. \tag{5.197}$$

The constant $B$ is irrelevant, so we'll ignore it. The two normal modes are:

- $A = 0$: In this case, $x = -\theta m\ell/(M+m)$. Both masses oscillate with the frequency $\omega$ given in eq. (5.196), always moving in opposite directions. The center of mass does not move.
- $C = 0$: In this case, $\theta = 0$ and $x = At$. The pendulum hangs vertically, with both masses moving horizontally at the same speed. The frequency of oscillations is zero in this mode.

REMARKS: If $M \gg m$, then $\omega = \sqrt{g/\ell}$, as expected, because the support essentially stays still.

If $m \gg M$, then $\omega \to \sqrt{m/M}\sqrt{g/\ell} \to \infty$. This makes sense, because the tension in the rod is so large. We can actually be quantitative about this limit. For small oscillations and for $m \gg M$, the tension of $mg$ in the rod produces a sideways force of $mg\theta$ on $M$. So the horizontal $F = Ma$ equation for $M$ is $mg\theta = M\ddot{x}$, But $x \approx -\ell\theta$ in this limit, so we have $mg\theta = -M\ell\ddot{\theta}$, from which the result follows. ♣

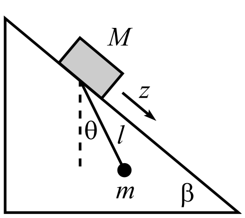


**Figure 5.42**

17. **Pendulum support on an inclined plane**

Let $z$ be the coordinate of $M$ along the plane, and let $\theta$ be the angle of the pendulum (see Fig. 5.42). In cartesian coordinates, the positions of $M$ and $m$ are

$$\begin{aligned}(x,y)_M &= (z\cos\beta, -z\sin\beta),\\ (x,y)_m &= (z\cos\beta + \ell\sin\theta, -z\sin\beta - \ell\cos\theta).\end{aligned} \tag{5.198}$$

Differentiating these positions, we find that the squares of the speeds are

$$\begin{aligned}v_M^2 &= \dot{z}^2,\\ v_m^2 &= \dot{z}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{z}\dot{\theta}(\cos\beta\cos\theta - \sin\beta\sin\theta).\end{aligned} \tag{5.199}$$

The Lagrangian is therefore

$$\frac{1}{2}M\dot{z}^2+\frac{1}{2}m\Big(\dot{z}^2+\ell^2\dot{\theta}^2+2\ell\dot{z}\dot{\theta}\cos(\theta+\beta)\Big)+Mgz\sin\beta+mg(z\sin\beta+\ell\cos\theta). \quad (5.200)$$

The equations of motion obtained from varying $z$ and $\theta$ are

$$\begin{aligned}(M+m)\ddot{z}+m\ell\Big(\ddot{\theta}\cos(\theta+\beta)-\dot{\theta}^2\sin(\theta+\beta)\Big) &= (M+m)g\sin\beta,\\ \ell\ddot{\theta}+\ddot{z}\cos(\theta+\beta) &= -g\sin\theta. \end{aligned} \quad (5.201)$$

Let us now consider small oscillations about the equilibrium point (where $\ddot{\theta}=\dot{\theta}=0$). We must first determine where this point is. The first equation above gives $\ddot{z}=g\sin\beta$. The second equation then gives $g\sin\beta\cos(\theta+\beta)=-g\sin\theta$. By expanding the cosine term, we find $\tan\theta=-\tan\beta$, so $\theta=-\beta$. ($\theta=\pi-\beta$ is also a solution, but this is an unstable equilibrium.) The equilibrium position of the pendulum is therefore where the string is perpendicular to the plane.[15]

To find the normal modes and frequencies of small oscillations, let $\theta\equiv-\beta+\delta$, and expand eqs. (5.201) to first order in $\delta$. Letting $\ddot{\eta}\equiv\ddot{z}-g\sin\beta$ for convenience, we have

$$\begin{aligned}(M+m)\ddot{\eta}+m\ell\ddot{\delta} &= 0,\\ \ddot{\eta}+\ell\ddot{\delta}+(g\cos\beta)\delta &= 0.\end{aligned} \quad (5.202)$$

Using the determinant method (or using the method in the previous problem; either way works), the frequencies of the normal modes are found to be

$$\omega_1=0, \qquad \text{and} \qquad \omega_2=\sqrt{1+\frac{m}{M}}\sqrt{\frac{g\cos\beta}{\ell}}\,. \quad (5.203)$$

These are the same as the frequencies in Problem 16 (where $M$ moves horizontally), but with $g\cos\beta$ in place of $g$.[16] (Compare eqs. (5.202) with eqs. (5.193).) Looking at eq. (5.197), and recalling the definition of $\eta$, we see that the general solutions for $\theta$ and $z$ are

$$\theta(t)=-\beta+C\cos(\omega t+\phi), \qquad z(t)=-\frac{Cm\ell}{M+m}\cos(\omega t+\phi)+\frac{g\sin\beta}{2}t^2+At+B. \quad (5.204)$$

The constant $B$ is irrelevant, so we'll ignore it. The basic difference between these normal modes and the ones in Problem 16 is the acceleration down the plane. If you go to a frame that accelerates down the plane at $g\sin\beta$, and if you tilt your head at an angle $\beta$ and accept the fact that $g'=g\cos\beta$ in your world, then the setup becomes identical to the one in Problem 16.

---

[15] This makes sense. Because the tension in the string is perpendicular to the plane, for all the pendulum bob knows, it may as well simply be sliding down a plane parallel to the given one, a distance $\ell$ away. Given the same initial speed, the two masses will slide down their two "planes" with equal speeds at all times.

[16] This makes sense, because in a frame that accelerates down the plane at $g\sin\beta$, the only external force on the masses is an effective gravity force of $g\cos\beta$ perpendicular to the plane. As far as $M$ and $m$ are concerned, they live in a world where gravity pulls "downward" (perpendicular to the plane) with strength $g'=g\cos\beta$.

18. **Tilting plane**

Relative to the support, the positions of the masses are

$$\begin{aligned} (x,y)_M &= (\ell\sin\theta, -\ell\cos\theta), \\ (x,y)_m &= (\ell\sin\theta + x\cos\theta, -\ell\cos\theta + x\sin\theta). \end{aligned} \tag{5.205}$$

Differentiating these positions, we find that the squares of the speeds are

$$v_M^2 = \ell^2\dot\theta^2, \qquad v_m^2 = (\ell\dot\theta + \dot x)^2 + x^2\dot\theta^2. \tag{5.206}$$

You can also obtain $v_m^2$ by noting that $(\ell\dot\theta + \dot x)$ is the speed along the long rod, and $x\dot\theta$ is the speed perpendicular to it. The Lagrangian is

$$L = \frac{1}{2}M\ell^2\dot\theta^2 + \frac{1}{2}m\Big((\ell\dot\theta + \dot x)^2 + x^2\dot\theta^2\Big) + Mg\ell\cos\theta + mg(\ell\cos\theta - x\sin\theta). \tag{5.207}$$

The equations of motion obtained from varying $x$ and $\theta$ are

$$\begin{aligned} \ell\ddot\theta + \ddot x &= x\dot\theta^2 - g\sin\theta, \\ M\ell^2\ddot\theta + m\ell(\ell\ddot\theta + \ddot x) + mx^2\ddot\theta + 2mx\dot x\dot\theta &= -(M+m)g\ell\sin\theta - mgx\cos\theta. \end{aligned} \tag{5.208}$$

Let us now consider the case where both $x$ and $\theta$ are small (or more precisely, $\theta \ll 1$ and $x/\ell \ll 1$). Expanding eqs. (5.208) to first order in $\theta$ and $x/\ell$ gives

$$\begin{aligned} (\ell\ddot\theta + \ddot x) + g\theta &= 0, \\ M\ell(\ell\ddot\theta + g\theta) + m\ell(\ell\ddot\theta + \ddot x) + mg\ell\theta + mgx &= 0. \end{aligned} \tag{5.209}$$

We can simplify these a bit. Using the first equation to substitute $-g\theta$ for $(\ell\ddot\theta + \ddot x)$, and also $-\ddot x$ for $(\ell\ddot\theta + g\theta)$, in the second equation gives

$$\begin{aligned} \ell\ddot\theta + \ddot x + g\theta &= 0, \\ -M\ell\ddot x + mgx &= 0. \end{aligned} \tag{5.210}$$

The normal modes can be found using the determinant method, or we can find them just by inspection. The second equation says that either $x(t) \equiv 0$, or $x(t) = A\cosh(\alpha t + \beta)$, where $\alpha = \sqrt{mg/M\ell}$. So we have two cases:

- If $x(t) = 0$, then the first equation in (5.210) says that the normal mode is

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(\omega t + \phi), \tag{5.211}$$

  where $\omega \equiv \sqrt{g/\ell}$. This mode is fairly clear. With the proper initial conditions, $m$ will stay right where $M$ is. The normal force from the long rod will be exactly what is needed in order for $m$ to undergo the same oscillatory motion as $M$.

- If $x(t) = A\cosh(\alpha t + \beta)$, then the first equation in (5.210) can be solved to give the normal mode,

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = C\begin{pmatrix} -m \\ \ell(M+m) \end{pmatrix}\cosh(\alpha t + \beta), \tag{5.212}$$

  where $\alpha = \sqrt{mg/M\ell}$. This mode is not as clear. And indeed, its range of validity is rather limited. The exponential behavior will quickly make $x$ and $\theta$ large, and thus outside the validity of our small-variable approximations. You can show that in this mode the center of mass remains fixed, directly below the pivot. This can occur, for example, by having $m$ move down to the right as the rods rotate and swing $M$ up to the left. There is no oscillation in this mode; the positions keep growing.

19. **Motion in a cone**

If the particle's distance from the axis is $r$, then its height is $r/\tan\alpha$, and it's distance up along the cone is $r/\sin\alpha$. Breaking the velocity into components up along the cone and around the cone, we see that the square of the speed is $v^2 = \dot{r}^2/\sin^2\alpha + r^2\dot{\theta}^2$. The Lagrangian is therefore

$$L = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}\,. \tag{5.213}$$

The equations of motion obtained from varying $\theta$ and $r$ are

$$\begin{aligned}\frac{d}{dt}(mr^2\dot{\theta}) &= 0\\ \ddot{r} &= r\dot{\theta}^2\sin^2\alpha - g\cos\alpha\sin\alpha.\end{aligned} \tag{5.214}$$

The first of these equations expresses conservation of angular momentum. The second equation is more transparent if we divide through by $\sin\alpha$. With $x \equiv r/\sin\alpha$ being the distance up along the cone, we have $\ddot{x} = (r\dot{\theta}^2)\sin\alpha - g\cos\alpha$. This is the $F = ma$ statement for the "$x$" direction.

Letting $mr^2\dot{\theta} \equiv L$, we may eliminate $\dot{\theta}$ from the second equation to obtain

$$\ddot{r} = \frac{L^2\sin^2\alpha}{m^2r^3} - g\cos\alpha\sin\alpha. \tag{5.215}$$

We will now calculate the two desired frequencies.

- Frequency of circular oscillations, $\omega$: For circular motion with $r = r_0$, we have $\dot{r} = \ddot{r} = 0$, so the second of eqs. (5.214) gives

$$\omega \equiv \dot{\theta} = \sqrt{\frac{g}{r_0\tan\alpha}}\,. \tag{5.216}$$

- Frequency of oscillations about a circle, $\Omega$: If the orbit were actually the circle $r = r_0$, then eq. (5.215) would give (with $\ddot{r} = 0$)

$$\frac{L^2\sin^2\alpha}{m^2r_0^3} = g\cos\alpha\sin\alpha. \tag{5.217}$$

This is equivalent to eq. (5.216), which can be seen by writing $L$ as $mr_0^2\dot{\theta}$.

We will now use our standard procedure of letting $r(t) = r_0 + \delta(t)$, where $\delta(t)$ is very small, and then plugging this into eq. (5.215) and expanding to first order in $\delta$. Using

$$\frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.218}$$

we have

$$\ddot{\delta} = \frac{L^2\sin^2\alpha}{m^2r_0^3}\left(1 - \frac{3\delta}{r_0}\right) - g\cos\alpha\sin\alpha. \tag{5.219}$$

Recalling eq. (5.217), we obtain a bit of cancellation and are left with

$$\ddot{\delta} = -\left(\frac{3L^2\sin^2\alpha}{m^2r_0^4}\right)\delta. \tag{5.220}$$

Using eq. (5.217) again to eliminate $L$ we have

$$\ddot{\delta} + \left(\frac{3g}{r_0}\sin\alpha\cos\alpha\right)\delta = 0. \tag{5.221}$$

Therefore,

$$\Omega = \sqrt{\frac{3g}{r_0}\sin\alpha\cos\alpha}. \tag{5.222}$$

Having found the two desired frequencies in eqs. (5.216) and (5.222), we see that their ratio is

$$\frac{\Omega}{\omega} = \sqrt{3}\sin\alpha. \tag{5.223}$$

This ratio $\Omega/\omega$ is independent of $r_0$.

The two frequencies are equal if $\sin\alpha = 1/\sqrt{3}$, that is, if $\alpha \approx 35.3^\circ \equiv \tilde{\alpha}$. If $\alpha = \tilde{\alpha}$, then after one revolution around the cone, $r$ returns to the value it had at the beginning of the revolution. So the particle undergoes periodic motion.

Remarks: In the limit $\alpha \to 0$, eq. (5.223) says that $\Omega/\omega \to 0$. In fact, eqs. (5.216) and (5.222) say that $\omega \to \infty$ and $\Omega \to 0$. So the particle spirals around many times during one complete $r$ cycle. This seems intuitive.

In the limit $\alpha \to \pi/2$ (that is, the cone is almost a flat plane) both $\omega$ and $\Omega$ go to zero, and eq. (5.223) says that $\Omega/\omega \to \sqrt{3}$. This result is not at all obvious (at least to me).

If $\Omega/\omega = \sqrt{3}\sin\alpha$ is a rational number, then the particle will undergo periodic motion. For example, if $\alpha = 60^\circ$, then $\Omega/\omega = 3/2$, so it takes two complete circles for $r$ to go through three cycles. Or, if $\alpha = \arcsin(1/2\sqrt{3}) \approx 16.8^\circ$, then $\Omega/\omega = 1/2$, so it takes two complete circles for $r$ to go through one cycle.

20. **Shortest distance in a plane**

Let the two given points be $(x_1, y_1)$ and $(x_2, y_2)$, and let the path be described by the function $y(x)$. (Yes, we'll assume it can be written as a function. Locally, we don't have to worry about any double-valued issues.) Then the length of the path is

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx. \tag{5.224}$$

The "Lagrangian" is $L = \sqrt{1 + y'^2}$, so the Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) &= \frac{\partial L}{\partial y} \\ \Longrightarrow \qquad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) &= 0. \end{aligned} \tag{5.225}$$

We see that $y'/\sqrt{1+y'^2}$ is constant. Therefore, $y'$ is also constant, so we have a straight line $y(x) = Ax + B$, where $A$ and $B$ are determined from the endpoint conditions.

21. **Index of refraction**

Let the path be described by $y(x)$. The speed at height $y$ is $v \propto y$. Therefore, the time to go from $(x_0, y_0)$ to $(x_1, y_1)$ is

$$T = \int_{x_0}^{x_1} \frac{ds}{v} \propto \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx. \tag{5.226}$$

Our goal is to find the function $y(x)$ that minimizes this integral, subject to the boundary conditions above. We can therefore apply the results of the variational technique, with a "Lagrangian" equal to

$$L \propto \frac{\sqrt{1+y'^2}}{y}\,. \tag{5.227}$$

At this point, we could apply the E-L equation to this $L$, but let's simply use Lemma 5.5, with $f(y) = 1/y$. Eq. (5.83) gives

$$1 + y'^2 = Bf(y)^2 \qquad \Longrightarrow \qquad 1 + y'^2 = \frac{B}{y^2}\,. \tag{5.228}$$

We must now integrate this. Solving for $y'$, and then separating variables and integrating, gives

$$\int dx = \pm \int \frac{y\,dy}{\sqrt{B-y^2}} \qquad \Longrightarrow \qquad x + A = \mp\sqrt{B-y^2}\,. \tag{5.229}$$

Therefore, $(x+A)^2 + y^2 = B$, which is the equation for a circle. Note that the circle is centered at $y = 0$, that is, at a point on the bottom of the slab. This is the point where the perpendicular bisector of the line joining the two given points intersects the bottom of the slab.

22. **The Brachistochrone**

**First solution:** In Fig. 5.43, the boundary conditions are $y(0) = 0$ and $y(x_0) = y_0$, with downward taken to be the positive $y$ direction. From conservation of energy, the speed as a function of $y$ is $v = \sqrt{2gy}$. The total time is therefore

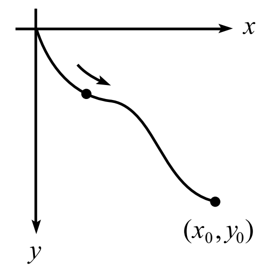



**Figure 5.43**

$$T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx. \tag{5.230}$$

Our goal is to find the function $y(x)$ that minimizes this integral, subject to the boundary conditions above. We can therefore apply the results of the variational technique, with a "Lagrangian" equal to

$$L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,. \tag{5.231}$$

At this point, we could apply the E-L equation to this $L$, but let's simply use Lemma 5.5, with $f(y) = 1/\sqrt{y}$. Eq. (5.83) gives

$$1 + y'^2 = Cf(y)^2 \qquad \Longrightarrow \qquad 1 + y'^2 = \frac{C}{y}\,, \tag{5.232}$$

as desired. We must now integrate one more time. Solving for $y'$ and separating variables gives

$$\frac{\sqrt{y}\,dy}{\sqrt{B-y}} = \pm\,dx. \tag{5.233}$$

A helpful change of variables to get rid of the square root in the denominator is $y \equiv B\sin^2\phi$. Then $dy = 2B\sin\phi\cos\phi\;d\phi$, and eq. (5.233) simplifies to

$$2B\sin^2\phi\;d\phi = \pm\,dx. \tag{5.234}$$

We can now make use of the relation $\sin^2\phi = (1-\cos 2\phi)/2$ to integrate this. The result is $B(2\phi - \sin 2\phi) = \pm\, 2x - C$, where $C$ is an integration constant.

Now note that we may rewrite our definition of $\phi$ (which was $y \equiv B\sin^2\phi$) as $2y = B(1-\cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have

$$x = \pm\, a(\theta - \sin\theta) \pm d, \qquad y = a(1-\cos\theta). \tag{5.235}$$

where $a \equiv B/2$, and $d \equiv C/2$.

The particle starts at $(x, y) = (0, 0)$. Therefore, $\theta$ starts at $\theta = 0$, since this corresponds to $y = 0$. The starting condition $x = 0$ then implies that $d = 0$. Also, we are assuming that the wire heads down to the right, so we choose the positive sign in the expression for $x$. Therefore, we finally have

$$x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta), \tag{5.236}$$

as desired. This is the parametrization of a *cycloid*, which is the path taken by a point on the rim of a rolling wheel. The initial slope of the $y(x)$ curve is infinite, as you can check.

REMARK: The above method derived the parametric form in (5.236) from scratch. But since eq. (5.236) was given in the statement of the problem, another route is to simply verify that this parametrization satisfies eq. (5.232). To this end, assume that $x = a(\theta - \sin\theta)$ and $y = a(1-\cos\theta)$, which gives

$$y' \equiv \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}\,. \tag{5.237}$$

Therefore,

$$1 + y'^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta} = \frac{2a}{y}\,, \tag{5.238}$$

which agrees with eq. (5.232), with $C \equiv 2a$. ♣

**Second solution:** Let's use a variational argument again, but now with $y$ as the independent variable. That is, let the chain be described by the function $x(y)$. The arclength is now given by $ds = \sqrt{1+x'^2}\,dy$. Therefore, instead of the Lagrangian in eq. (5.231), we now have

$$L \propto \frac{\sqrt{1+x'^2}}{\sqrt{y}}\,. \tag{5.239}$$

The Euler-Lagrange equation is

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad \frac{d}{dy}\left(\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}}\right) = 0. \tag{5.240}$$

The zero on the right-hand side makes things nice and easy, because it means that the quantity in parentheses is a constant. Call it $D$. We then have

$$\begin{aligned} \frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}} = D \qquad &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{dx/dy}{\sqrt{1+(dx/dy)^2}} = D \\ &\Longrightarrow \qquad \frac{1}{\sqrt{y}}\frac{1}{\sqrt{(dy/dx)^2+1}} = D. \end{aligned} \tag{5.241}$$

This is equivalent to eq. (5.232), and the solution proceeds as above.

**Third solution:** Note that the "Lagrangian" in the first solution above, which is given in eq. (5.231) as

$$L = \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,, \tag{5.242}$$

is independent of $x$. Therefore, in analogy with conservation of energy (which arises from a Lagrangian that is independent of $t$), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = \frac{-1}{\sqrt{y}\sqrt{1+y'^2}} \tag{5.243}$$

is a constant (that is, independent of $x$). We have therefore again reproduced eq. (5.232), and the solution proceeds as above.

23. **Minimal surface**

The tension throughout the surface is constant, because it is in equilibrium. (By "tension" in a surface, we mean the force per unit length in the surface.) The ratio of the circumferences of the circular boundaries of the ring is $y_2/y_1$. Therefore, the condition that the horizontal forces on the ring cancel is $y_1\cos\theta_1 = y_2\cos\theta_2$, where the $\theta$'s are the angles of the surface, as shown in Fig. 5.44. In other words, $y\cos\theta$ is constant throughout the surface. But $\cos\theta = 1/\sqrt{1+y'^2}$, so we have

$$\frac{y}{\sqrt{1+y'^2}} = C. \tag{5.244}$$

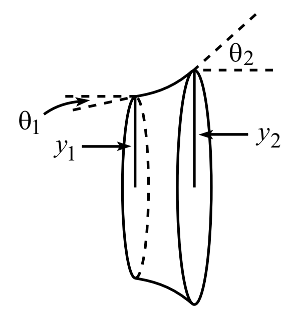


**Figure 5.44**

This is the same as eq. (5.75), and the solution proceeds as in Section 5.8.

24. **Existence of a minimal surface**

The general solution for $y(x)$ is given in eq. (5.76) as

$$y(x) = \frac{1}{b}\cosh b(x+d). \tag{5.245}$$

If we choose the origin to be midway between the rings, then $d = 0$. Both boundary condition are thus

$$r = \frac{1}{b}\cosh b\ell. \tag{5.246}$$

Let us now determine the maximum value of $\ell/r$ for which the minimal surface exists. If $\ell/r$ is too large, then we will see that there is no solution for $b$ in eq. (5.246); in short, the minimal "surface" turns out to be the two given circles, attached by a line, which isn't a nice two-dimensional surface. If you perform an experiment with soap bubbles (which want to minimize their area), and if you pull the rings too far apart, then the surface will break and disappear, as it tries to form the two circles.

Define the dimensionless quantities,

$$\eta \equiv \frac{\ell}{r}\,, \qquad \text{and} \qquad z \equiv br. \tag{5.247}$$

Then eq. (5.246) becomes

$$z = \cosh\eta z. \tag{5.248}$$

If we make a rough plot of the graphs of $w = z$ and $w = \cosh\eta z$ for a few values of $\eta$ (see Fig. 5.45), we see that there is no solution for $z$ if $\eta$ is too large. The limiting value of $\eta$ for which there exists a solution occurs when the curves $w = z$

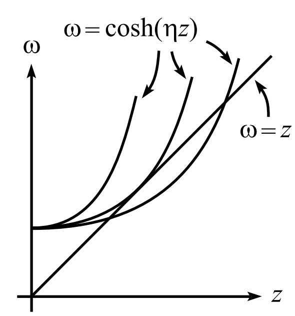


**Figure 5.45**

and $w = \cosh \eta z$ are tangent; that is, when the slopes are equal in addition to the functions being equal. Let $\eta_0$ be the limiting value of $\eta$, and let $z_0$ be the place where the tangency occurs. Then equality of the values and the slopes gives

$$z_0 = \cosh(\eta_0 z_0), \qquad \text{and} \qquad 1 = \eta_0 \sinh(\eta_0 z_0). \tag{5.249}$$

Dividing the second of these equations by the first gives

$$1 = (\eta_0 z_0) \tanh(\eta_0 z_0). \tag{5.250}$$

This must be solved numerically. The solution is

$$\eta_0 z_0 \approx 1.200. \tag{5.251}$$

Plugging this into the second of eqs. (5.249) gives

$$\left(\frac{\ell}{r}\right)_{\max} \equiv \eta_0 \approx 0.663. \tag{5.252}$$

Note also that $z_0 = 1.200/\eta_0 = 1.810$. We see that if $\ell/r$ is larger than 0.663, then there is no solution for $y(x)$ that is consistent with the boundary conditions. Above this value of $\ell/r$, the soap bubble minimizes its area by heading toward the shape of just two disks, but it will pop well before it reaches that configuration.

REMARKS:

(a) We glossed over one issue above, namely that there may be more than one solution for the constant $b$ in eq. (5.246). In fact, Fig. 5.45 shows that for any $\eta < 0.663$, there are two solutions for $z$ in eq. (5.248), and hence two solutions for $b$ in eq. (5.246). This means that there are two possible surfaces that might solve our problem. Which one do we want? It turns out that the surface corresponding to the smaller value of $b$ is the one that minimizes the area, while the surface corresponding to the larger value of $b$ is the one that (in some sense) maximizes the area.

We say "in some sense" because the large-$b$ surface is actually a saddle point for the area. It can't be a maximum, after all, because we can always make the area larger by adding little wiggles to it. It's a saddle point because there does exist a class of variations for which it has the maximum area, namely ones where the "dip" in the curve is continuously made larger (just imagine lowering the midpoint in a smooth manner). This surface arises because the Euler-Lagrange technique simply sets the "derivative" equal to zero and doesn't differentiate between maxima, minima, and saddle points.

(b) How does the area of the limiting surface (with $\eta_0 = 0.663$) compare with the area of the two circles? The area of the two circles is

$$A_{\text{c}} = 2\pi r^2. \tag{5.253}$$

The area of the limiting surface is

$$A_{\text{s}} = \int_{-\ell}^{\ell} 2\pi y \sqrt{1 + y'^2}\, dx. \tag{5.254}$$

Using eq. (5.246), this becomes

$$\begin{aligned} A_{\text{s}} &= \int_{-\ell}^{\ell} \frac{2\pi}{b} \cosh^2 bx\, dx \\ &= \int_{-\ell}^{\ell} \frac{\pi}{b}(1 + \cosh 2bx)\, dx \\ &= \frac{2\pi\ell}{b} + \frac{\pi \sinh 2b\ell}{b^2}\,. \end{aligned} \tag{5.255}$$

But from the definitions of $\eta$ and $z$, we have $\ell = \eta_0 r$ and $b = z_0/r$ for the limiting surface. Therefore, $A_s$ can be written as

$$A_{\rm s} = \pi r^2 \left( \frac{2\eta_0}{z_0} + \frac{\sinh 2\eta_0 z_0}{z_0^2} \right). \tag{5.256}$$

Plugging in the numerical values ($\eta_0 \approx 0.663$ and $z_0 \approx 1.810$) gives

$$A_{\rm c} \approx (6.28) r^2, \qquad \text{and} \qquad A_{\rm s} \approx (7.54) r^2. \tag{5.257}$$

The ratio of $A_s$ to $A_c$ is approximately 1.2 (it's actually $\eta_0 z_0$, as you can show). The limiting surface therefore has a larger area. This is expected, of course, because for $\ell/r > \eta_0$ the surface tries to run off to one with a smaller area, and there are no other stable configurations besides the cosh solution we found.

# Chapter 6

# Central Forces



A *central force* is by definition a force that points radially and whose magnitude depends only on the distance from the source (that is, not on the angle around the source).[1] Equivalently, we may say that a central force is one whose potential depends only on the distance from the source. That is, if the source is located at the origin, then the potential energy is of the form $V(\mathbf{r}) = V(r)$. Such a potential does indeed yield a central force, because

$$\mathbf{F}(\mathbf{r}) = -\nabla V(r) = -\frac{dV}{dr}\hat{\mathbf{r}}, \tag{6.1}$$

which points radially and depends only on $r$. Gravitational and electrostatic forces are central forces, with $V(r) \propto 1/r$. The spring force is also central, with $V(r) \propto (r-\ell)^2$, where $\ell$ is the equilibrium length.

There are two important facts concerning central forces: (1) they are ubiquitous in nature, so we had better learn how to deal with them, and (2) dealing with them is much easier than you might think, because crucial simplifications occur in the equations of motion when $V$ is a function of $r$ only. These simplifications will become evident in the following two sections.

## 6.1 Conservation of angular momentum

Angular momentum plays a key role in dealing with central forces because, as we will show, it is constant in time. For a point mass, we define the *angular momentum*, $\mathbf{L}$, by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{6.2}$$

The vector $\mathbf{L}$ depends, of course, on where you pick the origin of your coordinate system. Note that $\mathbf{L}$ is a vector, and that it is orthogonal to both $\mathbf{r}$ and $\mathbf{p}$, by nature of the cross product. You might wonder why we care enough about $\mathbf{r} \times \mathbf{p}$ to give it

[1] Taken literally, the term "central force" would imply only the radial nature of the force. But a physicist's definition also includes the dependence solely on the distance from the source.

a name. Why not look at $r^3 p^5 \mathbf{r} \times (\mathbf{r} \times \mathbf{p})$, or something else? The answer is that there are some very nice facts concerning $\mathbf{L}$, one of which is the following.[2]

**Theorem 6.1** *If a particle is subject to a central force only, then its angular momentum is conserved. That is,*

$$\text{If} \quad V(\mathbf{r}) = V(r), \qquad \text{then} \quad \frac{d\mathbf{L}}{dt} = 0. \tag{6.3}$$

**Proof:** We have

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \mathbf{F} \\ &= 0, \end{aligned} \tag{6.4}$$

because $\mathbf{F} \propto \mathbf{r}$, and the cross product of two parallel vectors is zero. ■

We will prove this theorem again in the next section, using the Lagrangian method. Let's now prove another theorem which is probably obvious, but good to show anyway.

**Theorem 6.2** *If a particle is subject to a central force only, then its motion takes place in a plane.*

**Proof:** At a given instant, $t_0$, consider the plane, $P$, containing the position vector $\mathbf{r}_0$ (with the source of the potential taken to be the origin) and the velocity vector $\mathbf{v}_0$. We claim that $\mathbf{r}$ lies in $P$ at all times.[3]

$P$ is defined as the plane orthogonal to the vector $\mathbf{n}_0 \equiv \mathbf{r}_0 \times \mathbf{v}_0$. But in the proof of Theorem 6.1, we showed that the vector $\mathbf{r} \times \mathbf{v} \equiv (\mathbf{r} \times \mathbf{p})/m$ does not change with time. Therefore, $\mathbf{r} \times \mathbf{v} = \mathbf{n}_0$ for all $t$. Since $\mathbf{r}$ is certainly orthogonal to $\mathbf{r} \times \mathbf{v}$, we see that $\mathbf{r}$ is orthogonal to $\mathbf{n}_0$ for all $t$. Hence, $\mathbf{r}$ must lie in $P$. ■

An intuitive look at this theorem is the following. Since the position, speed, and acceleration (which is proportional to $\mathbf{F}$, which in turn is proportional to the position vector, $\mathbf{r}$) vectors initially all lie in $P$, there is a symmetry between the two sides of $P$. Therefore, there is no reason for the particle to head out of $P$ on one side rather than the other. The particle therefore remains in $P$. We can then use this same reasoning again a short time later, and so on.

This theorem shows that we need only two coordinates, instead of the usual three, to describe the motion. But since we're on a roll, why stop there? We will show below that we really only need *one* variable. Not bad, three coordinates reduced down to one.

[2]This is a special case of the fact that torque equals the rate of change of angular momentum. We'll talk about this in great detail in Chapter 7.

[3]The plane $P$ is not well-defined if $\mathbf{v}_0 = \mathbf{0}$, or $\mathbf{r}_0 = \mathbf{0}$, or $\mathbf{v}_0$ is parallel to $\mathbf{r}_0$. But in these cases, you can easily show that the motion is always radial, which is even more restrictive than planar.

## 6.2 The effective potential

The *effective potential* provides a sneaky and useful method for simplifying a 3-dimensional central-force problem down to a 1-dimensional problem. Let's see how it works.

Consider a particle of mass $m$ subject to a central force only, described by the potential $V(r)$. Let $r$ and $\theta$ be the polar coordinates in the plane of the motion. In these polar coordinates, the Lagrangian (which we'll label as "$\mathcal{L}$", to save "$L$" for the angular momentum) is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{6.5}$$

The equations of motion obtained from varying $r$ and $\theta$ are

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - V'(r), \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{6.6}$$

The first equation is the force equation along the radial direction, complete with the centripetal acceleration, in agreement with the first of eqs. (2.52). The second equation is the statement of conservation of angular momentum, because $mr^2\dot{\theta} = r(mr\dot{\theta}) = rp_\theta$ (where $p_\theta$ is the magnitude of the momentum in the angular direction), which is the magnitude of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We therefore see that the magnitude of $\mathbf{L}$ is constant. And since the direction of $\mathbf{L}$ is always perpendicular to the fixed plane of the motion, the vector $\mathbf{L}$ is constant in time. We have therefore just given a second proof of Theorem 6.1. In the present Lagrangian language, the conservation of $\mathbf{L}$ follows from the fact that $\theta$ is a cyclic coordinate, as we saw in Example 2 in Section 5.5.1.

Since $mr^2\dot{\theta}$ does not change in time, let us denote its constant value by

$$L \equiv mr^2\dot{\theta}. \tag{6.7}$$

$L$ is determined by the initial conditions; it could be specified, for example, by giving the initial values of $r$ and $\dot{\theta}$. Using $\dot{\theta} = L/(mr^2)$, we may eliminate $\dot{\theta}$ from the first of eqs. (6.6). The result is

$$m\ddot{r} = \frac{L^2}{mr^3} - V'(r). \tag{6.8}$$

Multiplying by $\dot{r}$ and integrating with respect to time yields

$$\frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} + V(r)\right) = E, \tag{6.9}$$

where $E$ is a constant of integration. $E$ is simply the energy, which can be seen by noting that this equation could also have been obtained by simply using eq. (6.7) to eliminate $\dot{\theta}$ in the energy equation, $(m/2)(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$.

Eq. (6.9) is rather interesting. It involves only the variable $r$. And it looks a lot like the equation for a particle moving in one dimension (labeled by the coordinate $r$) under the influence of the potential

$$\boxed{V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r)}. \tag{6.10}$$

The subscript "eff" here stands for "effective". $V_{\text{eff}}(r)$ is called the *effective potential*. The "effective force" is easily read off from eq. (6.8) to be

$$F_{\text{eff}}(r) = \frac{L^2}{mr^3} - V'(r), \tag{6.11}$$

which agrees with $F_{\text{eff}} = -V'_{\text{eff}}(r)$, as it should.

This "effective" potential concept is a marvelous result and should be duly appreciated. It says that if we want to solve a two-dimensional problem (which could have come from a three-dimensional problem) involving a central force, we can recast the problem into a simple one-dimensional problem with a slightly modified potential. We can forget that we ever had the variable $\theta$, and we can solve this one-dimensional problem (as we'll demonstrate below) to obtain $r(t)$. Having found $r(t)$, we can use $\dot\theta(t) = L/mr^2$ to solve for $\theta(t)$ (in theory, at least).

Note that this whole procedure works only because there is a quantity involving $r$ and $\theta$ that is independent of time. The variables $r$ and $\theta$ are therefore *not* independent, so the problem is really one-dimensional instead of two-dimensional.

To get a general idea of how $r$ behaves with time, we simply have to graph $V_{\text{eff}}(r)$. Consider the example where $V(r) = Ar^2$. This is the potential for a spring with equilibrium length zero. Then

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + Ar^2. \tag{6.12}$$

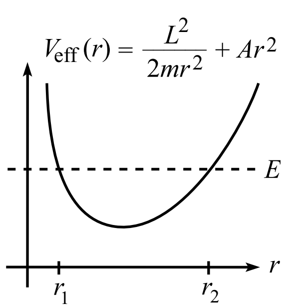


**Figure 6.1**

To graph $V_{\text{eff}}(r)$, we must be given $L$ and $A$. But the general shape looks like the curve in Fig. 6.1. The energy $E$ (which must be given, too) is also drawn. The coordinate $r$ will bounce back and forth between the turning points, $r_1$ and $r_2$, which satisfy $V_{\text{eff}}(r_{1,2}) = E$.[4] If $E$ equals the minimum of $V_{\text{eff}}(r)$, then $r_1 = r_2$, so $r$ is stuck at this one value, which means that the motion is a circle. Note that it is impossible for $E$ to be less than the minimum of $V_{\text{eff}}$.

Remark: The $L^2/2mr^2$ term in the effective potential is sometimes called the *angular momentum barrier*. It has the effect of keeping the particle from getting too close to the origin. Basically, the point is that $L \equiv mr^2\dot\theta$ is constant, so as $r$ gets smaller, $\dot\theta$ gets bigger. But $\theta$ increases at a greater rate than $r$ decreases, due to the square of the $r$ in $L = mr^2\dot\theta$. So eventually we end up with a tangential kinetic energy, $mr^2\dot\theta^2/2$, that is greater than what is allowed by conservation of energy.[5]

[4] It turns out that for our $Ar^2$ spring potential, the motion in space is an ellipse, with semi-axis lengths $r_1$ and $r_2$ (see Problem 5). But for a general potential, the motion isn't so nice.

[5] If $V(r)$ goes to $-\infty$ faster than $-1/r^2$, then this argument doesn't hold. You can see this by drawing the graph of $V_{\text{eff}}(r)$, which heads to $-\infty$ instead of $+\infty$ as $r \to 0$. $V(r)$ decreases fast enough to compensate for the increase in kinetic energy.

As he walked past the beautiful belle,
The attraction was easy to tell.
But despite his persistence,
He was kept at a distance
By that darn conservation of $L$. ♣

Note that it is by no means necessary to introduce the concept of the effective potential. You can simply solve the equations of motion, eqs. (6.6), as they are. But introducing $V_{\text{eff}}$ makes it much easier to see what's going on in a central-force problem.

When using potentials, effective,
Remember the one main objective:
The goal is to shun
All dimensions but one,
And then view things with 1-D perspective.

## 6.3 Solving the equations of motion

If we want to be quantitative, we must solve the equations of motion, eqs. (6.6). Equivalently, we must solve their integrated forms, eqs. (6.7) and (6.9), which are simply the conservation of $L$ and $E$ statements,

$$\begin{aligned} mr^2\dot{\theta} &= L, \\ \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) &= E. \end{aligned} \tag{6.13}$$

The word "solve" is a little ambiguous here, because we should specify what quantities we want to solve for in terms of what other quantities. There are essentially two things we can do. We can solve for $r$ and $\theta$ in terms of $t$. Or, we can solve for $r$ in terms of $\theta$. The former has the advantage of immediately yielding velocities and, of course, the information of where the particle is at time $t$. The latter has the advantage of explicitly showing what the trajectory looks like in space, even though we don't know how quickly it is being traversed. We will deal mainly with this latter case, particularly when we discuss the gravitational force and Kepler's Laws below. But let's look at both procedures now.

### 6.3.1 Finding $r(t)$ and $\theta(t)$

The value of $\dot{r}$ at any point is found from eq. (6.13) to be

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{L^2}{2mr^2} - V(r)}\,. \tag{6.14}$$

To get an actual $r(t)$ out of this, we must be supplied with $E$ and $L$ (which may be found using the initial values of $r$, $\dot{r}$, and $\dot{\theta}$), and also the function $V(r)$. To solve this differential equation, we "simply" have to separate variables and then (in theory) integrate:

$$\int \frac{dr}{\sqrt{E - \frac{L^2}{2mr^2} - V(r)}} = \pm \int \sqrt{\frac{2}{m}}\, dt = \pm\sqrt{\frac{2}{m}}\,(t - t_0). \tag{6.15}$$

We must perform this (rather unpleasant) integral on the left-hand side, to obtain $t$ as a function of $r$. Having found $t(r)$, we may then (in theory) invert the result to obtain $r(t)$. Finally, substituting this $r(t)$ into the relation $\dot\theta = L/mr^2$ from eq. (6.13), we have $\dot\theta$ as a function of $t$, which we can (in theory) integrate to obtain $\theta(t)$.

The bad news about this procedure is that for most $V(r)$'s the integral in eq. (6.15) is not calculable in closed form. There are only a few "nice" potentials $V(r)$ for which we can evaluate it. And even then, the procedure is a pain.[6] But the good news is that these "nice" potentials are precisely the ones we are most interested in. In particular, the gravitational potential, which goes like $1/r$ and which we will spend most of our time with in the remainder of this chapter, leads to a calculable integral (the spring potential $\sim r^2$ does also). But never mind; we're not going to apply this procedure to gravity. It's nice to know that the procedure exists, but we won't be doing anything else with it. Instead, we'll use the following strategy.

### 6.3.2 Finding $r(\theta)$

We may eliminate the $dt$ from eqs. (6.13) by getting the $\dot r^2$ term alone on the left side of the second equation, and then dividing by the square of the first equation. The $dt^2$ factors cancel, and we obtain

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2mV(r)}{L^2}\,. \tag{6.16}$$

At this point, we can (in theory) take a square root, separate variables, and then integrate to obtain $\theta$ as a function of $r$. We can then (in theory) invert to obtain $r$ as a function of $\theta$. To do this, of course, we must be given the function $V(r)$. So let's now finally give ourselves a $V(r)$ and do a problem all the way through. We'll study the most important potential of all (or perhaps the second most important one), gravity.[7]

## 6.4 Gravity, Kepler's Laws

### 6.4.1 Calculation of $r(\theta)$

Our goal in this subsection will be to obtain $r$ as a function of $\theta$, for a gravitational potential. Let's assume that we're dealing with the earth and the sun, with masses

[6] You can, of course, always evaluate the integral numerically. See Appendix D for a discussion of this.

[7] The two most important potentials in physics are certainly the gravitational and harmonic-oscillator ones. Interestingly, they both lead to doable integrals, and they both lead to elliptical orbits.

$M_\odot$ and $m$, respectively. The gravitational potential energy of the earth-sun system is

$$V(r) = -\frac{\alpha}{r}\,, \qquad \text{where } \alpha \equiv GM_\odot m. \tag{6.17}$$

In the present treatment, let us consider the sun to be bolted down at the origin of our coordinate system. Since $M_\odot \gg m$, this is approximately true for the earth-sun system.[8] Eq. (6.16) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} + \frac{2m\alpha}{rL^2}\,. \tag{6.18}$$

As stated above, we could take a square root, separate variables, integrate to find $\theta(r)$, and then invert to find $r(\theta)$. This method, although straightforward, is terribly messy. Let's solve for $r(\theta)$ in a slick way.

With all the $1/r$ terms floating around, it might be easier to solve for $1/r$ instead of $r$. Using $d(1/r)/d\theta = -(dr/d\theta)/r^2$, and letting $y \equiv 1/r$ for convenience, eq. (6.18) becomes

$$\left(\frac{dy}{d\theta}\right)^2 = -y^2 + \frac{2m\alpha}{L^2}y + \frac{2mE}{L^2}\,. \tag{6.19}$$

At this point, we could also use the separation-of-variables technique, but let's continue to be slick. Completing the square on the right-hand side, we obtain

$$\left(\frac{dy}{d\theta}\right)^2 = -\left(y - \frac{m\alpha}{L^2}\right)^2 + \frac{2mE}{L^2} + \left(\frac{m\alpha}{L^2}\right)^2. \tag{6.20}$$

Defining $z \equiv y - m\alpha/L^2$ for convenience, we have

$$\begin{aligned}\left(\frac{dz}{d\theta}\right)^2 &= -z^2 + \left(\frac{m\alpha}{L^2}\right)^2\left(1 + \frac{2EL^2}{m\alpha^2}\right)\\ &\equiv -z^2 + B^2, \qquad \text{where } B \equiv \left(\frac{m\alpha}{L^2}\right)\sqrt{1 + \frac{2EL^2}{m\alpha^2}}\,.\end{aligned} \tag{6.21}$$

At this point, in the spirit of being slick, we can just look at this equation and observe that

$$z = B\cos(\theta - \theta_0) \tag{6.22}$$

is the solution, because $\cos^2 x + \sin^2 x = 1$.

REMARK: Lest we feel guilty about not doing separation-of-variables at least once in this problem, let's solve eq. (6.21) that way, too. The integral is nice and doable, and we have

$$\begin{aligned}\int_{z_1}^{z} \frac{dz'}{\sqrt{B^2 - z'^2}} &= \int_{\theta_1}^{\theta} d\theta'\\ \Longrightarrow \quad \cos^{-1}\left(\frac{z'}{B}\right)\Bigg|_{z_1}^{z} &= (\theta - \theta_1)\\ \Longrightarrow \quad z &= B\cos\left((\theta - \theta_1) + \cos^{-1}\left(\frac{z_1}{B}\right)\right)\\ &\equiv B\cos(\theta - \theta_0). \ ♣\end{aligned} \tag{6.23}$$

[8] If we want to do the problem exactly, we must use the *reduced mass*. This topic is discussed in Section 6.4.5.

It is customary to pick the axes so that $\theta_0 = 0$, so we'll drop the $\theta_0$ from here on. Recalling our definition $z \equiv 1/r - m\alpha/L^2$ and also the definition of $B$ from eq. (6.21), eq. (6.22) becomes

$$\frac{1}{r} = \frac{m\alpha}{L^2}(1 + \epsilon\cos\theta), \tag{6.24}$$

where

$$\epsilon \equiv \sqrt{1 + \frac{2EL^2}{m\alpha^2}} \tag{6.25}$$

is the *eccentricity* of the particle's motion. We will see shortly exactly what $\epsilon$ signifies.

This completes the derivation of $r(\theta)$ for the gravitational potential, $V(r) \propto 1/r$. It was a little messy, but not unbearably painful. At any rate, we just discovered the basic motion of objects under the influence of gravity, which takes care of virtually all of the gazillion tons of stuff in the universe. Not bad for one page of work.

Newton said as he gazed off afar,
"From here to the most distant star,
These wond'rous ellipses
And solar eclipses
All come from a 1 over $r$."

What are the limits on $r$ in eq. (6.24)? The minimum value of $r$ is obtained when the right-hand side reaches its maximum value, which is $(m\alpha/L^2)(1+\epsilon)$. Therefore,

$$r_{\text{min}} = \frac{L^2}{m\alpha(1+\epsilon)}\,. \tag{6.26}$$

What is the maximum value of $r$? The answer depends on whether $\epsilon$ is greater than or less than 1. If $\epsilon < 1$ (which corresponds to circular or elliptical orbits, as we will see below), then the minimum value of the right-hand side of eq. (6.24) is $(m\alpha/L^2)(1 - \epsilon)$. Therefore,

$$r_{\text{max}} = \frac{L^2}{m\alpha(1-\epsilon)} \qquad (\text{if } \epsilon < 1). \tag{6.27}$$

If $\epsilon \geq 1$ (which corresponds to parabolic or hyperbolic orbits, as we will see below), then the right-hand side of eq. (6.24) can become zero (when $\cos\theta = -1/\epsilon$). Therefore,

$$r_{\text{max}} = \infty \qquad (\text{if } \epsilon \geq 1). \tag{6.28}$$

### 6.4.2 The orbits

Let's examine in detail the various cases for $\epsilon$.

- **Circle** ($\epsilon = 0$)

  If $\epsilon = 0$, then eq. (6.25) says that $E = -m\alpha^2/2L^2$. The negative $E$ simply means that the potential energy is more negative than the kinetic energy is positive. The particle is trapped in the potential well. Eqs. (6.26) and (6.27) give $r_{\text{min}} = r_{\text{max}} = L^2/m\alpha$. Therefore, the particle moves in a circular orbit with radius $L^2/m\alpha$. Equivalently, eq. (6.24) says that $r$ is independent of $\theta$.

  Note that it isn't necessary to do all the work of Section 6.4.1 if we just want to look at circular motion. For a given $L$, the energy $-m\alpha^2/2L^2$ is the minimum value that the $E$ given by eq. (6.13) can take. (To achieve the minimum, we certainly want $\dot{r} = 0$. And you can show that minimizing the effective potential, $L^2/2mr^2 - \alpha/r$, yields this value for $E$.) If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.2. The particle is trapped at the bottom of the potential well, so it has no motion in the $r$ direction.

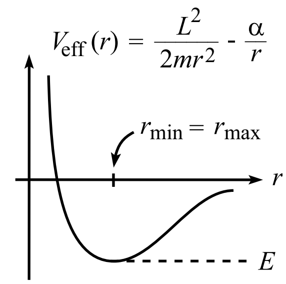



**Figure 6.2**

- **Ellipse** ($0 < \epsilon < 1$)

  If $0 < \epsilon < 1$, then eq. (6.25) says that $-m\alpha^2/2L^2 < E < 0$. Eqs. (6.26) and (6.27) give $r_{\text{min}}$ and $r_{\text{max}}$. It is not obvious that the resulting motion is an ellipse. We will demonstrate this below.

  If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.3. The particle oscillates between $r_{\text{min}}$ and $r_{\text{max}}$. The energy is negative, so the particle is trapped in the potential well.

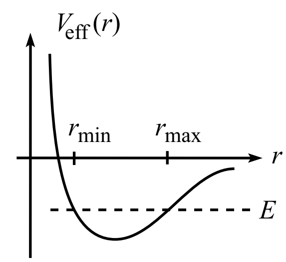



**Figure 6.3**

- **Parabola** ($\epsilon = 1$)

  If $\epsilon = 1$, then eq. (6.25) says that $E = 0$. This value of $E$ implies that the particle barely makes it out to infinity (its speed approaches zero as $r \to \infty$). Eq. (6.26) gives $r_{\text{min}} = L^2/2m\alpha$, and eq. (6.28) gives $r_{\text{max}} = \infty$. Again, it is not obvious that the resulting motion is a parabola. We will demonstrate this below.

  If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.4. The particle does not oscillate back and forth in the $r$-direction. It moves inward (or possibly not, if it was initially moving outward), turns around at $r_{\text{min}} = L^2/2m\alpha$, and then heads out to infinity forever.

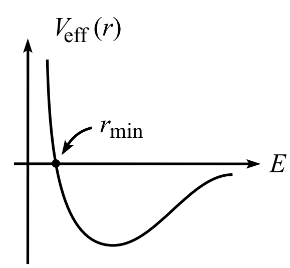



**Figure 6.4**

- **Hyperbola** ($\epsilon > 1$)

  If $\epsilon > 1$, then eq. (6.25) says that $E > 0$. This value of $E$ implies that the particle makes it out to infinity with energy to spare. (The potential goes to zero as $r \to \infty$, so the particle's speed approaches the nonzero value $\sqrt{2E/m}$ as $r \to \infty$.) Eq. (6.26) gives $r_{\text{min}}$, and eq. (6.28) gives $r_{\text{max}} = \infty$. Again, it is not obvious that the resulting motion is a hyperbola. We will demonstrate this below.

  If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.5. As in the parabola case, the particle does not oscillate back and forth in the $r$-direction. It moves inward (or possibly not, if it was initially moving outward), turns around at $r_{\text{min}}$, and then heads out to infinity forever.

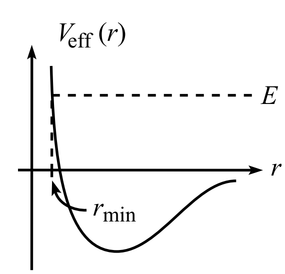



**Figure 6.5**

### 6.4.3 Proof of conic orbits

Let's now prove that eq. (6.24) does indeed describe the conic sections stated above. We will also show that the origin (the source of the potential) is a focus of the conic section. These proofs are straightforward, although the ellipse and hyperbola cases get a bit messy.

In what follows, we will find it easier to work with cartesian coordinates. For convenience, let

$$k \equiv \frac{L^2}{m\alpha}. \tag{6.29}$$

Multiplying eq. (6.24) through by $kr$, and using $\cos\theta = x/r$, gives

$$k = r + \epsilon x. \tag{6.30}$$

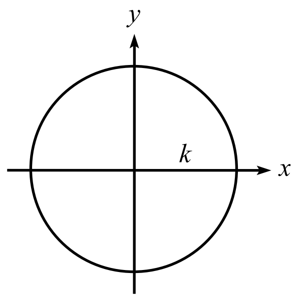



**Figure 6.6**

Solving for $r$ and squaring yields

$$x^2 + y^2 = k^2 - 2k\epsilon x + \epsilon^2 x^2. \tag{6.31}$$

Let's look at the various cases for $\epsilon$. We will invoke without proof various facts about conic sections (focal lengths, etc.).

- **Circle** ($\epsilon = 0$)

  In this case, eq. (6.31) becomes $x^2 + y^2 = k^2$. So we have a circle of radius $k = L^2/m\alpha$, with its center at the origin (see Fig. 6.6).

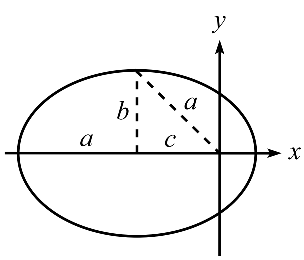



**Figure 6.7**

- **Ellipse** ($0 < \epsilon < 1$)

  In this case, eq. (6.31) may be written as (after completing the square for the $x$ terms, and expending some effort)

$$\frac{\left(x + \frac{k\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a = \frac{k}{1-\epsilon^2}, \quad \text{and } b = \frac{k}{\sqrt{1-\epsilon^2}}. \tag{6.32}$$

  This is the equation for an ellipse with its center located at $(-k\epsilon/(1-\epsilon^2), 0)$. The semi-major and semi-minor axes $a$ and $b$, respectively, and the focal length is $c = \sqrt{a^2 - b^2} = k\epsilon/(1-\epsilon^2)$. Therefore, one focus is located at the origin (see Fig. 6.7). Note that $c/a$ equals the eccentricity, $\epsilon$.

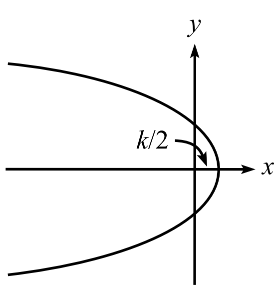



**Figure 6.8**

- **Parabola** ($\epsilon = 1$)

  In this case, eq. (6.31) becomes $y^2 = k^2 - 2kx$. This may be written as $y^2 = -2k(x - \frac{k}{2})$. This is the equation for a parabola with vertex at $(k/2, 0)$ and focal length $k/2$. (The focal length of a parabola written in the form $y^2 = 4ax$ is $a$.) So we have a parabola with its focus located at the origin (see Fig. 6.8).

- **Hyperbola** ($\epsilon > 1$)

  In this case, eq. (6.31) may be written (after completing the square for the $x$ terms)

$$\frac{\left(x - \frac{k\epsilon}{\epsilon^2-1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{where } a = \frac{k}{\epsilon^2 - 1}, \quad \text{and } b = \frac{k}{\sqrt{\epsilon^2 - 1}}. \quad (6.33)$$

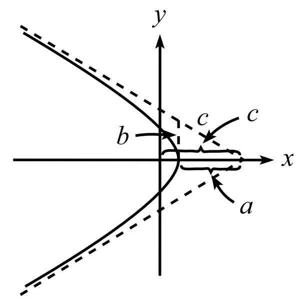


**Figure 6.9**

  This is the equation for a hyperbola with its center (defined to be the intersection of the asymptotes) located at $(k\epsilon/(\epsilon^2 - 1), 0)$. The focal length is $c = \sqrt{a^2 + b^2} = k\epsilon/(\epsilon^2 - 1)$. Therefore, the focus is located at the origin (see Fig. 6.9). Note that $c/a$ equals the eccentricity, $\epsilon$.

  The *impact parameter* (usually denoted by the letter $b$) of a trajectory is defined to be the closest distance to the origin the particle would achieve if it moved in the straight line determined by its initial velocity (that is, along the dotted line in the Fig. 6.9). You might think that choosing the letter $b$ here would cause a problem, because we already defined $b$ in eq. (6.33). However, it turns out that these two definitions are identical (see Exercise 6), so all is well.

  Remark: Eq. (6.33) actually describes an entire hyperbola, that is, it also describes a branch that opens up to the right. However, this right branch was introduced in the squaring operation that produced eq. (6.31). It is *not* a solution to the original equation we wanted to solve, eq. (6.30). What makes the left branch, and not the right branch, the relevant one? The left-right symmetry was broken when we arbitrarily chose a positive value for $B$ in eq. (6.21), or equivalently, a positive value for $\epsilon$ in eq. (6.25). If we had chosen $B$ and $\epsilon$ to be negative, then the hyperbola would be centered at a negative value of $x$ and would open up to the right, as you can check. The result would simply be Fig. 6.9, reflected across the $y$-axis.

  It turns out that the right-opening branch (or its reflection in the $y$-axis, depending on your choice of sign for $\epsilon$) is relevant in a certain physical situation; see Exercise 9. ♣

### 6.4.4 Kepler's Laws

We can now, with minimal extra work, write down Kepler's Laws. Kepler (1571–1630) lived prior to Newton (1642–1727). Kepler arrived at these laws via observational data, which was a rather impressive feat. It was known since the time of Copernicus (1473–1543) that the planets move around the sun, but it was Kepler and Newton who first gave a quantitative description of the orbits.

Kepler's laws assume that the sun is massive enough so that its position is essentially fixed in space. This is a very good approximation, but the following section on *reduced mass* will show how to modify them and solve things exactly.

- **First Law:** *The planets move in elliptical orbits with the sun at one focus.*

  We proved this in eq. (6.32). Of course, there are undoubtedly objects flying past the sun in hyperbolic orbits. But we don't call these things planets, because we never see the same one twice.

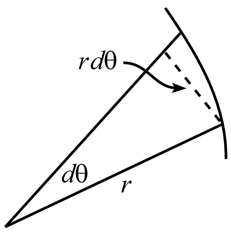



**Figure 6.10**

- **Second Law:** *The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit.*

  This law is nothing other than the statement of conservation of angular momentum. The area swept out by the radius vector during a short period of time is $dA = r(r\,d\theta)/2$, because $r\,d\theta$ is the base of the thin triangle in Fig. 6.10. Therefore, we have (using $L = mr^2\dot{\theta}$)

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m}\,, \tag{6.34}$$

  which is constant, because $L$ is constant for a central force.

- **Third Law:** *The square of the period of an orbit, $T$, is proportional to the cube of the semimajor-axis length, $a$. More precisely,*

$$T^2 = \frac{4\pi^2 ma^3}{\alpha} \equiv \frac{4\pi^2 a^3}{GM_\odot}\,, \tag{6.35}$$

  where $M_\odot$ is the mass of the sun. Note that the planet's mass, $m$, does not appear in this equation.

  *Proof:* Integrating eq. (6.34) over the time of a whole orbit gives

$$A = \frac{LT}{2m}\,. \tag{6.36}$$

  But the area of an ellipse is $A = \pi ab$, where $a$ and $b$ are the semi-major and semi-minor axes, respectively. Squaring (6.36) and using eq. (6.32) to write $b = a\sqrt{1-\epsilon^2}$ gives

$$\pi^2 a^4 = \left(\frac{L^2}{m(1-\epsilon^2)}\right)\frac{T^2}{4m}\,. \tag{6.37}$$

  We have grouped the right-hand side in this way because we may now use the $L^2 \equiv m\alpha k$ relation from eq. (6.29) to transform the term in parentheses into $\alpha k/(1-\epsilon^2) \equiv \alpha a$, where $a$ is given in eq. (6.32). But $\alpha a \equiv (GM_\odot m)a$, so we obtain

$$\pi^2 a^4 = \frac{(GM_\odot ma)T^2}{4m}\,, \tag{6.38}$$

  which gives eq. (6.35), as desired.

These three laws describe the motion of all the planets (and asteroids, comets, and such) in the solar system. But our solar system is only the tip of the iceberg. There's a lot more stuff out there, and it's all governed by gravity (although Newton's inverse square law must be supplanted by Einstein's General Relativity theory of gravitation). There's a whole universe around us, and with each generation we can see and understand a little more of it, both experimentally and theoretically. In recent years, we've even begun to look for friends we might have out there. Why? Because we can. There's nothing wrong with looking under the lamppost now and then. It just happens to be a very big one in this case.

As we grow up, we open an ear,
Exploring the cosmic frontier.
In this coming of age,
We turn in our cage,
All alone on a tiny blue sphere.

### 6.4.5 Reduced mass

We assumed in Section 6.4.1 that the sun is large enough so that it is only negligibly affected by the presence of the planets. That is, it is essentially fixed at the origin. But how do we solve a problem in which the masses of the two interacting bodies are comparable in size? Equivalently, how do we solve the earth-sun problem exactly? It turns out that the only modification required is a simple replacement of the earth's mass with the *reduced mass*, defined below. The following discussion actually holds for any central force, not just gravity.

The Lagrangian of a general central-force system consisting of the interacting masses $m_1$ and $m_2$ is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{6.39}$$

We have written the potential in this form, dependent only on the distance $|\mathbf{r}_1 - \mathbf{r}_2|$, because we are assuming a central force. Let us define

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad \text{and} \qquad \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2. \tag{6.40}$$

$\mathbf{R}$ and $\mathbf{r}$ are simply the position of the center of mass and the vector between the masses, respectively. Invert these equations to obtain

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \qquad \text{and} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}, \tag{6.41}$$

where $M \equiv m_1 + m_2$ is the total mass of the system. In terms of $\mathbf{R}$ and $\mathbf{r}$, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - V(|\mathbf{r}|) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - V(r) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r),
\end{aligned} \tag{6.42}$$

where the *reduced mass*, $\mu$, is defined by

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}. \tag{6.43}$$

We now note that the Lagrangian in eq. (6.42) depends on $\dot{\mathbf{R}}$, but not on $\mathbf{R}$. Therefore, the Euler-Lagrange equations say that $\dot{\mathbf{R}}$ is constant. That is, the CM moves at constant velocity (this is just the statement that there are no external forces). The CM motion is therefore trivial, so let's ignore it. Our Lagrangian therefore essentially becomes

$$\mathcal{L} \to \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r). \tag{6.44}$$

But this is simply the Lagrangian for a particle of mass $\mu$ which moves around a fixed origin under the influence of the potential $V(r)$.

For gravity, we have

$$\mathcal{L} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{\alpha}{r} \qquad \text{(where } \alpha \equiv GM_{\odot}m\text{)}. \tag{6.45}$$

To solve the earth-sun system exactly, we therefore simply need to replace (in the calculation in Section 6.4.1) the earth's mass, $m$, with the reduced mass, $\mu$, given by

$$\frac{1}{\mu} \equiv \frac{1}{m} + \frac{1}{M_{\odot}}\,. \tag{6.46}$$

The resulting value of $r$ in eq. (6.24) is the distance between the earth and sun. The earth and sun are therefore distances of $(M_{\odot}/M)r$ and $(m/M)r$, respectively, away from the CM, from eq. (6.41). These distances are simply scaled-down versions of the distance $r$, which represents and ellipse, so we see that the earth and sun move in elliptical orbits (whose sizes are in the ratio $M_{\odot}/m$) with the CM as a focus. Note that the $m$'s that are buried in the $L$ and $\epsilon$ in eq. (6.24) must be changed to $\mu$'s. But $\alpha$ is still defined to be $GM_{\odot}m$, so the $m$ in this definition does *not* get replaced with $\mu$.

For the earth-sun system, the $\mu$ in eq. (6.46) is essentially equal to $m$, because $M_{\odot}$ is so large. Using $m = 5.98 \cdot 10^{24}$ kg, and $M_{\odot} = 1.99 \cdot 10^{30}$ kg, we find that $\mu$ is smaller than $m$ by only one part in $3 \cdot 10^5$. Our fixed-sun approximation is therefore a very good one. You can show that the CM is $5 \cdot 10^5$ m from the center of the sun, which is well within the sun (about a thousandth of the radius).

How are Kepler's laws modified when we solve for the orbits exactly using the reduced mass?

- **First Law:** The elliptical statement in the first law is still true, but with the CM (not the sun) located at a focus. The sun also travels in an ellipse with the CM at a focus.[9] Whatever is true for the earth must also be true for the sun, because they come into eq. (6.42) symmetrically. The only difference is in the size of various quantities.

- **Second Law:** In the second law, we need to consider the position vector from the CM (not the sun) to the planet. This vector sweeps out equal areas in equal times, because the angular momentum of the earth (and the sun, too) relative to the CM is fixed. This is true because the gravitational force always

[9] Well, this statement is true only if there is just one planet. With many planets, the tiny motion of the sun is very complicated. This is perhaps the best reason to work in the approximation where it is essentially bolted down.

points through the CM, so the force is a central force with the CM as the origin.

- **Third Law:** Eq. (6.45) describes a particle of mass $\mu$ moving in a potential of $-\alpha/r$. The reasoning we used in obtaining eq. (6.35) still holds, provided that we change all the $m$'s to $\mu$'s, except the one in $\alpha \equiv GM_\odot m$. In other words, we still arrive at eq. (6.38), except with the bottom $m$ (but not the top one) replaced with $\mu$. Therefore, we obtain

$$T^2 = \frac{4\pi^2 a^3 \mu}{GM_\odot m} = \frac{4\pi^2 a^3}{G(M_\odot + m)}\,, \tag{6.47}$$

where we have used $\mu \equiv M_\odot m/(M_\odot + m)$. Eq. (6.47) reduces to eq. (6.35) when $\mu \approx m$ (that is, when $M_\odot \gg m$), as it should. Note the symmetry between $M_\odot$ and $m$.

The $T$ in eq. (6.47) is the time for the hypothetical particle of mass $\mu$ to complete an orbit. But this is the same as the time for the earth (and the sun) to complete an orbit. So it is indeed the time we are looking for. The $a$ in eq. (6.47) is the semi-major axis of the hypothetical particle's orbit. In other words, it is half of the maximum distance between the earth and the sun. If you want to write the third law using the semi-major axis of the earth's elliptical orbit, which is $a_e = (M_\odot/M)a$, then simply plug $a = (M/M_\odot)a_e$ into eq. (6.47).

## 6.5 Exercises

*Section 6.1: Conservation of angular momentum*

1. **Wrapping around a pole** $*$

   A puck of mass $m$ on frictionless ice is attached by a horizontal string of length $\ell$ to a very thin vertical pole of radius $R$. The puck is given a kick and circles around the pole with initial speed $v_0$. The string wraps around the pole, and the puck gets drawn in and eventually hits the pole. What quantity is conserved during the motion? What is the puck's speed right before it hits the pole?

*Section 6.2: The effective potential*

2. **Power-law spiral** $**$

   Given $L$, find the form of $V(r)$ so that the path of a particle is given by the spiral $r = C\theta^k$, where $C$ and $k$ are constants. *Hint:* Obtain an expression for $\dot{r}$ that contains no $\theta$'s, and then use eq. (6.9).

*Section 6.4: Gravity, Kepler's Laws*

3. **Circular orbit** $*$

   For a circular orbit, derive Kepler's third law from scratch, using $\mathbf{F} = m\mathbf{a}$.

4. **Falling into the sun** $*$

   Imagine that the earth is suddenly (and tragically) stopped in its orbit, and then allowed to fall radially into the sun. How long will this take? Use data from Appendix J. *Hint*: Consider the radially path to be half of a very thin ellipse.

5. **Closest approach** $**$

   A particle with speed $v_0$ and impact parameter $b$ starts far away from a planet of mass $M$.

   (a) Starting from scratch (that is, without using any of the results from Section 6.4), find the distance of closest approach to the planet.

   (b) Use the results of the hyperbola discussion in Section 6.4.3 to show that the distance of closest approach to the planet is $k/(\epsilon + 1)$, and then show that this agrees with your answer to part (a).

6. **Impact parameter** $**$

   Show that the distance $b$ defined in eq. (6.33) and Fig. 6.9 is equal to the impact parameter. Do this:

   (a) Geometrically, by showing that $b$ is the distance from the origin to the dotted line in Fig. 6.9.

(b) Analytically, by letting the particle come in from infinity at speed $v_0$ and impact parameter $b'$, and then showing that the $b$ in eq. (6.33) equals $b'$.

7. **Skimming a planet** $**$

A particle travels in a parabolic orbit in a planet's gravitational field and skims the surface at its closest approach. The planet has mass density $\rho$. Relative to the center of the planet, what is the angular velocity of the particle as it skims the surface?

8. **Parabola** $L$ $**$

Consider a parabolic orbit of the form $y = x^2/(4\ell)$, which has focal length $\ell$. Let the speed at closest approach be $v_0$. The angular momentum is then $mv_0\ell$. Show explicitly (by finding the speed and the "lever arm") that this is also the angular momentum when the particle is very far from the origin (as it must be, because $L$ is conserved).

9. **Repulsive potential** $**$

Consider an "anti-gravitational" potential (or more mundanely, the electrostatic potential between two like charges),

$$V(r) = \frac{\alpha}{r}, \qquad \text{where } \alpha > 0. \tag{6.48}$$

What is the basic change in the analysis of Section 6.4.3? Draw the figure analogous to Fig. 6.9 for the hyperbolic orbit. Show that circular, elliptical, and parabolic orbits do not exist.

10. **Ellipse axes** $**$

Taking it as given that eq. (6.24) describes as ellipse for $0 < \epsilon < 1$, calculate the lengths of the semi-major and semi-minor axes, and show that your results agree with eq. (6.32).

11. **Zero potential** $**$

A particle is subject to a constant potential, which we will take to be zero. Following the general strategy in Sections 6.4.1 and 6.4.3, show that the particle's path is a straight line.

## 6.6 Problems

*Section 6.2: The effective potential*

1. **Maximum $L$** ***

   A particle moves in a potential $V(r) = -V_0 e^{-\lambda^2 r^2}$.

   (a) Given $L$, find the radius of the stable circular orbit. An implicit equation is fine here.

   (b) It turns out that if $L$ is too large, then a circular orbit actually doesn't exist. What is the largest value of $L$ for which a circular orbit does indeed exist? What is the value of $V_{\text{eff}}(r)$ in this case?

2. **Cross section** **

   A particle moves in a potential $V(r) = -C/(3r^3)$.

   (a) Given $L$, find the maximum value of the effective potential.

   (b) Let the particle come in from infinity with speed $v_0$ and impact parameter $b$. In terms of $C$, $m$, and $v_0$, what is the largest value of $b$ (call it $b_{\text{max}}$) for which the particle is captured by the potential? In other words, what is the "cross section" for capture, $\pi b_{\text{max}}^2$, for this potential?

3. **Exponential spiral** **

   Given $L$, find the form of $V(r)$ so that the path of a particle is given by the spiral $r = Ae^{a\theta}$, where $A$ and $a$ are constants. *Hint:* Obtain an expression for $\dot{r}$ that contains no $\theta$'s, and then use eq. (6.9).

*Section 6.4: Gravity, Kepler's Laws*

4. **$r^k$ potential** ***

   A particle of mass $m$ moves in a potential given by $V(r) = \beta r^k$. Let the angular momentum be $L$.

   (a) Find the radius, $r_0$, of a circular orbit.

   (b) If the particle is given a tiny kick so that the radius oscillates around $r_0$, find the frequency, $\omega_r$, of these small oscillations in $r$.

   (c) What is the ratio of the frequency $\omega_r$ to the frequency of the (nearly) circular motion, $\omega_\theta \equiv \dot{\theta}$? Give a few values of $k$ for which the ratio is rational, that is, for which the path of the nearly circular motion closes back on itself.

5. **Spring ellipse** ***

   A particle moves in a $V(r) = \beta r^2$ potential. Following the general strategy in Sections 6.4.1 and 6.4.3, show that the particle's path is an ellipse.

6. **$\beta/r^2$ potential** ***

   A particle is subject to a $V(r) = \beta/r^2$ potential. Following the general strategy in Section 6.4.1, find the shape of the particle's path. You will need to consider various cases for $\beta$.

7. **Rutherford scattering** ***

   A particle of mass $m$ travels in a hyperbolic orbit past a mass $M$, whose position is assumed to be fixed. The speed at infinity is $v_0$, and the impact parameter is $b$ (see Exercise 6).

   (a) Show that the angle through which the particle is deflected is

   $$\phi = \pi - 2\tan^{-1}(\gamma b) \quad \Longrightarrow \quad b = \frac{1}{\gamma}\cot\left(\frac{\phi}{2}\right), \qquad \text{where } \gamma \equiv \frac{v_0^2}{GM}. \tag{6.49}$$

   (b) Let $d\sigma$ be the cross-sectional area (measured when the particle is initially at infinity) that gets deflected into a solid angle of size $d\Omega$ at angle $\phi$.[10] Show that

   $$\frac{d\sigma}{d\Omega} = \frac{1}{4\gamma^2 \sin^2(\phi/2)}. \tag{6.50}$$

   This quantity is called the *differential cross section*. The term *Rutherford scattering* actually refers to the scattering of charged particles, but since the electrostatic and gravitational forces are both inverse-square laws, the scattering formulas look the same, except for a few constants.

[10] The *solid angle* of a patch on a sphere is the area of the patch divided by the square of the sphere's radius. So a whole sphere subtends a solid angle of $4\pi$ *steradians* (the name for one unit of solid angle).

## 6.7 Solutions

1. **Maximum $L$**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - V_0 e^{-\lambda^2 r^2}. \tag{6.51}$$

A circular orbit exists at the value(s) of $r$ for which $V'_{\text{eff}}(r) = 0$. Setting the derivative equal to zero and solving for $L^2$ gives, as you can show,

$$L^2 = (2mV_0\lambda^2)r^4 e^{-\lambda^2 r^2}. \tag{6.52}$$

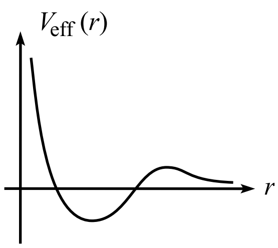


**Figure 6.11**

This implicitly determines $r$. As long as $L$ isn't too large, $V_{\text{eff}}(r)$ looks something like the graph in Fig. 6.11 (although it doesn't necessarily dip down to negative values; see the remark below), so there are two solutions for $r$. The smaller solution is the one with the stable orbit. However, if $L$ is too large, then there are no solutions to $V'_{\text{eff}}(r) = 0$, because $V_{\text{eff}}(r)$ decreases monotonically to zero (because $L^2/2mr^2$ does so). We'll be quantitative about this in part (b).

(b) The function $r^4 e^{-\lambda^2 r^2}$ on the right-hand side of eq. (6.52) has a maximum value, because it goes to zero for both $r \to 0$, and $r \to \infty$. Therefore, there is a maximum value of $L$ for which a solution for $r$ exists. The maximum of $r^4 e^{-\lambda^2 r^2}$ occurs when

$$(r^4 e^{-\lambda^2 r^2})' = e^{-\lambda^2 r^2}[4r^3 + r^4(-2\lambda^2 r)] = 0 \qquad \Longrightarrow \qquad r^2 = \frac{2}{\lambda^2} \equiv r_0^2. \tag{6.53}$$

Plugging $r_0$ into eq. (6.52) gives

$$L_{\text{max}}^2 = \frac{8mV_0}{\lambda^2 e^2}\,. \tag{6.54}$$

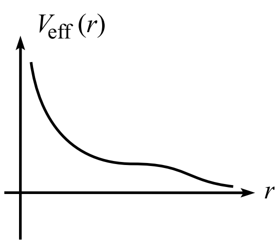


**Figure 6.12**

Plugging $r_0$ and $L_{\text{max}}^2$ into (6.51) gives

$$V_{\text{eff}}(r_0) = \frac{V_0}{e^2} \qquad (\text{for } L = L_{\text{max}}). \tag{6.55}$$

Note that this is greater than zero. For the $L = L_{\text{max}}$ case, the graph of $V_{\text{eff}}$ is shown in Fig. 6.12. This is the cutoff case between having a dip in the graph, and decreasing monotonically to zero.

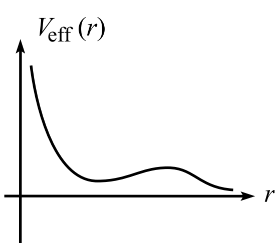


**Figure 6.13**

Remark: A common error in this problem is to say that the condition for a circular orbit to exist is that $V_{\text{eff}}(r) < 0$ at the point where $V_{\text{eff}}(r)$ is minimum. The logic here is that the goal is to have a well in which the particle can be trapped, so it seems like we just need $V_{\text{eff}}$ to achieve a value less than the value at $r = \infty$, namely 0. However, this gives the wrong answer ($L_{\text{max}}^2 = 2mV_0/\lambda^2 e$, as you can show), because $V_{\text{eff}}(r)$ can look like the graph in Fig. 6.13. This has a local minimum with $V_{\text{eff}}(r) > 0$. ♣

2. **Cross section**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{C}{3r^3}. \tag{6.56}$$

Setting the derivative equal to zero gives $r = mC/L^2$. Plugging this into $V_{\text{eff}}(r)$ gives

$$V_{\text{eff}}^{\text{max}} = \frac{L^6}{6m^3C^2}\,. \tag{6.57}$$

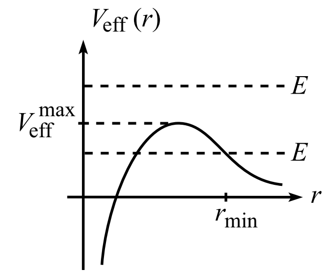



**Figure 6.14**

(b) If the energy of the particle, $E$, is less than $V_{\text{eff}}^{\text{max}}$, then the particle will reach a minimum value of $r$, and then head back out to infinity (see Fig. 6.14). If $E$ is greater than $V_{\text{eff}}^{\text{max}}$, then the particle will head in to $r = 0$, never to return. The condition for capture is therefore $V_{\text{eff}}^{\text{max}} < E$. Using $L = mv_0 b$ and $E = E_\infty = mv_0^2/2$, this condition becomes

$$\frac{(mv_0 b)^6}{6m^3C^2} < \frac{mv_0^2}{2}$$
$$\Longrightarrow \quad b < \left(\frac{3C^2}{m^2v_0^4}\right)^{1/6} \equiv b_{\text{max}}. \tag{6.58}$$

The cross section for capture is therefore

$$\sigma = \pi b_{\text{max}}^2 = \pi\left(\frac{3C^2}{m^2v_0^4}\right)^{1/3}. \tag{6.59}$$

It makes sense that this should increase with $C$ and decrease with $m$ and $v_0$.

3. **Exponential spiral**

The given information $r = Ae^{a\theta}$ yields (using $\dot\theta = L/mr^2$)

$$\dot r = aAe^{a\theta}\dot\theta = ar\left(\frac{L}{mr^2}\right) = \frac{aL}{mr}. \tag{6.60}$$

Plugging this into eq. (6.9) gives

$$\frac{m}{2}\left(\frac{aL}{mr}\right)^2 + \frac{L^2}{2mr^2} + V(r) = E. \tag{6.61}$$

Therefore,

$$V(r) = E - \frac{(1+a^2)L^2}{2mr^2}. \tag{6.62}$$

The total energy, $E$, may be arbitrarily chosen to equal zero, if desired.

4. **$r^k$ potential**

(a) A circular orbit exists at the value of $r$ for which the derivative of the effective potential (which is the negative of the effective force) is zero. This is simply the statement that the right-hand side of eq. (6.8) equals zero, so that $\ddot r = 0$. Since $V'(r) = \beta k r^{k-1}$, eq. (6.8) gives

$$\frac{L^2}{mr^3} - \beta k r^{k-1} = 0 \qquad \Longrightarrow \qquad r_0 = \left(\frac{L^2}{m\beta k}\right)^{1/(k+2)}. \tag{6.63}$$

Note that if $k$ is negative, then $\beta$ must also be negative if there is to be a real solution for $r_0$.

(b) The long method of finding the frequency is to set $r(t) \equiv r_0 + \epsilon(t)$, where $\epsilon$ represents the small deviation from the circular orbit, and to then plug this expression for $r$ into eq. (6.8). The result (after making some approximations) is a harmonic-oscillator equation of the form $\ddot\epsilon = -\omega_r^2\epsilon$. This general procedure, which is described in detail in Section 5.7, will work fine here (as you are encouraged to show), but let's use an easier method.

By introducing the effective potential, we have reduced the problem to a one-dimensional problem in the variable $r$. Therefore, we can make use of the result in Section 4.2, where we found in eq. (4.15) that to find the frequency of small oscillations, we simply need to calculate the second derivative of the potential. For the problem at hand, we must use the effective potential, because that is what determines the motion of the variable $r$. We therefore have

$$\omega_r = \sqrt{\frac{V''_{\text{eff}}(r_0)}{m}}\,. \tag{6.64}$$

If you work through the $r \equiv r_0 + \epsilon$ method described above, you will find that you are basically calculating the second derivative of $V_{\text{eff}}$, but in a rather cumbersome way.

Using the form of the effective potential, we have

$$\begin{aligned} V''_{\text{eff}}(r_0) &= \frac{3L^2}{mr_0^4} + \beta k(k-1)r_0^{k-2} \\ &= \frac{1}{r_0^4}\left(\frac{3L^2}{m} + \beta k(k-1)r_0^{k+2}\right). \end{aligned} \tag{6.65}$$

Using the $r_0$ from eq. (6.63), this simplifies to

$$V''_{\text{eff}}(r_0) = \frac{L^2(k+2)}{mr_0^4} \qquad \Longrightarrow \qquad \omega_r = \sqrt{\frac{V''_{\text{eff}}(r_0)}{m}} = \frac{L\sqrt{k+2}}{mr_0^2}\,. \tag{6.66}$$

We could get rid of the $r_0$ here by using eq. (6.63), but this form of $\omega_r$ will be more useful in part (c).

Note that we must have $k > -2$ for $\omega_r$ to be real. If $k < -2$, then $V''_{\text{eff}}(r_0) < 0$, which means that we have a local maximum of $V_{\text{eff}}$, instead of a local minimum. In other words, the circular orbit is unstable. Small perturbations grow, instead of oscillating around zero.

(c) Since $L = mr_0^2\dot{\theta}$ for the circular orbit, we have

$$\omega_\theta \equiv \dot{\theta} = \frac{L}{mr_0^2}\,. \tag{6.67}$$

Combining this with eq. (6.66), we find

$$\frac{\omega_r}{\omega_\theta} = \sqrt{k+2}. \tag{6.68}$$

A few values of $k$ that yield rational values for this ratio are (the plots of the orbits are shown below):

- $k = -1 \Longrightarrow \omega_r/\omega_\theta = 1$. This is the gravitational potential. The variable $r$ makes one oscillation for each complete revolution of the (nearly) circular orbit.
- $k = 2 \Longrightarrow \omega_r/\omega_\theta = 2$: This is the spring potential. The variable $r$ makes two oscillations for each complete revolution.
- $k = 7 \Longrightarrow \omega_r/\omega_\theta = 3$: The variable $r$ makes three oscillations for each complete revolution.
- $k = -7/4 \Longrightarrow \omega_r/\omega_\theta = 1/2$: The variable $r$ makes half of an oscillation for each complete revolution. So we need to have two revolutions to get back to the same value of $r$.

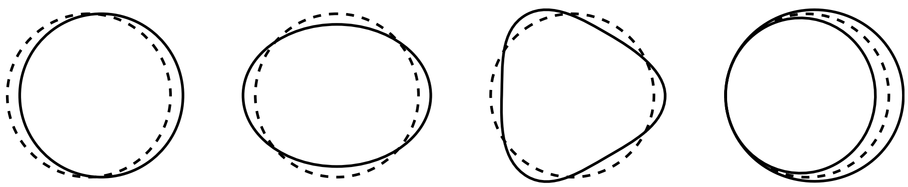



**Figure 6.15**

There is an infinite number of $k$ values that yield closed orbits. But note that this statement applies only to orbits that are nearly circular. The "closed" nature of the orbits is only approximate, because it is based on eq. (6.64) which is an approximate result based on small oscillations. The only $k$ values that lead to exactly closed orbits for any initial conditions are $k = -1$ (gravity) and $k = 2$ (spring), and in both cases the orbits are ellipses. This result is known as Bertrand's Theorem.

5. **Spring ellipse**

With $V(r) = \beta r^2$, eq. (6.16) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2m\beta r^2}{L^2}\,. \tag{6.69}$$

As stated in Section 6.4.1, we could take a square root, separate variables, integrate to find $\theta(r)$, and then invert to find $r(\theta)$. But let's solve for $r(\theta)$ in a slick way, as we did for the gravitational case, where we made the change of variables, $y \equiv 1/r$. Since there are lots of $r^2$ terms floating around in eq. (6.69), it is reasonable to try the change of variables, $y \equiv r^2$ or $y \equiv 1/r^2$. The latter turns out to be the better choice. So, using $y \equiv 1/r^2$ and $dy/d\theta = -2(dr/d\theta)/r^3$, and multiplying eq. (6.69) through by $1/r^2$, we obtain

$$\begin{aligned}\left(\frac{1}{2}\frac{dy}{d\theta}\right)^2 &= \frac{2mEy}{L^2} - y^2 - \frac{2m\beta}{L^2}\,. \\ &= -\left(y - \frac{mE}{L^2}\right)^2 - \frac{2m\beta}{L^2} + \left(\frac{mE}{L^2}\right)^2. \end{aligned} \tag{6.70}$$

Defining $z \equiv y - mE/L^2$ for convenience, we have

$$\begin{aligned}\left(\frac{dz}{d\theta}\right)^2 &= -4z^2 + 4\left(\frac{mE}{L^2}\right)^2\left(1 - \frac{2\beta L^2}{mE^2}\right) \\ &\equiv -4z^2 + 4B^2. \end{aligned} \tag{6.71}$$

As in Section 6.4.1, we can just look at this equation and observe that

$$z = B\cos 2(\theta - \theta_0) \tag{6.72}$$

is the solution. We can rotate the axes so that $\theta_0 = 0$, so we'll drop the $\theta_0$ from here on. Recalling our definition $z \equiv 1/r^2 - mE/L^2$ and also the definition of $B$ from eq. (6.71), eq. (6.72) becomes

$$\frac{1}{r^2} = \frac{mE}{L^2}(1 + \epsilon\cos 2\theta), \tag{6.73}$$

where

$$\epsilon \equiv \sqrt{1 - \frac{2\beta L^2}{mE^2}}\,. \tag{6.74}$$

It turns out, as we will see below, that $\epsilon$ is *not* the eccentricity of the ellipse, as it was in the gravitational case.

We will now use the procedure in Section 6.4.3 to show that eq. (6.74) represents an ellipse. For convenience, let

$$k \equiv \frac{L^2}{mE}\,. \tag{6.75}$$

Multiplying eq. (6.73) through by $kr^2$, and using

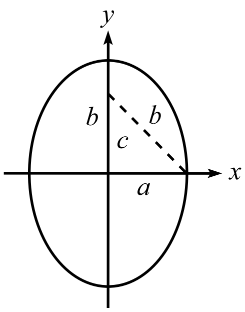



**Figure 6.16**

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{x^2}{r^2} - \frac{y^2}{r^2}\,, \tag{6.76}$$

and also $r^2 = x^2 + y^2$, we obtain $k = (x^2 + y^2) + \epsilon(x^2 - y^2)$. This can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a = \sqrt{\frac{k}{1+\epsilon}}\,, \quad \text{and } b = \sqrt{\frac{k}{1-\epsilon}}\,. \tag{6.77}$$

This is the equation for an ellipse with its center located at the origin (as opposed to its focus located at the origin, as it was in the gravitational case). The semi-major and semi-minor axes are $b$ and $a$, respectively, and the focal length is $c = \sqrt{b^2 - a^2} = \sqrt{2k\epsilon/(1-\epsilon^2)}$ (see Fig. 6.16). The eccentricity is $c/b = \sqrt{2\epsilon/(1+\epsilon)}$.

Remark: If $\epsilon = 0$, then $a = b$, which means that the ellipse is actually a circle. Let's see if this makes sense. Looking at eq. (6.74), we see that we want to show that circular motion implies $2\beta L^2 = mE^2$. For circular motion, the radial $F = ma$ equation is $mv^2/r = 2\beta r \Longrightarrow v^2 = 2\beta r^2/m$. The energy is therefore $E = mv^2/2 + \beta r^2 = 2\beta r^2$. Also, the square of the angular momentum is $L^2 = m^2v^2r^2 = 2m\beta r^4$. Therefore, $2\beta L^2 = 2\beta(2m\beta r^4) = m(2\beta r^2)^2 = mE^2$, as we wanted to show. ♣

6. $\beta/r^2$ **potential**

With $V(r) = \beta/r^2$, eq. (6.16) becomes

$$\begin{aligned}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 &= \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2m\beta}{r^2L^2} \\ &= \frac{2mE}{L^2} - \frac{1}{r^2}\left(1 + \frac{2m\beta}{L^2}\right).\end{aligned} \tag{6.78}$$

Letting $y \equiv 1/r$, this becomes

$$\left(\frac{dy}{d\theta}\right)^2 + a^2y^2 = \frac{2mE}{L^2}\,, \qquad \text{where } a^2 \equiv 1 + \frac{2m\beta}{L^2}\,. \tag{6.79}$$

We must now consider various possibilities for $a^2$. These possibilities depend on how $\beta$ compares to $L^2$ (which depends on the initial conditions of the motion). In what follows, note that the effective potential equals

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{\beta}{r^2} = \frac{a^2L^2}{2mr^2}\,. \tag{6.80}$$

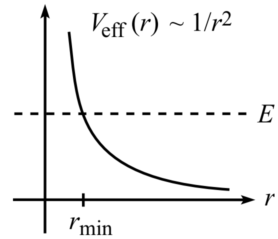



**Figure 6.17**

- $a^2 > 0$, or equivalently, $\beta > -L^2/2m$: In this case, the effective potential looks like the graph in Fig. 6.17. The solution for $y$ in eq. (6.79) is a trig function, which we will take to be a "sin" by appropriately rotating the axes. Using $y \equiv 1/r$, we obtain

$$\frac{1}{r} = \frac{1}{a}\sqrt{\frac{2mE}{L^2}} \sin a\theta. \tag{6.81}$$

$\theta = 0$ and $\theta = \pi/a$ make the right-hand side equal to zero, so they correspond to $r = \infty$. And $\theta = \pi/2a$ makes the right-hand side maximum, so it corresponds to the minimum value of $r$, which is $r_{\text{min}} = a\sqrt{L^2/2mE}$. This minimum $r$ can also be obtained in a much quicker manner by finding where $V_{\text{eff}}(r) = E$.

If the particle comes in from infinity (at $\theta = 0$), we see that it eventually heads back out to infinity (at $\theta = \pi/a$). The angle that the incoming path makes with the outgoing path is therefore $\pi/a$. So if $a$ is large (that is, if $\beta$ is large and positive, or if $L$ is small), then the particle bounces nearly straight backwards. If $a$ is small (that is, if $\beta$ is negative, and if $L^2$ is only slightly larger than $-2m\beta$), then the particle spirals around many times before popping back out to infinity.

A few special cases are: (1) $\beta = 0 \Longrightarrow a = 1$, which means that the total angle is $\pi$, that is, there is no net deflection. In fact, the particle's path is a straight line, because the potential is zero; see Exercise 11. (2) $L^2 = -8m\beta/3 \Longrightarrow a = 1/2$, which means that the total angle is $2\pi$, that is, the particle eventually comes back out along the same line that it went in.

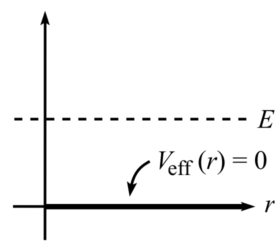



**Figure 6.18**

- $a = 0$, or equivalently, $\beta = -L^2/2m$: In this case, the effective potential is identically zero, as shown in Fig. 6.18. Eq. (6.79) becomes

$$\left(\frac{dy}{d\theta}\right)^2 = \frac{2mE}{L^2}\,. \tag{6.82}$$

The solution to this is $y = \theta\sqrt{2mE/L^2} + C$, which gives

$$r = \frac{1}{\theta}\sqrt{\frac{L^2}{2mE}}\,, \tag{6.83}$$

where we have set the integration constant, $C$, equal to zero by choosing $\theta = 0$ to be the angle that corresponds to $r = \infty$. Note that we can use $\beta = -L^2/2m$ to write $r$ as $r = \sqrt{-\beta/E}/\theta$.

Since the effective potential is flat, the rate of change of $r$ is constant. If the particle has $\dot{r} < 0$, it will therefore reach the origin in finite time, even though eq. (6.83) say that it will spiral around the origin an infinite number of times (because $\theta \to \infty$ as $r \to 0$).

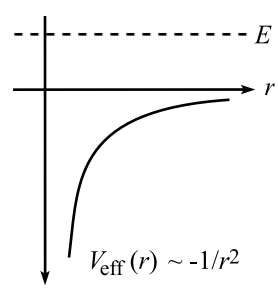



**Figure 6.19**

- $a^2 < 0$, or equivalently, $\beta < -L^2/2m$: In this case, the effective potential looks like the graph in either Fig. 6.19 or Fig. 6.20, depending on the sign of $E$. For convenience, let $b$ be the positive real number such that $b^2 = -a^2$. Then eq. (6.79) becomes

$$\left(\frac{dy}{d\theta}\right)^2 - b^2y^2 = \frac{2mE}{L^2}\,. \tag{6.84}$$

The solution to this equation is a hyperbolic trig function. But we must consider two cases:

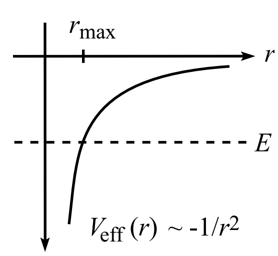



**Figure 6.20**

(a) $E > 0$: Using the identity $\cosh^2 z - \sinh^2 z = 1$, and recalling $y \equiv 1/r$, we see that the solution to eq. (6.84) is[11]

$$\frac{1}{r} = \frac{1}{b}\sqrt{\frac{2mE}{L^2}}\sinh b\theta. \tag{6.85}$$

Unlike the $a^2 > 0$ case above, the sinh function has no maximum value. Therefore, the right-hand side can head to infinity, which means that $r$ can head to zero. Note that for large $z$, we have $\sinh z \approx e^z/2$. So $r$ heads to zero like $e^{-b\theta}$, in other words, exponentially quickly.

(b) $E < 0$: In this case, eq. (6.84) can be rewritten as

$$b^2y^2 - \left(\frac{dy}{d\theta}\right)^2 = \frac{2m|E|}{L^2}\,. \tag{6.86}$$

The solution to this equation is[12]

$$\frac{1}{r} = \frac{1}{b}\sqrt{\frac{2m|E|}{L^2}}\cosh b\theta. \tag{6.87}$$

As in the sinh case, the cosh function has no maximum value. Therefore, the right-hand side can head to infinity, which means that $r$ can head to zero. But in the present cosh case, the right-hand side does achieve a nonzero minimum value, when $\theta = 0$. So $r$ achieves a maximum value (this is clear from Fig. 6.20) equal to $r_{\text{max}} = b\sqrt{L^2/2m|E|}$. This maximum $r$ can also be obtained by simply finding where $V_{\text{eff}}(r) = E$. After reaching $r_{\text{max}}$, the particle heads back down to the origin.

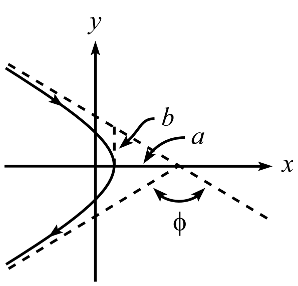


**Figure 6.21**

7. **Rutherford scattering**

(a) From Exercise 6, we know that the impact parameter, $b$, equals the distance $b$ shown in Fig. 6.9. Therefore, Fig. 6.21 tells us that the angle of deflection (the angle between the initial and final velocity vectors) is

$$\phi = \pi - 2\tan^{-1}\left(\frac{b}{a}\right). \tag{6.88}$$

But from eqs. (6.33) and (6.25), we have

$$\frac{b}{a} = \sqrt{\epsilon^2 - 1} = \sqrt{\frac{2EL^2}{m\alpha^2}} = \sqrt{\frac{2(mv_0^2/2)(mv_0b)^2}{m(GMm)^2}} = \frac{v_0^2 b}{GM}\,. \tag{6.89}$$

Substituting this into eq. (6.88), with $\gamma \equiv v_0^2/(GM)$, gives the first expression in eq. (6.49). Dividing by 2 and taking the cotangent of both sides then gives the second expression,

$$b = \frac{1}{\gamma}\cot\left(\frac{\phi}{2}\right). \tag{6.90}$$

[11]More generally, we should write $\sinh(\theta - \theta_0)$ here. But we can eliminate the need for $\theta_0$ by picking $\theta = 0$ to be the angle that corresponds to $r = \infty$.

[12]Again, we should write $\cosh(\theta - \theta_0)$ here. But we can eliminate the need for $\theta_0$ by picking $\theta = 0$ to be the angle that corresponds to the maximum value of $r$.

Note that it actually isn't necessary to go through all the work of Section 6.4.3 to obtain this result, by determining $a$ and $b$. We can simply use eq. (6.24), which says that $r \to \infty$ when $\cos\theta \to -1/\epsilon$. This then implies that the dotted lines in Fig. 6.21 have slope $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\epsilon^2 - 1}$, which reproduces eq. (6.89).

(b) Imagine a wide beam of particles moving in the positive $x$-direction, toward the mass $M$. Consider a thin cross-sectional ring in this beam, with radius $b$ and thickness $db$. Now consider a large sphere centered at $M$. Any particle that passed through the cross-sectional ring of radius $b$ will hit this sphere in a ring located at an angle $\phi$ relative to the $x$-axis, with an angular spread of $d\phi$. The relation between $db$ and $d\phi$ is found from eq. (6.90). Using $d(\cot\beta)/d\beta = -1/\sin^2\beta$, we have

$$\left|\frac{db}{d\phi}\right| = \frac{1}{2\gamma\sin^2(\phi/2)}\,. \tag{6.91}$$

The area of the incident cross-sectional ring is $d\sigma = 2\pi b\,|db|$. What is the solid angle subtended by a ring at angle $\phi$ with thickness $d\phi$? Taking the radius of the sphere to be $R$ (which will cancel out), the radius of the ring is $R\sin\phi$, and the linear thickness is $R\,|d\phi|$. The area of the ring is therefore $2\pi(R\sin\phi)(R\,|d\phi|)$, and so the solid angle subtended by the ring is $d\Omega = 2\pi\sin\phi\,|d\phi|$ steradians. Therefore, the differential cross section is

$$\begin{aligned}\frac{d\sigma}{d\Omega} = \frac{2\pi b\,|db|}{2\pi\sin\phi\,|d\phi|} &= \left(\frac{b}{\sin\phi}\right)\left|\frac{db}{d\phi}\right| \\ &= \left(\frac{(1/\gamma)\cot(\phi/2)}{2\sin(\phi/2)\cos(\phi/2)}\right)\left(\frac{1}{2\gamma\sin^2(\phi/2)}\right) \\ &= \frac{1}{4\gamma^2\sin^4(\phi/2)}\,. \end{aligned} \tag{6.92}$$

Remarks: What does this "differential cross section" result tell us? It tells us that if we want to find out how much cross-sectional area gets mapped into the solid angle $d\Omega$ at the angle $\phi$, then we can simply use eq. (6.92) to say (recalling $\gamma \equiv v_0^2/(GM)$),

$$d\sigma = \frac{G^2M^2}{4v_0^4\sin^4(\phi/2)}\,d\Omega\,. \tag{6.93}$$

Let's look at some special cases. If $\phi \approx 180^\circ$ (that is, backward scattering), then the amount of area that gets scattered into a nearly backward solid angle of $d\Omega$ equals $d\sigma = (G^2M^2/4v_0^4)\,d\Omega$. If $v_0$ is small, then we see that $d\sigma$ is large, that is, a large area gets deflected nearly straight backwards. This makes sense, because with $v_0 \approx 0$, the orbits are essentially parabolic, which means that the initial and final velocities at infinity are (anti)parallel. (If you release a particle from rest far away from a gravitational source, it will come back to you. Assuming it doesn't bump into the source, of course.) If $v_0$ is large, then we see that $d\sigma$ is small, that is, only a small area gets deflected backwards. This makes sense, because the particles are more likely to fly past $M$ without any deflection if they are moving fast, because the force has less time to act.

Another special case is $\phi \approx 0^\circ$ (that is, there is negligible deflection). In this case, eq. (6.93) tells us that the amount of area that gets scattered into a nearly forward solid angle of $d\Omega$ equals $d\sigma \approx \infty$. This makes sense, because if the impact parameter is large (and there is an infinite cross-sectional area for which this is true), then the particle will hardly feel the mass $M$, and will therefore continue to move essentially in a straight line.

What if we consider the electrostatic force, instead of the gravitational force? What is the differential cross section is that case? To answer this, note that we may rewrite $\gamma$ as

$$\gamma = \frac{v_0^2}{GM} = \frac{2(mv_0^2/2)}{GMm} \equiv \frac{2E}{\alpha}\,. \tag{6.94}$$

In the case of electrostatics, the force takes the form, $F_{\rm e} = kq_1q_2/r^2$. This looks like the gravitational force, $F_{\rm g} = Gm_1m_2/r^2$, except that the constant $\alpha$ is now $kq_1q_2$, instead of $Gm_1m_2$. Therefore, the $\gamma$ in eq. (6.94) becomes $\gamma_{\rm e} = 2E/(kq_1q_2)$. Substituting this into eq. (6.92), we see that the differential cross section for electrostatic scattering is

$$\frac{d\sigma}{d\Omega} = \frac{k^2q_1^2q_2^2}{16E^2\sin^4(\phi/2)}\,. \tag{6.95}$$

This is the Rutherford scattering differential cross section formula. Around 1910, Rutherford and his students bombarded metal foils with alpha particles. Their results for the distribution of scattering angles were consistent with the above formula. In particular, they observed back-scattering of the alpha particles. Since the above formula is based on the assumption of a point source for the potential, this led Rutherford to his theory that atoms contained a dense positively-charged nucleus, as opposed to being made of a spread-out "plum pudding" distribution of charge, which (as a special case of not yielding the correct distribution of scattering angles) doesn't yield back-scattering. ♣

# Chapter 7

# Angular Momentum, Part I (Constant $\hat{\mathrm{L}}$)



The angular momentum of a point mass, relative to a given origin, is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{7.1}$$

For a collection of particles, the total $\mathbf{L}$ is simply the sum of the $\mathbf{L}$'s of each particle.

The quantity $\mathbf{r} \times \mathbf{p}$ is a useful thing to study because it has many nice properties. One of these is the conservation law presented in Theorem 6.1, which allowed us to introduce the "effective potential" in Section 6.2. And later in this chapter we will introduce the concept of *torque*, $\boldsymbol{\tau}$, which appears in the bread-and-butter statement, $\boldsymbol{\tau} = d\mathbf{L}/dt$ (analogous to Newton's $\mathbf{F} = d\mathbf{p}/dt$ law).

There are two basic types of angular momentum problems in the world. Since the solution to any rotational problem invariably comes down to using $\boldsymbol{\tau} = d\mathbf{L}/dt$, we must determine how $\mathbf{L}$ changes in time. And since $\mathbf{L}$ is a vector, it can change because (1) its length changes, or (2) its direction changes (or through some combination of these effects). In other words, if we write $\mathbf{L} = L\hat{\mathbf{L}}$, where $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction, then $\mathbf{L}$ can change because $L$ changes, or because $\hat{\mathbf{L}}$ changes, or both.

The first of these cases, that of constant $\hat{\mathbf{L}}$, is the easily understood one. Consider a spinning record. The vector $\mathbf{L} = \sum \mathbf{r} \times \mathbf{p}$ is perpendicular to the record. If we give the record a tangential force in the proper direction, then it will speed up (in a precise way which we will soon determine). There is nothing mysterious going on here. If we push on the record, it goes faster. $\mathbf{L}$ points in the same direction as before, but it now simply has a larger magnitude. In fact, in this type of problem, we can completely forget that $\mathbf{L}$ is a vector. We can just deal with its magnitude $L$, and everything will be fine. This first case is the subject of the present chapter.

The second case however, where $\mathbf{L}$ changes direction, can get rather confusing. This is the subject of the following chapter, where we will talk about gyroscopes, tops, and other such spinning objects that have a tendency to make one's head spin also. In these situations, the entire point is that $\mathbf{L}$ is actually a vector. And unlike in the constant-$\hat{\mathbf{L}}$ case, we really have to visualize things in three dimensions to see what's going on.[1]

The angular momentum of a point mass is given by the simple expression in eq. (7.1). But in order to deal with setups in the real world, which invariably consist of many particles, we must learn how to calculate the angular momentum of an extended object. This is the task of the Section 7.1. We will deal only with motion in the $x$-$y$ plane in this chapter. Any rotations we talk about will therefore be around the $z$-axis (or an axis parallel to the $z$-axis). We'll save the general 3-D case for Chapter 8.

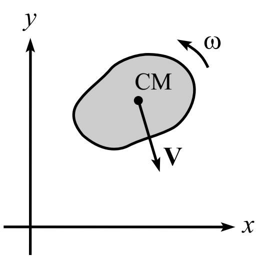


**Figure 7.1**

## 7.1 Pancake object in $x$-$y$ plane

Consider a flat, rigid body undergoing arbitrary motion (both translating and spinning) in the $x$-$y$ plane; see Fig. 7.1. What is the angular momentum of this body, relative to the origin of the coordinate system?[2]

If we imagine the body to consist of particles of mass $m_i$, then the angular momentum of the entire body is the sum of the angular momenta of each $m_i$, which are $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$. So the total angular momentum is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i. \tag{7.2}$$

For a continuous distribution of mass, we would have an integral instead of a sum.

$\mathbf{L}$ depends on the locations and momenta of the masses. The momenta in turn depend on how fast the body is translating and spinning. Our goal here is to find the dependence of $\mathbf{L}$ on the distribution and motion of its constituent masses. The result will involve the geometry of the body in a specific way, as we will show.

In this section, we will deal only with pancake-like objects that move in the $x$-$y$ plane (or simple extensions of these). We will find $\mathbf{L}$ relative to the origin, and we will also derive an expression for the kinetic energy. We will deal with non-pancake objects in Section 7.2.

Note that since both $\mathbf{r}$ and $\mathbf{p}$ for our pancake-like objects always lie in the $x$-$y$ plane, the vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ always points in the $\hat{\mathbf{z}}$ direction. This fact is what makes these pancake cases easy to deal with; $\mathbf{L}$ changes only because its length changes, not its direction. So when we eventually get to the $\boldsymbol{\tau} = d\mathbf{L}/dt$ equation, it will take on a simple form.

Let's first look at a special case, and then we'll look at general motion in the $x$-$y$ plane.

[1]The difference between these two cases is essentially the same as the difference between the two basic $\mathbf{F} = d\mathbf{p}/dt$ cases. The vector $\mathbf{p}$ can change simply because its magnitude changes, in which case we have $F = ma$. Or, $\mathbf{p}$ can change because its direction changes, in which case we have the centripetal-acceleration statement, $F = mv^2/r$. (Or, there could be a combination of these effects.) The former case seems a bit more intuitive than the latter.

[2]Remember, $\mathbf{L}$ is defined relative to a chosen origin (because it has the vector $\mathbf{r}$ in it), so it makes no sense to ask what $\mathbf{L}$ is, without specifying what origin you've chosen.

### 7.1.1 Rotation about the $z$-axis

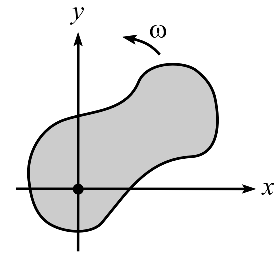



**Figure 7.2**

The pancake in Fig. 7.2 rotates with angular speed $\omega$ around the $z$-axis, in the counterclockwise direction (as viewed from above). Consider a little piece of the body, with mass $dm$ and position $(x, y)$. Let $r = \sqrt{x^2 + y^2}$. This little piece travels in a circle around the origin with speed $v = \omega r$. Therefore, the angular momentum of this piece (relative to the origin) is equal to $\mathbf{L} = \mathbf{r} \times \mathbf{p} = r(v\, dm)\hat{\mathbf{z}} = dm\, r^2\omega\hat{\mathbf{z}}$. The $\hat{\mathbf{z}}$ direction arises from the cross product of the (orthogonal) vectors $\mathbf{r}$ and $\mathbf{p}$. The angular momentum of the entire body is therefore

$$\begin{aligned}\mathbf{L} &= \int r^2\omega\hat{\mathbf{z}}\, dm \\ &= \int (x^2 + y^2)\omega\hat{\mathbf{z}}\, dm,\end{aligned} \tag{7.3}$$

where the integration runs over the area of the body. If the density of the object is constant, as is usually the case, then we have $dm = \rho\, dx\, dy$. If we define the *moment of inertia* around the $z$-axis to be

$$I_z \equiv \int r^2\, dm = \int (x^2 + y^2)\, dm, \tag{7.4}$$

then the $z$-component of $\mathbf{L}$ is

$$L_z = I_z\omega, \tag{7.5}$$

and $L_x$ and $L_y$ are both equal to zero. In the case where the rigid body is made up of a collection of point masses $m_i$ in the $x$-$y$ plane, the moment of inertia in eq. (7.4) simply takes the discretized form,

$$I_z \equiv \sum_i m_i r_i^2. \tag{7.6}$$

Given any rigid body in the $x$-$y$ plane, we can calculate $I_z$. And given $\omega$, we can then multiply it by $I_z$ to find $L_z$. In Section 7.3.1, we will get some practice calculating various moments of inertia.

What is the kinetic energy of our object? We need to add up the energies of all the little pieces. A little piece has energy $dm\, v^2/2 = dm(r\omega)^2/2$. So the total kinetic energy is

$$T = \int \frac{r^2\omega^2}{2}\, dm. \tag{7.7}$$

With our definition of $I_z$ in eq. (7.4), we have

$$T = \frac{I_z\omega^2}{2}\,. \tag{7.8}$$

This is easy to remember, because it looks a lot like the kinetic energy of a point mass, $mv^2/2$.

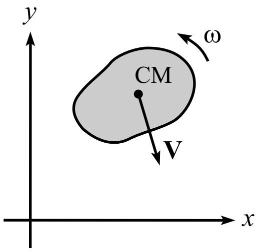



**Figure 7.3**

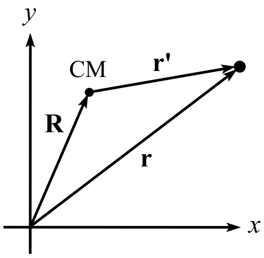



**Figure 7.4**

### 7.1.2 General motion in $x$-$y$ plane

How do we deal with general motion in the $x$-$y$ plane? For the motion in Fig. 7.3, where the object is both translating and spinning, the various pieces of mass do not travel in circles around the origin, so we cannot write $v = \omega r$ as we did above.

It turns out to be highly advantageous to write the angular momentum, **L**, and the kinetic energy, $T$, in terms of the center-of-mass (CM) coordinates and the coordinates relative to the CM. The expressions for **L** and $T$ take on very nice forms when written this way, as we now show.

Let the coordinates of the CM be $\mathbf{R} = (X, Y)$, and let the coordinates of a given point relative to the CM be $\mathbf{r}' = (x', y')$. Then the given point has coordinates $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 7.4). Let the velocity of the CM be $\mathbf{V}$, and let the velocity relative to the CM be $\mathbf{v}'$. Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$. Let the body rotate with angular speed $\omega'$ around the CM (around an instantaneous axis parallel to the $z$-axis, so that the pancake remains in the $x$-$y$ plane at all times).[3] Then $v' = \omega' r'$.

Let's look at **L** first. The angular momentum relative to the origin is

$$
\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times (\mathbf{V} + \mathbf{v}')\, dm \\
&= M\mathbf{R} \times \mathbf{V} + \int \mathbf{r}' \times \mathbf{v}'\, dm \qquad \text{(cross terms vanish; see below)} \\
&= M\mathbf{R} \times \mathbf{V} + \left( \int r'^2 \omega' dm \right) \hat{\mathbf{z}} \\
&\equiv M\mathbf{R} \times \mathbf{V} + \left( I_z^{\text{CM}} \omega' \right) \hat{\mathbf{z}},
\end{aligned}
\tag{7.9}
$$

where $M$ is the mass of the pancake. In going from the second to the third line above, the cross terms, $\int \mathbf{r}' \times \mathbf{V}\, dm$ and $\int \mathbf{R} \times \mathbf{v}'\, dm$, vanish by definition of the CM, which says that $\int \mathbf{r}'\, dm = 0$ (see eq. (4.69)), and hence $\int \mathbf{v}'\, dm = d(\int \mathbf{r}'\, dm)/dt = 0$. The quantity $I_z^{\text{CM}}$ is the moment of inertia around an axis through the CM, parallel to the $z$-axis. Eq. (7.9) is a very nice result, and it is important enough to be called a theorem. In words, it says:

**Theorem 7.1** *The angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the CM and finding the angular momentum of this point mass relative to the origin, and by then adding on the angular momentum of the body relative to the CM.* [4]

---

[3] What we mean here is the following. Consider a coordinate system whose origin is the CM and whose axes are parallel to the fixed $x$- and $y$-axes. Then the pancake rotates with angular speed $\omega'$ in this reference frame.

[4] This theorem only works if we use the CM as the location of the imagined point mass. True, in the above analysis we could have chosen a point $P$ other than the CM, and then written things in terms of the coordinates of $P$ and the coordinates relative to $P$ (which could also be described by a rotation). But then the cross terms in eq. (7.9) wouldn't vanish, and we'd end up with an unenlightening mess.

Note that if we have the special case where the CM travels around the origin in a circle, with angular speed $\Omega$ (so that $V = \Omega R$), then eq. (7.9) becomes $\mathbf{L} = \left(MR^2\Omega + I_z^{\text{CM}}\omega'\right)\hat{\mathbf{z}}$.

Now let's look at $T$. The kinetic energy is

$$\begin{aligned} T &= \int \frac{1}{2}v^2\,dm \\ &= \int \frac{1}{2}|\mathbf{V}+\mathbf{v}'|^2\,dm \\ &= \frac{1}{2}MV^2 + \int \frac{1}{2}v'^2\,dm \qquad \text{(cross term vanishes; see below)} \\ &= \frac{1}{2}MV^2 + \int \frac{1}{2}r'^2\omega'^2 dm \\ &\equiv \frac{1}{2}MV^2 + \frac{1}{2}I_z^{\text{CM}}\omega'^2. \end{aligned} \tag{7.10}$$

In going from the second to third line above, the cross term $\int \mathbf{V}\cdot\mathbf{v}'\,dm$ vanishes by definition of the CM, as in the above calculation of $\mathbf{L}$. Again, eq. (7.10) is a very nice result. In words, it says:

**Theorem 7.2** *The kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to the motion relative to the CM.*

To calculate $E$, my dear class,
Just add up two things, and you'll pass.
Take the CM point's $E$,
And then add on with glee,
The $E$ 'round the center of mass.

### 7.1.3 The parallel-axis theorem

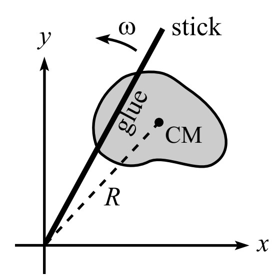


**Figure 7.5**

Consider the special case where the CM rotates around the origin at the same rate as the body rotates around the CM. This may be achieved, for example, by gluing a stick across the pancake and pivoting one end of the stick at the origin; see Fig. 7.5. In this special case, we have the simplified situation where all points in the pancake travel in circles around the origin. Let their angular speed be $\omega$.

In this situation, the speed of the CM is $V = \omega R$, so eq. (7.9) says that the angular momentum around the origin is

$$L_z = (MR^2 + I_z^{\text{CM}})\omega. \tag{7.11}$$

In other words, the moment of inertia around the origin is

$$\boxed{I_z = MR^2 + I_z^{\text{CM}}}. \tag{7.12}$$

This is the *parallel-axis theorem.* It says that once you've calculated the moment of inertia of an object around the axis passing through the CM (namely $I_z^{\text{CM}}$), then if you want to calculate the moment of inertia around a parallel axis passing through an arbitrary point in the plane of the pancake, you simply have to add on $MR^2$, where $R$ is the distance from the point to the CM, and $M$ is the mass of the pancake.

Note that the parallel-axis theorem is simply a special case of the more general result in eq. (7.9), so it is valid *only* with the CM, and not with any other point.

We can also look at the kinetic energy in this special case where the CM rotates around the origin at the same rate as the body rotates around the CM. Using $V = \omega R$ in eq. (7.10), we find

$$T = \frac{1}{2}(MR^2 + I_z^{\mathrm{CM}})\omega^2. \tag{7.13}$$

---

**Example (A stick):** Let's verify the parallel-axis theorem for a stick of mass $m$ and length $\ell$, in the case where we want to compare the moment of inertia around an axis through an end with the moment of inertia around an axis through the CM. Both of the axes are perpendicular to the stick, and parallel to each other, of course.

For convenience, let $\rho = m/\ell$ be the density. The moment of inertia around an axis through an end is

$$I^{\mathrm{end}} = \int_0^\ell x^2\,dm = \int_0^\ell x^2\rho\,dx = \frac{1}{3}\rho\ell^3 = \frac{1}{3}(\rho\ell)\ell^2 = \frac{1}{3}m\ell^2. \tag{7.14}$$

The moment of inertia around an axis through the CM is

$$I^{\mathrm{CM}} = \int_{-\ell/2}^{\ell/2} x^2\,dm = \int_{-\ell/2}^{\ell/2} x^2\rho\,dx = \frac{1}{12}\rho\ell^3 = \frac{1}{12}m\ell^2. \tag{7.15}$$

This is consistent with the parallel-axis theorem, eq. (7.12), because

$$I^{\mathrm{end}} = m\left(\frac{\ell}{2}\right)^2 + I^{\mathrm{CM}}. \tag{7.16}$$

Remember that this works only with the CM. If we instead want to compare $I^{\mathrm{end}}$ with the $I$ around a point, say, $\ell/6$ from that end, then we cannot say that they differ by $m(\ell/6)^2$. But we *can* compare each of them to $I^{\mathrm{CM}}$ and say that they differ by $(\ell/2)^2 - (\ell/3)^2 = 5\ell^2/36$.

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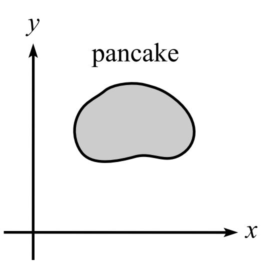


**Figure 7.6**

### 7.1.4 The perpendicular-axis theorem

This theorem is valid *only* for pancake objects. Consider a pancake object in the $x$-$y$ plane (see Fig. 7.6). Then the *perpendicular-axis theorem* says that

$$I_z = I_x + I_y, \tag{7.17}$$

where $I_x$ and $I_y$ are defined analogously to the $I_z$ in eq. (7.4). That is, to find $I_x$, imagine spinning the object around the $x$-axis at angular speed $\omega$, and then define $I_x \equiv L_x/\omega$. Likewise for $I_y$. In other words,

$$I_x \equiv \int (y^2 + z^2)\, dm, \qquad I_y \equiv \int (z^2 + x^2)\, dm, \qquad I_z \equiv \int (x^2 + y^2)\, dm. \quad (7.18)$$

To prove this theorem, we simply use the fact that $z = 0$ for our pancake object. Eq. (7.18) then gives $I_z = I_x + I_y$.

In the limited number of situations where this theorem is applicable, it can save you some trouble. A few examples are given in Section 7.3.1

## 7.2 Non-planar objects

In Section 7.1, we restricted the discussion to pancake objects in the $x$-$y$ plane. However, nearly all the results we derived carry over to non-planar objects, provided that the axis of rotation is parallel to the $z$-axis, and provided that we are concerned only with $L_z$, and not $L_x$ or $L_y$. So let's drop the pancake assumption and run through the results we obtained above.

First, consider an object rotating around the $z$-axis. Let the object have extension in the $z$ direction. If we imagine slicing the object into pancakes parallel to the $x$-$y$ plane, then eqs. (7.4) and (7.5) correctly give $L_z$ for each pancake. And since the $L_z$ of the whole object is simply the sum of the $L_z$'s of all the pancakes, we see that the $I_z$ of the whole object is simply the sum of the $I_z$'s of all the pancakes. The difference in the $z$ values of the pancakes is irrelevant. Therefore, for *any* object, we have

$$I_z = \int (x^2 + y^2)\, dm, \qquad \text{and} \qquad L_z = I_z\omega, \quad (7.19)$$

where the integration runs over the entire volume of the body. In Section 7.3.1 we will calculate $I_z$ for many non-planar objects.

Even though eq. (7.19) gives the $L_z$ for an arbitrary object, the analysis in this chapter is still not completely general because (1) we are restricting the axis of rotation to be the (fixed) $z$-axis, and (2) even with this restriction, an object outside the $x$-$y$ plane might have nonzero $x$ and $y$ components of $\mathbf{L}$; we found only the $z$-component in eq. (7.19) This second fact is strange but true. Ponder it for now; we'll deal with it in Section 8.2.

As far as the kinetic energy goes, the $T$ for a non-planar object rotating around the $z$-axis is still given by eq. (7.8), because we can obtain the total $T$ by simply adding up the $T$'s of each of the pancake slices.

Also, eqs. (7.9) and (7.10) continue to hold for a non-planar object in the case where the CM is translating while the object is spinning around it (or more precisely, spinning around an axis parallel to the $z$-axis and passing through the CM). The velocity $\mathbf{V}$ of the CM can actually point in any direction, and these two equations will still be valid. But we'll assume in this chapter that all velocities are in the $x$-$y$ plane.

Lastly, the parallel-axis theorem still holds for non-planar object. But as mentioned in Section 7.1.4, the perpendicular-axis theorem does *not*. This is the one instance where we need the planar assumption.

**Finding the CM**

The center of mass has come up repeatedly in this Chapter. For example, when we used the parallel-axis theorem, we needed to know where the CM was. In some cases, such as with a stick or a disk, the location is obvious. But in other cases, it isn't so clear. So let's get a little practice calculating the location of the CM. Depending on whether the mass distribution is discrete or continuous, the position of the CM is defined by (see eq. (4.69))

$$\mathbf{R}_{\text{CM}} = \frac{\sum \mathbf{r}_i m_i}{M}, \qquad \text{or} \qquad \mathbf{R}_{\text{CM}} = \frac{\int \mathbf{r}\, dm}{M}, \tag{7.20}$$

where $M$ is the total mass.

Let's do an example with a continuous mass distribution. As with many problems involving an integral, the main step in the solution is deciding how you want to slice up the object to do the integral.

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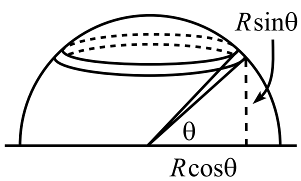


**Figure 7.7**

**Example (Hemispherical shell):** Find the location of the CM of a hollow hemispherical shell, with uniform mass density and radius $R$.

**Solution:** By symmetry, the CM is located on the line above the center of the base. So our task reduces to finding the height, $y_{\text{CM}}$. Let the mass density be $\sigma$. We'll slice the hemisphere up into horizontal rings, described by the angle $\theta$ above the horizontal, as shown in Fig. 7.7. If the angular thickness of a ring is $d\theta$, then its mass is

$$dm = \sigma\, dA = \sigma(\text{length})(\text{width}) = \sigma(2\pi R\cos\theta)(R\, d\theta). \tag{7.21}$$

All points on the ring have a $y$ value of $R\sin\theta$. Therefore,

$$\begin{aligned} y_{\text{CM}} = \frac{1}{M}\int y\, dm &= \frac{1}{(2\pi R^2)\sigma}\int_0^{\pi/2} (R\sin\theta)(2\pi R^2\sigma\cos\theta\, d\theta) \\ &= R\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \\ &= \frac{R\sin^2\theta}{2}\bigg|_0^{\pi/2} \\ &= \frac{R}{2}. \end{aligned} \tag{7.22}$$

The simple factor of 1/2 here is nice, but it's not all that obvious. It comes from the fact that each value of $y$ is represented equally. If you solved the problem by doing a $dy$ integral instead of a $d\theta$ one, you would find that there is the same area (and hence the same mass) in each ring of height $dy$. You are encouraged to work this out.

---

The calculation of a CM is very similar to the calculation of a moment of inertia. Both involve an integration over the mass of an object, but the former has one power of a length in the integrand, whereas the latter has two powers.

## 7.3 Calculating moments of inertia

### 7.3.1 Lots of examples

Let's now calculate the moments of inertia of various objects, around specified axes. We will use $\rho$ to denote mass density (per unit length, area, or volume, as appropriate). We will assume that this density is uniform throughout the object. For the more complicated of the objects below, it is generally a good idea to slice the object up into pieces for which $I$ is already known. The problem then reduces to integrating over these known $I$'s. There is usually more than one way to do this slicing. For example, a sphere may be looked at as a series of concentric shells or a collection of disks stacked on top of each other. In the examples below, you may want to play around with slicings other than the ones given. Consider at least a few of these examples to be problems and try to work them out for yourself.

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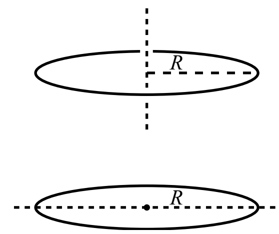


**Figure 7.8**

1. A ring of mass $M$ and radius $R$ (axis through center, perpendicular to plane; Fig. 7.8):

$$I = \int r^2\,dm = \int_0^{2\pi} R^2 \rho R\,d\theta = (2\pi R\rho)R^2 = \boxed{MR^2}, \tag{7.23}$$

as it should be, because all of the mass is a distance $R$ from the axis.

2. A ring of mass $M$ and radius $R$ (axis through center, in plane; Fig. 7.8):
The distance from the axis is (the absolute value of) $R\sin\theta$. Therefore,

$$I = \int r^2\,dm = \int_0^{2\pi} (R\sin\theta)^2 \rho R\,d\theta = \frac{1}{2}(2\pi R\rho)R^2 = \boxed{\tfrac{1}{2}MR^2}, \tag{7.24}$$

where we have used $\sin^2\theta = (1-\cos 2\theta)/2$. You can also find $I$ by using the perpendicular-axis theorem. In the notation of section 7.1.4, we have $I_x = I_y$, by symmetry. Therefore, $I_z = 2I_x$. Using $I_z = MR^2$ from Example 1 then gives $I_x = MR^2/2$.

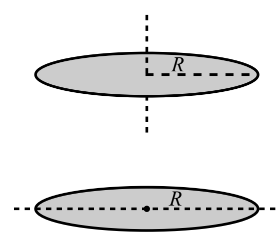


**Figure 7.9**

3. A disk of mass $M$ and radius $R$ (axis through center, perpendicular to plane; Fig. 7.9):

$$I = \int r^2\,dm = \int_0^{2\pi}\int_0^R r^2 \rho r\,dr\,d\theta = (R^4/4)2\pi\rho = \frac{1}{2}(\rho\pi R^2)R^2 = \boxed{\tfrac{1}{2}MR^2}. \tag{7.25}$$

You can save one (trivial) integration step by considering the disk to be made up of many concentric rings, and invoking Example 1. The mass of each ring is $\rho 2\pi r\,dr$. Integrating over the rings gives $I = \int_0^R (\rho 2\pi r\,dr)r^2 = \pi R^4\rho/2 = MR^2/2$, as above. Slicing up the disk is fairly inconsequential in this example, but it will save you some trouble in others.

4. A disk of mass $M$ and radius $R$ (axis through center, in plane; Fig. 7.9):

   Slice the disk up into rings, and use Example 2.

$$I = \int_0^R (1/2)(\rho 2\pi r\,dr)r^2 = (R^2/4)\rho\pi = \frac{1}{4}(\rho\pi R^2)R^2 = \boxed{\tfrac{1}{4}MR^2}. \tag{7.26}$$

   Or, just use Example 3 and the perpendicular-axis theorem.

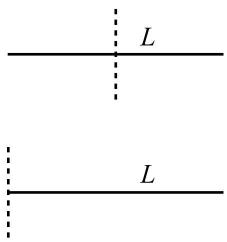


Figure 7.10

5. A thin uniform rod of mass $M$ and length $L$ (axis through center, perpendicular to rod; Fig. 7.10):

$$I = \int x^2 dm = \int_{-L/2}^{L/2} x^2 \rho\,dx = \frac{1}{12}(\rho L)L^2 = \boxed{\tfrac{1}{12}ML^2}. \tag{7.27}$$

6. A thin uniform rod of mass $M$ and length $L$ (axis through end, perpendicular to rod; Fig. 7.10):

$$I = \int x^2 dm = \int_0^L x^2 \rho\,dx = \frac{1}{3}(\rho L)L^2 = \boxed{\tfrac{1}{3}ML^2}. \tag{7.28}$$

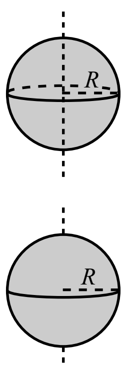


Figure 7.11

7. A spherical shell of mass $M$ and radius $R$ (any axis through center; Fig. 7.11):

   Let's slice the sphere into horizontal ring-like strips. In spherical coordinates, the radius of a ring is given by $r = R\sin\theta$, where $\theta$ is the angle down from the north pole. The area of a strip is then $2\pi(R\sin\theta)R\,d\theta$. Using $\int \sin^3\theta = \int \sin\theta(1-\cos^2\theta) = -\cos\theta + \cos^3\theta/3$, we have

$$\begin{aligned} I = \int r^2\,dm &= \int_0^\pi (R\sin\theta)^2\, 2\pi\rho(R\sin\theta)R\,d\theta = 2\pi\rho R^4 \int_0^\pi \sin^3\theta \\ &= 2\pi\rho R^4(4/3) = \frac{2}{3}(4\pi R^2\rho)R^2 = \boxed{\tfrac{2}{3}MR^2}. \end{aligned} \tag{7.29}$$

8. A solid sphere of mass $M$ and radius $R$ (any axis through center; Fig. 7.11):

   A sphere is made up of concentric spherical shells. The volume of a shell is $4\pi r^2 dr$. Using Example 7, we have

$$I = \int_0^R (2/3)(4\pi\rho r^2 dr)r^2 = (R^5/5)(8\pi\rho/3) = \frac{2}{5}(4/3\pi R^3\rho)R^2 = \boxed{\tfrac{2}{5}MR^2}. \tag{7.30}$$

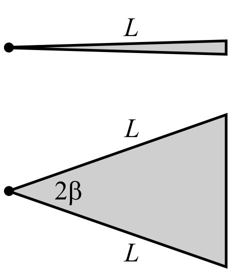


Figure 7.12

9. An infinitesimally thin triangle of mass $M$ and length $L$ (axis through tip, perpendicular to plane; Fig. 7.12):

   Let the base have length $a$, where $a$ is infinitesimally small. Then a slice at a distance $x$ from the tip has length $a(x/L)$. If the slice has thickness $dx$, then it is essentially a point mass of mass $dm = \rho ax\,dx/L$. Therefore,

$$I = \int x^2 dm = \int_0^L x^2 \rho a x/L\,dx = \frac{1}{2}(\rho aL/2)L^2 = \boxed{\tfrac{1}{2}ML^2}, \tag{7.31}$$

   because $aL/2$ is the area of the triangle. This of course has the same form as the disk in Example 3, because a disk is made up of many of these triangles.

10. An isosceles triangle of mass $M$, vertex angle $2\beta$, and common-side length $L$ (axis through tip, perpendicular to plane; Fig. 7.12):

Let $h$ be the altitude of the triangle (so $h = L\cos\beta$). Slice the triangle into thin strips parallel to the base. Let $x$ be the distance from the vertex to a thin strip. Then the length of a strip is $\ell = 2x\tan\beta$, and its mass is $dm = \rho(2x\tan\beta\,dx)$. Using Example 5 above, along with the parallel-axis theorem, we have

$$\begin{aligned} I &= \int_0^h dm\left(\frac{\ell^2}{12}+x^2\right) = \int_0^h (\rho 2x\tan\beta\,dx)\left(\frac{(2x\tan\beta)^2}{12}+x^2\right) \\ &= \int_0^h 2\rho\tan\beta\left(1+\frac{\tan^2\beta}{3}\right)x^3\,dx = 2\rho\tan\beta\left(1+\frac{\tan^2\beta}{3}\right)\frac{h^4}{4}. \end{aligned} \tag{7.32}$$

But the area of the whole triangle is $h^2\tan\beta$, so we have $I = (Mh^2/2)(1+\tan^2\beta/3)$. In terms of $L$, this is

$$I = (ML^2/2)(\cos^2\beta + \sin^2\beta/3) = \boxed{\tfrac{1}{2}ML^2\big(1-\tfrac{2}{3}\sin^2\beta\big)}. \tag{7.33}$$

11. A regular $N$-gon of mass $M$ and "radius" $R$ (axis through center, perpendicular to plane; Fig. 7.13):

    The $N$-gon is made up of $N$ isosceles triangles, so we can use Example 10, with $\beta = \pi/N$. The masses of the triangles simply add, so if $M$ is the mass of the whole $N$-gon, we have

$$I = \boxed{\tfrac{1}{2}MR^2\big(1-\tfrac{2}{3}\sin^2\tfrac{\pi}{N}\big)}. \tag{7.34}$$

    Let's list the values of $I$ for a few $N$. We'll use the shorthand notation $(N,\ I/MR^2)$. Eq. 7.34 gives $(3, \frac{1}{4})$, $(4, \frac{1}{3})$, $(6, \frac{5}{12})$, $(\infty, \frac{1}{2})$. These values of $I$ form a nice arithmetic progression.

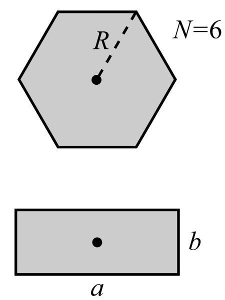



**Figure 7.13**

12. A rectangle of mass $M$ and sides of length $a$ and $b$ (axis through center, perpendicular to plane; Fig. 7.13):

    Let the $z$-axis be perpendicular to the plane. We know that $I_x = Mb^2/12$ and $I_y = Ma^2/12$, so the perpendicular-axis theorem tells us that

$$I_z = I_x + I_y = \boxed{\tfrac{1}{12}M(a^2+b^2)}. \tag{7.35}$$

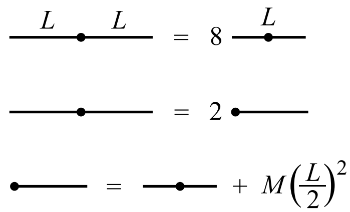
---

### 7.3.2 A neat trick

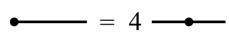
For some objects with certain symmetries, it is possible to calculate $I$ without doing any integrals. All that is needed is a scaling argument and the parallel-axis theorem. We will illustrate this technique by finding $I$ for a stick (Example 5 above). Other applications can be found in the problems for this chapter.

In the present example, the basic trick is to compare $I$ for a stick of length $L$ with $I$ for a stick of length $2L$. A simple scaling argument shows the latter is eight times the former. This is true because the integral $\int x^2\,dm = \int x^2\rho\,dx$ has three powers of $x$ in it. So a change of variables, $y = 2x$, brings in a factor of $2^3 = 8$. Equivalently, if we imagine expanding the smaller stick into the larger one, then a corresponding piece will be twice as far from the axis, and also twice as massive. The integral $\int x^2\,dm$ therefore increases by a factor of $2^2\cdot 2 = 8$.

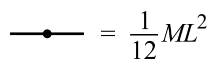
The technique is most easily illustrated with pictures. If we denote the moment of inertia of an object by a picture of the object, with a dot signifying the axis, then we have:



The first line comes from the scaling argument, the second line comes from the fact that moments of inertia simply add (the left-hand side is two copies of the right-hand side, attached at the pivot), and the third line comes from the parallel-axis theorem. Equating the right-hand sides of the first two equations gives



Plugging this expression for  into the third equation gives the desired result,



Note that sooner or later we must use real live numbers, which enter here through the parallel-axis theorem. Using only scaling arguments isn't sufficient, because they provide only linear equations homogeneous in the $I$'s, and therefore give no means of picking up the proper dimensions.

Once you've mastered this trick and applied it to the fractal objects in Problem 6, you can impress your friends by saying that you can "use scaling arguments, along with the parallel-axis theorem, to calculate moments of inertia of objects with fractal dimension." And you never know when that might come in handy!

## 7.4 Torque

We will now show that (under certain conditions, stated below) the rate of change of angular momentum is equal to a certain quantity, $\boldsymbol{\tau}$, which we call the *torque*. That is, $\boldsymbol{\tau} = d\mathbf{L}/dt$. This is the rotational analog of our old friend $\mathbf{F} = d\mathbf{p}/dt$ involving linear momentum. The basic idea here is straightforward, but there are two subtle issues. One deals with internal forces within a collection of particles. The other deals with origins (the points relative to which the angular momentum is calculated) that are not fixed. To keep things straight, we'll prove the general theorem by dealing with three increasingly complicated situations.

Our derivation of $\boldsymbol{\tau} = d\mathbf{L}/dt$ here holds for completely general motion; we can take the result and use it in the following chapter, too. If you wish, you can construct a more specific proof of $\boldsymbol{\tau} = d\mathbf{L}/dt$ for the special case where the axis of rotation is parallel to the $z$-axis. But since the general proof is no more difficult, we'll present it here in this chapter and get it over with.

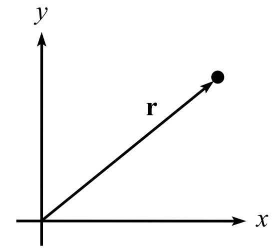



**Figure 7.14**

### 7.4.1 Point mass, fixed origin

Consider a point mass at position $\mathbf{r}$ relative to a fixed origin (see Fig. 7.14). The time derivative of the angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \mathbf{F} \\ &= 0 + \mathbf{r} \times \mathbf{F}, \end{aligned} \tag{7.36}$$

where $\mathbf{F}$ is the force acting on the particle. This is the same proof as in Theorem 6.1, except that here we are considering an arbitrary force instead of a central one. If we define the *torque* on the particle as

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{7.37}$$

then eq. (7.36) becomes

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}\,. \tag{7.38}$$

### 7.4.2 Extended mass, fixed origin

In an extended object, there are internal forces acting on the various pieces of the object, in addition to whatever external forces exist. For example, the external force on a given atom in a body might come from gravity, while the internal forces come from the adjacent atoms. How do we deal with these different types of forces?

In what follows, we will deal only with internal forces that are central forces, so that the force between two objects is directed along the line between them. This is a valid assumption for the pushing and pulling forces between molecules in a solid. (It isn't valid, for example, when dealing with magnetic forces. But we won't be interested in such things here.) We will invoke Newton's third law, which says that the force that particle 1 applies to particle 2 is equal and opposite to the force that particle 2 applies to particle 1.

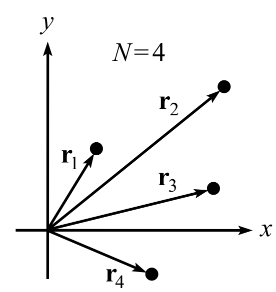



**Figure 7.15**

For concreteness, let us assume that we have a collection of $N$ discrete particles labeled by the index $i$ (see Fig. 7.15). In the continuous case, we would need to replace the following sums with integrals. The total angular momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i. \tag{7.39}$$

The force acting on each particle is $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} = d\mathbf{p}_i/dt$. Therefore,

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i \\ &= \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \end{aligned}$$

$$
\begin{aligned}
&= \sum_i \mathbf{v}_i \times (m\mathbf{v}_i) + \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}) \\
&= 0 + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} \\
&= \sum_i \boldsymbol{\tau}_i^{\text{ext}}. \qquad (7.40)
\end{aligned}
$$

The second-to-last line follows because $\mathbf{v}_i \times \mathbf{v}_i = 0$, and also because $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$, as you can show in Problem 8. In other words, the internal forces provide no net torque. This is quite reasonable. It basically says that a rigid object with no external forces won't spontaneously start rotating.

Note that the right-hand side involves the *total* external torque acting on the body, which may come from forces acting at many different points. Note also that nowhere did we assume that the particles were rigidly connected to each other. Eq. (7.40) still holds even if there is relative motion among the particles.

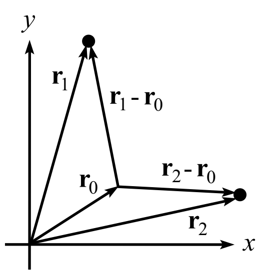


**Figure 7.16**

### 7.4.3 Extended mass, non-fixed origin

Let the position of the origin be $\mathbf{r}_0$ (see Fig. 7.16), and let the positions of the particles be $\mathbf{r}_i$. The vectors $\mathbf{r}_0$ and $\mathbf{r}_i$ are measured with respect to a given fixed coordinate system. The total angular momentum of the system, relative to the (possibly moving) origin $\mathbf{r}_0$, is

$$
\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0). \qquad (7.41)
$$

Therefore,

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)\right) \\
&= \sum_i (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_0) \\
&= 0 + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} - m_i\ddot{\mathbf{r}}_0), \qquad (7.42)
\end{aligned}
$$

because $m_i\ddot{\mathbf{r}}_i$ is the net force (namely $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$) acting on the $i$th particle. But a quick corollary of Problem 8 is that the term involving $\mathbf{F}_i^{\text{int}}$ vanishes (show this). And since $\sum m_i\mathbf{r}_i = M\mathbf{R}$ (where $M = \sum m_i$ is the total mass, and $\mathbf{R}$ is the position of the center of mass), we have

$$
\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} - M(\mathbf{R} - \mathbf{r}_0) \times \ddot{\mathbf{r}}_0. \qquad (7.43)
$$

The first term is the external torque, relative to the origin $\mathbf{r}_0$. The second term is something we wish would go away. And indeed, it usually does. It vanishes if any of the following three conditions is satisfied.

1. $\mathbf{R} = \mathbf{r}_0$. That is, the origin is the CM.

2. $\ddot{\mathbf{r}}_0 = 0$. That is, the origin is not accelerating.

3. $(\mathbf{R} - \mathbf{r}_0)$ is parallel to $\ddot{\mathbf{r}}_0$. This condition is rarely invoked.

If any of these conditions is satisfied, then we are free to write

$$\boxed{\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} \equiv \sum_i \boldsymbol{\tau}_i^{\text{ext}}}. \tag{7.44}$$

In other words, we can equate the total torque with the rate of change of the total angular momentum. An immediate corollary of this result is:

**Corollary 7.3** *If the total torque on a system is zero, then its angular momentum is conserved. In particular, the angular momentum of an isolated system (one that is subject to no external forces) is conserved.*

Everything up to this point is valid for arbitrary motion. The particles can be moving relative to each other, and the various $\mathbf{L}_i$'s can point in different directions, etc. But let's now restrict the motion. In the present chapter, we are dealing only with cases where $\hat{\mathbf{L}}$ is constant (taken to point in the $z$-direction). Therefore, $d\mathbf{L}/dt = d(L\hat{\mathbf{L}})/dt = (dL/dt)\hat{\mathbf{L}}$. If in addition we consider only rigid objects (where the relative distances among the particles is fixed) that undergo pure rotation around a given point, then $L = I\omega$, which gives $dL/dt = I\dot{\omega} \equiv I\alpha$. Taking the magnitude of both sides of eq. (7.44) then gives

$$\tau = I\alpha. \tag{7.45}$$

Invariably, we will calculate angular momentum and torque around either a fixed point or the CM. These are "safe" origins, in the sense that eq. (7.44) holds. As long as you vow to always use one of these safe origins, you can simply apply eq. (7.44) and not worry much about its derivation.

Remarks on the third condition: You'll probably never end up invoking the third condition above, but it's interesting to note that there is a simple way of understanding it in terms of accelerating reference frames. This is the topic of Chapter 9, so we're getting a little ahead of ourselves here, but the reasoning is as follows. Let $\mathbf{r}_0$ be the origin of a reference frame that is accelerating with acceleration $\ddot{\mathbf{r}}_0$. Then all objects in this accelerated frame feel a mysterious fictitious force of $-m\ddot{\mathbf{r}}_0$. For example, on a train accelerating to the right with acceleration $a$, you feel a strange force to the left of $ma$. If you don't counter this with another force, you will fall over. The fictitious force acts just like a gravitational force. Therefore, it effectively acts at the CM, producing a torque of $(\mathbf{R} - \mathbf{r}_0) \times (-M\ddot{\mathbf{r}}_0)$. This is the second term in eq. (7.43). This term will vanish if the CM is directly "above" (as far as the fictitious gravitational force is concerned) the origin, in other words, if $(\mathbf{R} - \mathbf{r}_0)$ is parallel to $\ddot{\mathbf{r}}_0$.

There is one common situation where the third condition can be invoked. Consider a wheel rolling without slipping on the ground. Mark a point on the rim. At the instant this point in in contact with the ground, it is a valid choice for the origin. This is true because $(\mathbf{R} - \mathbf{r}_0)$ points vertically. And $\ddot{\mathbf{r}}_0$ also points vertically. A point on a rolling wheel traces out a cycloid. Right before the point hits the ground, it is moving straight downward; right after it hits the ground, it is moving straight upward. But never mind, it's still a good idea to pick your origin to be the CM or a fixed point, even if the third condition holds. ♣

For conditions that number but three,
We say, "Torque is $dL$ by $dt$."
But though they're all true,
I'll stick to just two;
It's CM's and fixed points for me.

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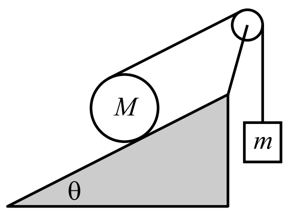


**Figure 7.17**

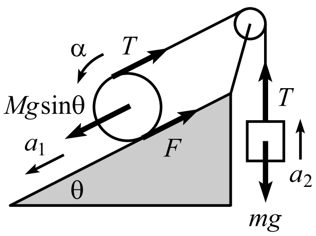


**Figure 7.18**

**Example:** A string wraps around a uniform cylinder of mass $M$, which rests on a fixed plane. The string passes up over a massless pulley and is connected to a mass $m$, as shown in Fig. 7.17. Assume that the cylinder rolls without slipping on the plane, and that the string is parallel to the plane. What is the acceleration of the mass $m$? What is the minimum value of $M/m$ for which the cylinder accelerates down the plane?

**Solution:** The friction, tension, and gravitational forces are shown in Fig. 7.18. Define positive $a_1$, $a_2$, and $\alpha$ as shown. These three accelerations, along with $T$ and $F$, are five unknowns. We therefore need to produce five equations. They are:

(1) $F = ma$ on $m \implies T - mg = ma_2$.
(2) $F = ma$ on $M \implies Mg\sin\theta - T - F = Ma_1$.
(3) $\tau = I\alpha$ on $M$ (around the CM) $\implies FR - TR = (MR^2/2)\alpha$.
(4) Non-slipping condition $\implies \alpha = a_1/R$.
(5) Conservation of string $\implies a_2 = 2a_1$.

A few comments on these equations: The normal force and the gravitational force perpendicular to the plane cancel, so we can ignore them. We have picked positive $F$ to point up the plane, but if it happens to point down the plane and thereby turn out to be negative, that's fine (but it won't); we don't need to worry about which way it really points. In (3), we are using the CM of the cylinder as our origin, but we can also use a fixed point; see the remark below. In (5), we have used the fact that the top of a rolling wheel moves twice as fast as the center. This is true because it has the same speed relative to the center as the center had relative to the ground.

We can go about solving these five equations in various ways. Three of the equations involve only two variables, so it's not so bad. (3) and (4) give $F - T = Ma_1/2$. Adding this to (2) gives $Mg\sin\theta - 2T = 3Ma_1/2$. Using (1) to eliminate $T$, and using (5) to write $a_1$ in terms of $a_2$, then gives

$$Mg\sin\theta - 2(mg + ma_2) = \frac{3Ma_2}{4} \qquad \Longrightarrow \qquad a_2 = \frac{(M\sin\theta - 2m)g}{\frac{3}{4}M + 2m}. \tag{7.46}$$

And $a_1 = a_2/2$. We see that $a_1$ is positive (that is, the cylinder rolls down the plane) if $M/m > 2/\sin\theta$.

Remark: In using $\tau = dL/dt$, we can also pick a fixed point as our origin, instead of the CM. The most sensible point is one located somewhere along the plane. The $Mg\sin\theta$ force now provides a torque, but the friction does not. The angular momentum of the cylinder with respect to a point on the plane is $I\omega + MvR$, where the second term comes from the $L$ due to the object being treated like a point mass at the CM. So $\tau = dL/dt$ becomes $(Mg\sin\theta)R - T(2R) = I\alpha + Ma_1R$. This is simply the sum of the third equation plus $R$ times the second equation above. We therefore obtain the same result. ♣

---

## 7.5 Collisions

In Section 4.7, we looked at collisions involving point particles (or otherwise non-rotating objects). The fundamental ingredients we used to solve a collision problem were conservation of momentum and conservation of energy (if the collision was elastic). With conservation of angular momentum now at our disposal, we can extend our study of collisions to ones that contain rotating objects. The additional fact of conservation of $L$ will be compensated for by the new degree of freedom for the rotation. Therefore, provided that the problem is set up properly, we will still have the same number of equations as unknowns.

In an isolated system, conservation of energy can be used only if the collision is elastic (by definition). But conservation of angular momentum is similar to conservation of momentum, in that it can *always* be used. However, conservation of $L$ is a little different from conservation of $p$, because we have to pick an origin before we can proceed. In view of the three conditions that are necessary for Corollary 7.3 to hold, we must pick our origin to be either a fixed point or the CM of the system (we'll ignore the third condition, since it's rarely used). If we choose some other point, then $\boldsymbol{\tau} = d\mathbf{L}/dt$ does *not* hold, so we have no right to claim that $d\mathbf{L}/dt$ equals zero just because the torque is zero (as it is for an isolated system).

There is, of course, some freedom in choosing an origin from among the legal possibilities of fixed points or the CM. And since it is generally the case that one choice is better than the others (in that it makes the calculations easier), you should take advantage of this freedom.

Let's to two examples. First, and elastic collision, and then an inelastic one.

---

**Example (Elastic collision):** A mass $m$ travels perpendicularly to a stick of mass $m$ and length $\ell$, which is initially at rest. At what location should the mass collide elastically with the stick, so that the mass and the center of the stick move with equal speeds after the collision?

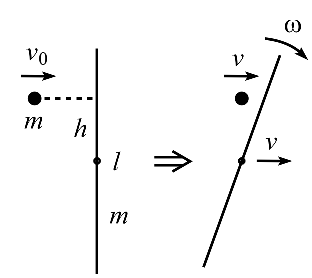


**Figure 7.19**

**Solution:** Let the initial speed of the mass be $v_0$. We have three unknowns in the problem (see Fig. 7.19), namely the desired distance from the middle of the stick, $h$; the final (equal) speeds of the stick and the mass, $v$; and the final angular speed of the stick, $\omega$. We can solve for these three unknowns by using our three available conservation laws:

- Conservation of $p$:

$$mv_0 = mv + mv \qquad \Longrightarrow \qquad v = \frac{v_0}{2}. \tag{7.47}$$

- Conservation of $E$:

$$\frac{mv_0^2}{2} = \frac{m}{2}\left(\frac{v_0}{2}\right)^2 + \left[\frac{m}{2}\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}\left(\frac{m\ell^2}{12}\right)\omega^2\right]$$
$$\Longrightarrow \qquad \omega = \frac{\sqrt{6}v_0}{\ell}. \tag{7.48}$$

- Conservation of $L$: Let's pick our origin to be the fixed point in space that coincides with the initial location of the center of the stick. Then conservation of $L$ gives

$$mv_0h = m\left(\frac{v_0}{2}\right)h + \left[\left(\frac{m\ell^2}{12}\right)\omega + 0\right]. \tag{7.49}$$

The zero here comes from the fact that the CM of the stick moves directly away from the origin, so there is no contribution to $L$ from the first of the two parts in Theorem 7.1. Plugging the $\omega$ from eq. 7.48 into eq. 7.49 gives

$$\frac{1}{2}mv_0h = \left(\frac{m\ell^2}{12}\right)\left(\frac{\sqrt{6}v_0}{\ell}\right) \qquad \Longrightarrow \qquad h = \frac{\ell}{\sqrt{6}}. \tag{7.50}$$

---

Let's now do an inelastic problem, where one object sticks to another. We won't be able to use conservation of $E$ now. But conservation of $p$ and $L$ will be sufficient, because there is one fewer degree of freedom in the final motion, due to the fact that the objects do not move independently.

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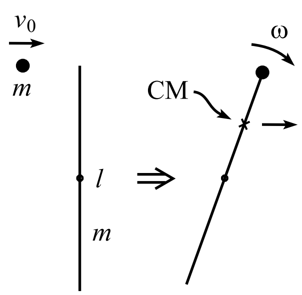


**Figure 7.20**

**Example (Inelastic collision):** A mass $m$ travels at speed $v_0$ perpendicularly to a stick of mass $m$ and length $\ell$, which is initially at rest. The mass collides completely inelastically with the stick at one of its ends, and sticks to it. What is the resulting angular velocity of the system?

**Solution:** The first thing to note is that the CM of the system is $\ell/4$ from the end, as shown in Fig. 7.20. The system will rotate about the CM as the CM moves in a straight line. Conservation of momentum quickly tells us that the speed of the CM is $v_0/2$. Also, using the parallel-axis theorem, the moment of inertia of the system about the CM is

$$I_{\text{CM}} = I_{\text{CM}}^{\text{stick}} + I_{\text{CM}}^{\text{mass}} = \left[\frac{m\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2\right] + m\left(\frac{\ell}{4}\right)^2 = \frac{5}{24}m\ell^2. \tag{7.51}$$

There are now many ways to proceed, depending on what point we choose as our origin.

**First method:** Choose the origin to be the fixed point that coincides with the location of the CM right when the collision happens (that is, the point $\ell/4$ from the end of the stick). Conservation of $L$ says that the initial $L$ of the ball must equal the final $L$ of the system. This gives

$$mv_0\left(\frac{\ell}{4}\right) = \left(\frac{5}{24}m\ell^2\right)\omega + 0. \qquad \Longrightarrow \qquad \omega = \frac{6v_0}{5\ell}. \tag{7.52}$$

The zero here comes from the fact that the CM of the stick moves directly away from the origin, so there is no contribution to $L$ from the first of the two parts in Theorem 7.1. Note that we didn't need to use conservation of $p$ in this method.

**Second method:** Choose the origin to be the fixed point that coincides with the initial center of the stick. Then conservation of $L$ gives

$$mv_0\left(\frac{\ell}{2}\right) = \left(\frac{5}{24}m\ell^2\right)\omega + (2m)\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right). \qquad \Longrightarrow \qquad \omega = \frac{6v_0}{5\ell}. \tag{7.53}$$

The right-hand side is the angular momentum of the system relative to the CM, plus the angular momentum (relative to the origin) of a point mass of mass $2m$ located at the CM.

**Third method:** Choose the origin to be the CM of the system. This point moves to the right with speed $v_0/2$, along the line a distance $\ell/4$ below the top of the stick. Relative to the CM, the mass $m$ moves to the right, and the stick moves to the left, both with speed $v_0/2$. Conservation of $L$ gives

$$m\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right) + \left[0 + m\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right)\right] = \left(\frac{5}{24}m\ell^2\right)\omega \qquad \Longrightarrow \qquad \omega = \frac{6v_0}{5\ell}\,. \tag{7.54}$$

The zero here comes from the fact that the stick initially has no $L$ around its center. A fourth reasonable choice for the origin is the fixed point that coincides with the initial location of the top of the stick. You can work this one out for practice.

## 7.6 Angular impulse

In Section 4.5.1, we defined the *impulse*, $\mathcal{I}$, to be the time integral of the force applied to an object, which is the net change in linear momentum. That is,

$$\mathcal{I} \equiv \int_{t_1}^{t_2} \mathbf{F}(t)\,dt = \Delta\mathbf{p}. \tag{7.55}$$

We now define the *angular impulse*, $\mathcal{I}_\theta$, to be the time integral of the torque applied to an object, which is the net change in angular momentum. That is,

$$\mathcal{I}_\theta \equiv \int_{t_1}^{t_2} \boldsymbol{\tau}(t)\,dt = \Delta\mathbf{L}. \tag{7.56}$$

These are just definitions, devoid of any content. The place where the physics comes in is the following. Consider a situation where $\mathbf{F}(t)$ is always applied at the same position relative to the origin around which $\boldsymbol{\tau}(t)$ is calculated. Let this position be $\mathbf{R}$. Then we have $\boldsymbol{\tau}(t) = \mathbf{R} \times \mathbf{F}(t)$. Plugging this into eq. (7.56), and taking the constant $\mathbf{R}$ outside the integral, gives $\mathcal{I}_\theta = \mathbf{R} \times \mathcal{I}$. That is,

$$\Delta\mathbf{L} = \mathbf{R} \times (\Delta\mathbf{p}) \qquad \text{(for } \mathbf{F}(t) \text{ applied at one position).} \tag{7.57}$$

This is a very useful result. It deals with the net changes in $\mathbf{L}$ and $\mathbf{p}$, and not with their changes at any particular instant. Even if $\mathbf{F}$ is changing in some arbitrary manner as time goes by, so that we have no idea what $\Delta\mathbf{p}$ and $\Delta\mathbf{L}$ are, we still know that they are related by eq. (7.57). Also, note that the derivation of eq. (7.57) was completely general, so we can apply it in the next chapter, too.

In many cases, we don't have to worry about the cross product in eq. (7.57), because the lever arm $\mathbf{R}$ is perpendicular to the change in momentum $\Delta\mathbf{p}$. In such cases, we have

$$|\Delta L| = R|\Delta p|. \tag{7.58}$$

Also, in many cases the object starts at rest, so we don't have to bother with the $\Delta$'s. The following example is a classic application of angular impulse and eq. (7.58).

**Example (Striking a stick):** A stick of mass $m$ and length $\ell$, initially at rest, is struck with a hammer. The blow is made perpendicular to the stick, at one end. Let the blow occur quickly, so that the stick doesn't move much while the hammer is in contact. If the CM of the stick ends up moving at speed $v$, what are the velocities of the ends, right after the blow?

**Solution:** We have no idea exactly what $F(t)$ looks like, or for how long it is applied, but we do know from eq. (7.58) that $\Delta L = (\ell/2)\Delta p$, where $L$ is calculated relative to the CM (so the lever arm is $\ell/2$). Therefore, $(m\ell^2/12)\omega = (\ell/2)mv$. Hence, the final $v$ and $\omega$ are related by $\omega = 6v/\ell$.

The velocities of the ends are obtained by adding (or subtracting) the rotational motion to the CM's translational motion. The rotational velocities of the ends are $\pm\omega(\ell/2) = \pm(6v/\ell)(\ell/2) = \pm 3v$. Therefore, the end that was hit moves with velocity $v+3v = 4v$, and the other end moves with velocity $v-3v = -2v$ (that is, backwards).

What $L$ was, he just couldn't tell.
And $p$? He was clueless as well.
But despite his distress,
He wrote down the right guess
For their quotient: the lever-arm's $\ell$.

Impulse is also useful for "collisions" that occur over extended times (see, for example, Problem 18).

## 7.7 Exercises

*Section 7.2: Non-planar objects*

1. **Semicircle CM** $*$

   A wire is bent into a semicircle of radius $R$. Find the location of the center of mass.

2. **Triangle CM** $*$

   Find the CM of an isosceles triangle.

3. **Hemisphere CM** $*$

   Find the CM of a solid hemisphere.

*Section 7.3: Calculating moments of inertia*

4. **A cone** $*$

   Find the moment of inertia of a solid cone (mass $M$, base radius $R$) around its symmetry axis.

5. **Triangle, the slick way** $*$

   In the spirit of Section 7.3.2, find the moment of inertia of a uniform equilateral triangle of mass $m$ and side $\ell$, around a line joining a vertex to the opposite side (see Fig. 7.21).

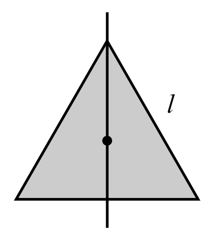



**Figure 7.21**

6. **Fractal triangle** $**$

   Take an equilateral triangle of side $\ell$, and remove the "middle" triangle (1/4 of the area). Then remove the "middle" triangle from each of the remaining three triangles, and so on, forever. Let the final fractal object have mass $m$. In the spirit of Section 7.3.2, find the moment of inertia around a line joining a vertex to the opposite side (see Fig. 7.22). Be careful how the mass scales.

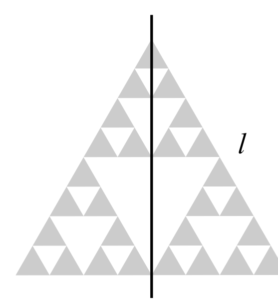



**Figure 7.22**

*Section 7.4: Torque*

7. **Swinging your arms** $*$

   You are standing on the edge of a step on some stairs, facing up the stairs. You feel yourself starting to fall backwards, so you start swinging your arms around in vertical circles, like a windmill. This is what people tend to do in such a situation, but does it actually help you not to fall, or does it simply make you look silly? Explain your reasoning.

8. **Wrapping around the pole** $*$

   A hockey puck, sliding on frictionless ice, is attached by a piece of string (lying along the surface) to a thin vertical pole. The puck is given a tangential velocity, and as the string wraps around the pole, the puck gradually spirals in. Is the following statement correct? "From conservation of angular momentum, the speed of the puck will increase as the distance to the pole decreases."

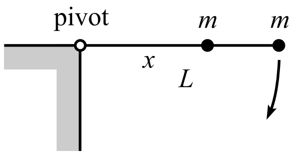


Figure 7.23

9. **Falling quickly** $*$

A massless stick of length $L$ is pivoted at one end and has a mass $m$ attached to its other end. It is held in a horizontal position, as shown in Fig. 7.23. Where should a second mass $m$ be attached to the stick, so that the stick falls as fast as possible when dropped?

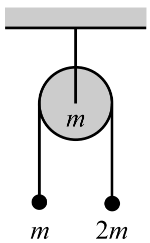


Figure 7.24

10. **Massive pulley** $*$

Consider the Atwood's machine shown in Fig. 7.24. The masses are $m$ and $2m$, and the pulley is a uniform disk of mass $m$ and radius $r$. The string is massless and does not slip with respect to the pulley. Find the acceleration of the masses.

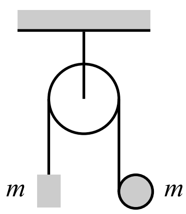


Figure 7.25

11. **Atwood's with a cylinder** $**$

A massless string of negligible thickness is wrapped around a uniform cylinder of mass $m$ and radius $R$. The string passes up over a massless pulley and is tied to a block of mass $m$ at its other end, as shown in Fig. 7.25. What are the accelerations of the block and the cylinder? Assume that the string does not slip with respect to the cylinder.

12. **Maximum frequency** $*$

A pendulum is made of a uniform stick of length $\ell$. A pivot is placed somewhere along the stick, which is allowed to swing in a vertical plane. Where should the pivot be placed on the stick so that the frequency of (small) oscillations is maximum?

13. **Rolling down the plane** $*$

An circular object with moment of inertia $\beta mr^2$ rolls without slipping down a plane inclined at angle $\theta$. What is its linear acceleration?

14. **Coin on a plane** $*$

A coin rolls down a plane inclined at angle $\theta$. If the coefficient of static friction between the coin and the plane is $\mu$, what is the largest angle $\theta$ for which the coin doesn't slip?

15. **Bowling ball on paper** $*$

A bowling ball sits on a piece of paper on the floor. You grab the paper and pull it horizontally along the floor, with acceleration $a$. What is the acceleration of the center of the ball? Assume that the ball does not slip with respect to the paper.

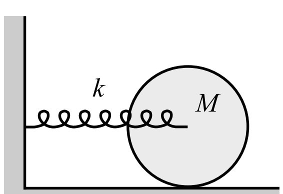


Figure 7.26

16. **Spring and cylinder** $*$

The axle of a solid cylinder (mass $M$, radius $R$) is connected to a spring with spring-constant $k$, as shown in Fig. 7.26. If the cylinder rolls without slipping, what is the frequency of oscillations?

17. **Another spring and cylinder** $**$

The top of a solid cylinder (mass $M$, radius $R$) is connected to a spring (at its equilibrium length) with spring-constant $k$, as shown in Fig. 7.27. If the cylinder rolls without slipping, what is the frequency of (small) oscillations?

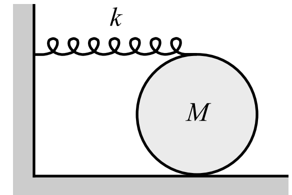



**Figure 7.27**

18. **The spool** $**$

A spool of mass $m$ and moment of inertia $I$ (around the center) is free to roll without slipping on a table. It has an inner radius $r$, and an outer radius $R$. If you pull on the string with tension $T$ at an angle $\theta$ (see Fig. 7.28), what is the acceleration of the spool? Which way does it move?

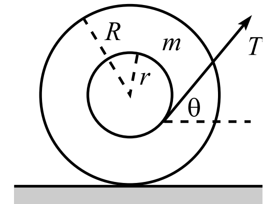



**Figure 7.28**

19. **Stopping the coin** $**$

A coin stands vertically on a table. It is projected forward (in the plane of itself) with speed $v$ and angular speed $\omega$, as shown in Fig. 7.29. The coefficient of kinetic friction between the coin and the table is $\mu$. What should $v$ and $\omega$ be so that the coin comes to rest (both translationally and rotationally) a distance $d$ from where it started?

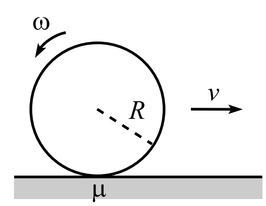



**Figure 7.29**

20. **Accelerating plane** $*$

A ball with $I = (2/5)MR^2$ is placed on a plane inclined at angle $\theta$. The plane is accelerated upwards (along its direction) with acceleration $a$; see Fig. 7.30. For what value of $a$ will the CM of the ball not move? Assume that there is sufficient friction so that the ball doesn't slip with respect to the plane.

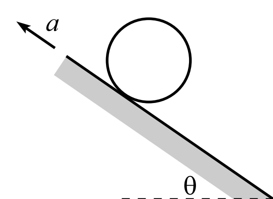



**Figure 7.30**

21. **Raising the hoop** $**$

A bead of mass $m$ is positioned at the top of a frictionless hoop of mass $M$ and radius $R$, which stands vertically on the ground. A wall touches the hoop on its left, and a short wall of height $R$ touches the hoop on its right, as shown in Fig. 7.31. All surfaces are frictionless. The bead is given a tiny kick, and it slides down the hoop, as shown. What is the smallest value of $m/M$ for which the hoop will rise up off the ground at some time during the motion? (*Note*: It is possible to solve this problem using only force, but solve it here by using torque.)

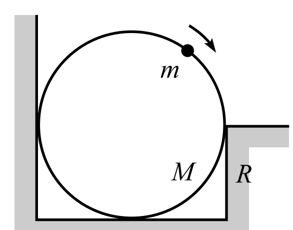



**Figure 7.31**

22. **Coin and plank** $**$

A coin of mass $M$ and radius $R$ stands vertically on the right end of a horizontal plank of mass $M$ and length $L$, as shown in Fig. 7.32. The plank is pulled to the right with a constant force $F$. Assume that the coin does not slip with respect to the plank. What are the accelerations of the plank and coin? How far to the right does the coin move by the time the left end of the plank reaches it?

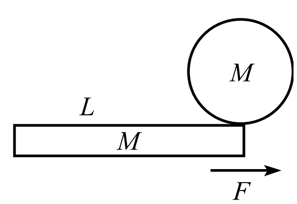



**Figure 7.32**

**23. Board and cylinders** ***

A board lies on top of two uniform cylinders which lie on a fixed plane inclined at angle $\theta$, as shown in Fig. 7.33. The board has mass $m$, and each of the cylinders has mass $m/2$. If there is no slipping between any of the surfaces, what is the acceleration of the board?

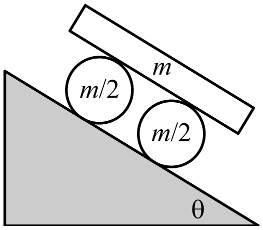



**Figure 7.33**

**24. Moving plane** ****

A disk of mass $m$ and moment of inertia $I = \beta m r^2$ is held motionless on a plane of mass $M$ and angle of inclination $\theta$ (see Fig. 7.34). The plane rests on a frictionless horizontal surface. The disk is released. Assuming that it rolls without slipping on the plane, what is the horizontal acceleration of the plane? *Hint*: You probably want to do Problem 2.2 first.

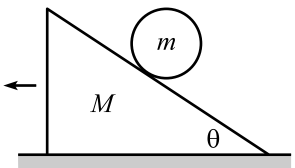



**Figure 7.34**

**25. Tower of cylinders** ****

Consider the infinitely tall system of identical massive cylinders and massless planks shown in Fig. 7.35. The moment of inertia of the cylinders is $I = MR^2/2$. There are two cylinders at each level, and the number of levels is infinite. The cylinders do not slip with respect to the planks, but the bottom plank is free to slide on a table. If you pull on the bottom plank so that it accelerates horizontally with acceleration $a$, what is the horizontal acceleration of the bottom row of cylinders?

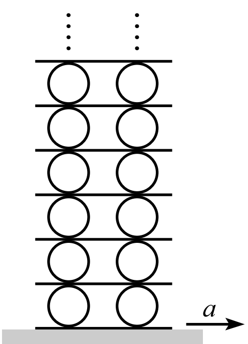



**Figure 7.35**

*Section 7.5: Collisions*

**26. Pendulum collision** *

A stick of mass $m$ and length $\ell$ is pivoted at an end. It is held horizontally and then released. It swings down, and when it is vertical, the free end elastically collides with a ball, as shown in Fig. 7.36. (Assume that the ball is initially held, and then released a split second before the stick strikes it.) If the stick loses half of its angular velocity during the collision, what is the mass of the ball? What is the speed of the ball right after the collision?

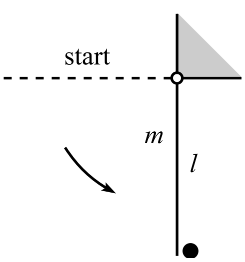



**Figure 7.36**

**27. Spinning stick** **

A stick of mass $m$ and length $\ell$ spins around on a frictionless table, with its CM stationary (but not fixed by a pivot). A mass $M$ is placed on the plane, and the end of the stick collides elastically with it, as shown in Fig. 7.37. What should $M$ be so that after the collision the stick has translational motion, but no rotational motion?

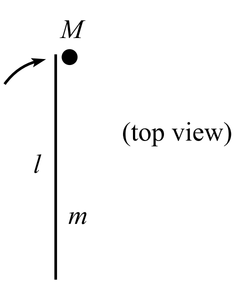



**Figure 7.37**

28. **Another spinning stick** $**$

A stick of mass $m$ and length $\ell$ initially rotates with frequency $\omega$ on a frictionless table, with its CM at rest (but not fixed by a pivot). A ball of mass $m$ is placed on the table, and the end of the stick collides elastically with it, as shown in Fig. 7.38. What is the resulting angular velocity of the stick?

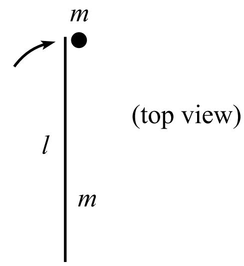



**Figure 7.38**

29. **Same final speeds** $*$

A stick slides (without rotating) across a frictionless table and collides elastically at one of its ends with a ball. Both the stick and the ball have mass $m$. The mass of the stick is distributed in such a way that the moment of inertia around the CM (which is at the center of the stick) equals $I = Am\ell^2$, where $A$ is some number. What should $A$ be so that the ball moves at the same speed as the center of the stick after the collision?

30. **No final rotation** $*$

A stick of mass $m$ and length $\ell$ spins around on a frictionless table, with its CM stationary (but not fixed by a pivot). It collides elastically with a mass $m$, as shown in Fig. 7.39. At what location should the collision occur (specify this by giving the distance from the center of the stick) so that the stick has no rotational motion afterwards?

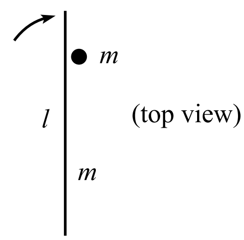



**Figure 7.39**

*Section 7.6: Angular Impulse*

31. **Center of percussion** $*$

You hold one end of a uniform stick of length $L$. The stick is struck with a hammer. Where should this blow occur so that the end you are holding doesn't move (immediately after the blow)? In other words, where should the blow occur so that you don't feel a "sting" in your hand? This point is called the *center of percussion*.

32. **Another center of percussion** $*$

You hold the top vertex of a solid equilateral triangle of side length $L$. The plane of the triangle is vertical. It is struck with a hammer, somewhere along the vertical axis. Where should this blow occur so that the point you are holding doesn't move (immediately after the blow)? The moment of inertia about any axis through the CM of an equilateral triangle is $ML^2/24$.

33. **Not hitting the pole** $*$

A (possibly non-uniform) stick of mass $m$ and length $\ell$ lies on frictionless ice. Its midpoint (which is also its CM) touches a thin pole sticking out of the ice. One end of the stick is struck with a quick blow perpendicular to the stick, so that the CM moves away from the pole. What is the minimum value of the stick's moment of inertia that allows the stick not to hit the pole?

34. **Up, down, and twisting** $**$

A uniform stick is held horizontally and then released. At the same instant, one end is struck with a quick upwards blow. If the stick ends up horizontal when it returns to its original height, what are the possible values for the maximum height to which the stick's center rises?

35. **Striking a pool ball** $**$

At what height should you horizontally strike a pool ball so that it immediately rolls without slipping?

36. **Doing work** $*$

(a) A pencil of mass $m$ and length $\ell$ lies at rest on a frictionless table. You push on it at its midpoint (perpendicular to it), with a constant force $F$ for a time $t$. Find the final speed and the distance traveled. Verify that the work you do equals the final kinetic energy.

(b) Assume that you apply the same $F$ for the same $t$ as above, but that you now apply it at one of the pencil's ends (perpendicular to the pencil). Assume that $t$ is small, so that the pencil doesn't have much time to rotate.[5] Find the final CM speed, the final angular speed, and the distance your hand moves. Verify that the work you do equals the final kinetic energy.

37. **Repetitive bouncing** $*$

Using the result of Problem 19, what must the relation between $v_x$ and $R\omega$ be so that the superball continually bounces back and forth between the same two points of contact on the ground?

38. **Bouncing under a table** $**$

You throw a superball so that it bounces off the floor, then off the underside of a table, then off the floor again. What must the initial relation between $v_x$ and $R\omega$ be so that the ball returns to your hand (with the return and outward paths the same)? Use the result of Problem 19, and modifications thereof.[6]

39. **Bouncing between walls** $***$

A stick of length $\ell$ slides on frictionless ice. It bounces between two parallel walls, a distance $L$ apart, in such a way that the same end touches both walls, and the stick hits the walls at an angle $\theta$ each time. What is $\theta$, in terms of $L$ and $\ell$? What does the situation look like in the limit $L \ll \ell$?

[5] This means that you can assume that your force is always essentially perpendicular to the pencil, as far as the torque is concerned.

[6] You are strongly encouraged to bounce a ball in such a manner and have it magically come back to your hand. It turns out that the required value of $\omega$ is small, so a natural throw with $\omega \approx 0$ will essentially get the job done.

What should $\theta$ be, in terms of $L$ and $\ell$, if the stick makes an additional $n$ full revolutions between the walls? What is the minimum value of $L/\ell$ for which this is possible?

## 7.8 Problems

*Section 7.1: Pancake object in x-y plane*

1. **Leaving the sphere** ∗∗

   A ball with moment of inertia $\eta m r^2$ rests on top of a fixed sphere. There is friction between the ball and the sphere. The ball is given an infinitesimal kick and rolls down without slipping. Assuming that $r$ is much smaller than the radius of the sphere, at what point does the ball lose contact with the sphere? How does your answer change if the size of the ball is comparable to, or larger than, the size of the sphere? You may want to solve Problem 4.3 first, if you haven't already done so.

2. **Sliding ladder** ∗∗∗

   A ladder of length $\ell$ and uniform mass density stands on a frictionless floor and leans against a frictionless wall. It is initially held motionless, with its bottom end an infinitesimal distance from the wall. It is then released, whereupon the bottom end slides away from the wall, and the top end slides down the wall (see Fig. 7.40). When it loses contact with the wall, what is the horizontal component of the velocity of the center of mass?

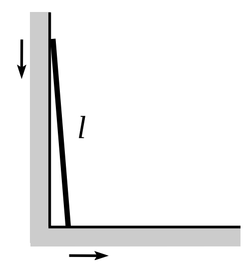



**Figure 7.40**

3. **Leaning rectangle** ∗∗∗

   A rectangle of height $2a$ and width $2b$ rests on top of a fixed cylinder of radius $R$ (see Fig. 7.41). The moment of inertia of the rectangle around its center is $I$. The rectangle is given an infinitesimal kick, and then "rolls" on the cylinder without slipping. Find the equation of motion for the tilt angle of the rectangle. Under what conditions will the rectangle fall off the cylinder, and under what conditions will it oscillate back and forth? Find the frequency of these small oscillations.

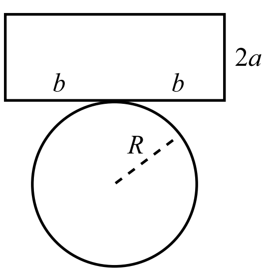



**Figure 7.41**

4. **Mass in a tube** ∗∗∗

   A tube of mass $M$ and length $\ell$ is free to swing by a pivot at one end. A mass $m$ is positioned inside the tube at this end. The tube is held horizontal and then released (see Fig. 7.42). Let $\theta$ be the angle of the tube with respect to the horizontal, and let $x$ be the distance the mass has traveled along the tube. Find the Euler-Lagrange equations for $\theta$ and $x$, and then write them in terms of $\theta$ and $\eta \equiv x/\ell$ (the fraction of the distance along the tube).

   These equations can only be solved numerically, and you must pick a numerical value for the ratio $r \equiv m/M$ in order to do this. Write a program (see Appendix D) that produces the value of $\eta$ when the tube is vertical ($\theta = \pi/2$). Give this value of $\eta$ for a few values of $r$.

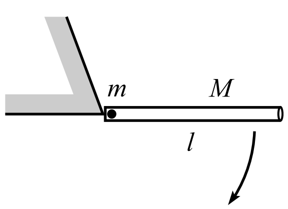



**Figure 7.42**

*Section 7.3: Calculating moments of inertia*

5. **Slick calculations of $I$** $**$

   In the spirit of Section 7.3.2, find the moments of inertia of the following objects (see Fig. 7.43).

   (a) A uniform square of mass $m$ and side $\ell$ (axis through center, perpendicular to plane).

   (b) A uniform equilateral triangle of mass $m$ and side $\ell$ (axis through center, perpendicular to plane).

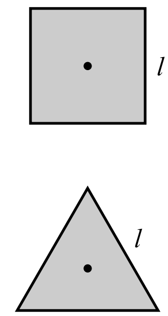



**Figure 7.43**

6. **Slick calculations of $I$ for fractal objects** $***$

   In the spirit of Section 7.3.2, find the moments of inertia of the following fractal objects. (Be careful how the mass scales.)

   (a) Take a stick of length $\ell$, and remove the middle third. Then remove the middle third from each of the remaining two pieces. Then remove the middle third from each of the remaining four pieces, and so on, forever. Let the final object have mass $m$ (axis through center, perpendicular to stick; see Fig. 7.44).[7]

   (b) Take a square of side $\ell$, and remove the "middle" square (1/9 of the area). Then remove the "middle" square from each of the remaining eight squares, and so on, forever. Let the final object have mass $m$ (axis through center, perpendicular to plane; see Fig. 7.45).

   (c) Take an equilateral triangle of side $\ell$, and remove the "middle" triangle (1/4 of the area). Then remove the "middle" triangle from each of the remaining three triangles, and so on, forever. Let the final object have mass $m$ (axis through center, perpendicular to plane; Fig. 7.46).

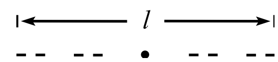



**Figure 7.44**

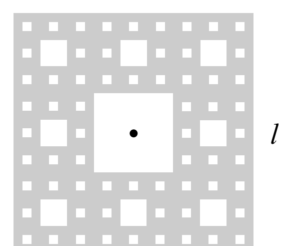



**Figure 7.45**

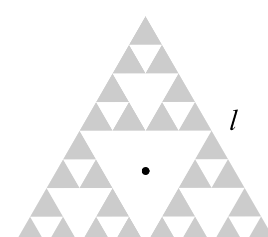



**Figure 7.46**

7. **Minimum $I$**

   A moldable blob of matter of mass $M$ is to be situated between the planes $z = 0$ and $z = 1$ (see Fig. 7.47) so that the moment of inertia around the $z$-axis be as small as possible. What shape should the blob take?

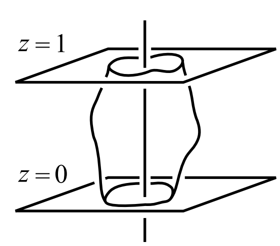



**Figure 7.47**

[7] This object is the Cantor set, for those who like such things. It has no length, so the density of the remaining mass is infinite. If you suddenly develop an aversion to point masses with infinite density, simply imagine the above iteration being carried out only, say, a million times.

*Section 7.4: Torque*

8. **Zero torque from internal forces** $**$

Given a collection of particles with positions $\mathbf{r}_i$, let the force on the $i$th particle, due to all the others, be $\mathbf{F}_i^{\text{int}}$. Assuming that the force between any two particles is directed along the line between them, use Newton's third law to show that $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$.

9. **Removing a support** $*$

(a) A uniform rod of length $\ell$ and mass $m$ rests on supports at its ends. The right support is quickly removed (see Fig. 7.48). What is the force on the left support immediately thereafter?

(b) A rod of length $2r$ and moment of inertia $\eta m r^2$ rests on top of two supports, each of which is a distance $d$ away from the center. The right support is quickly removed (see Fig. 7.48). What is the force on the left support immediately thereafter?

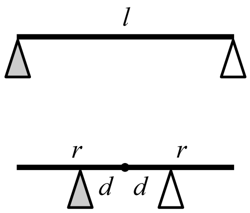


**Figure 7.48**

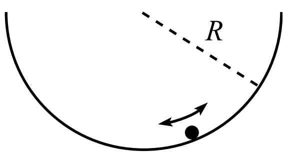


**Figure 7.49**

10. **Oscillating ball** $**$

A small ball with radius $r$ and uniform density rolls without slipping near the bottom of a fixed cylinder of radius $R$ (see Fig. 7.49). What is the frequency of small oscillations? Assume $r \ll R$.

11. **Oscillating cylinders** $**$

A hollow cylinder of mass $M_1$ and radius $R_1$ rolls without slipping on the inside surface of another hollow cylinder of mass $M_2$ and radius $R_2$. Assume $R_1 \ll R_2$. Both axes are horizontal, and the larger cylinder is free to rotate about its axis. What is the frequency of small oscillations?

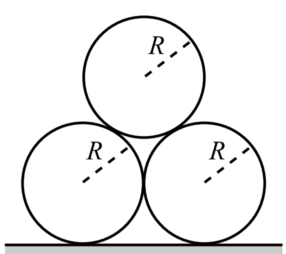


**Figure 7.50**

12. **A triangle of cylinders** $***$

Three identical cylinders with moments of inertia $I = \eta M R^2$ are situated in a triangle as shown in Fig. 7.50. Find the initial downward acceleration of the top cylinder for the following two cases. Which case has a larger acceleration?

(a) There is friction between the bottom two cylinders and the ground (so they roll without slipping), but there is no friction between any of the cylinders.

(b) There is no friction between the bottom two cylinders and the ground, but there is friction between the cylinders (so they don't slip with respect to each other).

13. **Falling stick** $*$

A massless stick of length $b$ has one end attached to a pivot and the other end glued perpendicularly to the middle of a stick of mass $m$ and length $\ell$.

(a) If the two sticks are held in a horizontal plane (see Fig. 7.51) and then released, what is the initial acceleration of the CM?

(b) If the two sticks are held in a vertical plane (see Fig. 7.51) and then released, what is the initial acceleration of the CM?

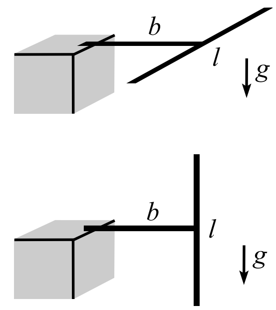


**Figure 7.51**

14. **Lengthening the string** $**$

A mass hangs from a massless string and swings around in a horizontal circle, as shown in Fig. 7.52. The length of the string is very slowly increased (or decreased). Let $\theta$, $\ell$, $r$, and $h$ be defined as shown.

(a) Assuming $\theta$ is very small, how does $r$ depend on $\ell$?

(b) Assuming $\theta$ is very close to $\pi/2$, how does $h$ depend on $\ell$?

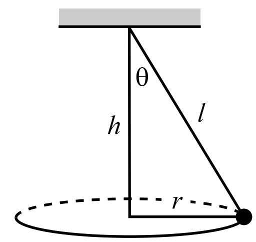


**Figure 7.52**

15. **Falling Chimney** $****$

A chimney initially stands upright. It is given a tiny kick, and it topples over. At what point along its length is it most likely to break?

In doing this problem, work with the following two-dimensional simplified model of a chimney. Assume that the chimney consists of boards stacked on top of each other, and that each board is attached to the two adjacent ones with tiny rods at each end (see Fig. 7.53). The goal is to determine which rod in the chimney has the maximum tension. (Work in the approximation where the width of the chimney is very small compared to its height.)

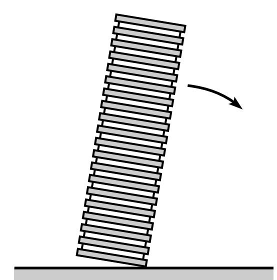


**Figure 7.53**

*Section 7.5: Collisions*

16. **Ball hitting stick** $**$

A ball of mass $M$ collides with a stick with moment of inertia $I = \eta m\ell^2$ (relative to its center, which is its CM). The ball is initially traveling with velocity $V_0$ perpendicular to the stick. The ball strikes the stick at a distance $d$ from the center (see Fig. 7.54). The collision is elastic. Find the resulting translational and rotational speeds of the stick, and also the resulting speed of the ball.

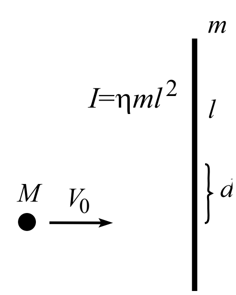


**Figure 7.54**

17. **A ball and stick theorem** $**$

Consider the setup in Problem 16. Show that the relative speed of the ball and the point of contact on the stick is the same before and immediately after the collision. (This result is analogous to the "relative speed" result for a 1-D collision, Theorem 4.3.)

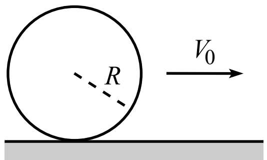



**Figure 7.55**

*Section 7.6: Angular Impulse*

18. **Sliding to rolling** $**$

A ball initially slides without rotating on a horizontal surface with friction (see Fig. 7.55). The initial speed of the ball is $V_0$, and the moment of inertia about its center is $I = \eta m R^2$.

(a) Without knowing anything about the nature of the friction force, find the speed of the ball when it begins to roll without slipping. Also, find the kinetic energy lost while sliding.

(b) Now consider the special case where the coefficient of kinetic friction is $\mu$, independent of position. At what time, and at what distance, does the ball begin to roll without slipping? Verify that the work done by friction equals the energy loss calculated in part (a). (Be careful on this.)

19. **The superball** $**$

A ball with radius $R$ and $I = (2/5)mR^2$ is thrown through the air. It spins around the axis perpendicular to the plane of the motion (call this the $x$-$y$ plane). The ball bounces off a floor without slipping during the time of contact. Assume that the collision is elastic, and that the magnitude of the vertical $v_y$ is the same before and after the bounce. Show that $v'_x$ and $\omega'$ after the bounce are related to $v_x$ and $\omega$ before the bounce by

$$\begin{pmatrix} v'_x \\ R\omega' \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} v_x \\ R\omega \end{pmatrix}, \tag{7.59}$$

where positive $v_x$ is to the right, and positive $\omega$ is counterclockwise.

20. **Many bounces** $*$

Using the result of Problem 19, describe what happens over the course of many superball bounces.

21. **Rolling over a bump** $**$

A ball with radius $R$ and $I = (2/5)mR^2$ rolls with speed $V_0$ without slipping on the ground. It encounters a step of height $h$ and rolls up over it. Assume that the ball sticks to the corner of the step briefly (until the center of the ball is directly above the corner). Show that if the ball is to climb over the step, then $V_0$ must satisfy

$$V_0 \geq \sqrt{\frac{10gh}{7}}\left(1 - \frac{5h}{7R}\right)^{-1}. \tag{7.60}$$

22. **Lots of sticks** $***$

Consider a collection of rigid sticks of length $2r$, masses $m_i$, and moments of inertia $\eta m_i r^2$, with $m_1 \gg m_2 \gg m_3 \gg \cdots$. The CM of each stick is located at the center. The sticks are placed on a horizontal frictionless surface, as shown in Fig. 7.56. The ends overlap a negligible distance, and the ends are a negligible distance apart.

The first (heaviest) stick is given an instantaneous blow (as shown) which causes it to translate and rotate. The first stick will strike the second stick, which will then strike the third stick, and so on. Assume all the collisions are elastic.

Depending on the size of $\eta$, the speed of the $n$th stick will either (1) approach zero, (2) approach infinity, or (3) be independent of $n$, as $n \to \infty$. Show that the special value of $\eta$ corresponding to the third of these three scenarios is $\eta = 1/3$, which happens to correspond to a uniform stick.

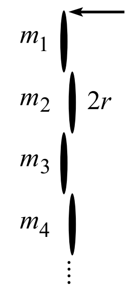



**Figure 7.56**

## 7.9 Solutions

1. **Leaving the sphere**

In this setup, as in Problem 4.3, the ball leaves the sphere when the normal force becomes zero, that is, when

$$\frac{mv^2}{R} = mg\cos\theta. \tag{7.61}$$

The only change from the solution to Problem 4.3 comes in the calculation of $v$. The ball now has rotational energy, so conservation of energy gives $mgR(1-\cos\theta) = mv^2/2 + I\omega^2/2 = mv^2/2 + \eta mr^2\omega^2/2$. But $r\omega = v$, so we have

$$\frac{1}{2}(1+\eta)mv^2 = mgR(1-\cos\theta) \qquad \Longrightarrow \qquad v = \sqrt{\frac{2gR(1-\cos\theta)}{1+\eta}}\,. \tag{7.62}$$

Plugging this into eq. (7.61), we see that the ball leaves the sphere when

$$\cos\theta = \frac{2}{3+\eta}\,. \tag{7.63}$$

Remarks: For $\eta = 0$, this equals 2/3, as in Problem 4.3. For a uniform ball with $\eta = 2/5$, we have $\cos\theta = 10/17$, so $\theta \approx 54^\circ$. For $\eta \to \infty$ (for example, a spool with a very thin axle rolling down the rim of a circle), we have $\cos\theta \to 0$, so $\theta \approx 90^\circ$. This makes sense because $v$ is always very small, because most of the energy takes the form of rotational energy. ♣

If the size of the ball is comparable to, or larger than, the size of the sphere, then we must take into account the fact that the CM of the ball does not move along a circle of radius $R$. Instead, it moves along a circle of radius $R + r$, so eq. (7.61) becomes

$$\frac{mv^2}{R+r} = mg\cos\theta. \tag{7.64}$$

Also, the conservation-of-energy equation takes the form, $mg(R+r)(1-\cos\theta) = mv^2/2 + \eta mr^2\omega^2/2$. But $r\omega$ still equals $v$ (prove this), so the kinetic energy still equals $(1+\eta)mv^2/2$.[8] We therefore have the same equations as above, except that $R$ is replaced everywhere by $R + r$. But $R$ didn't appear in the result for $\theta$ in eq. (7.63), so the answer is unchanged.

Remark: Note that the method of the second solution to Problem 4.3 will *not* work in this problem, because there *is* a force available to make $v_x$ decrease, namely the friction force. And indeed, $v_x$ does decrease before the rolling ball leaves the sphere. The $v$ in the present problem is simply $1/\sqrt{1+\eta}$ times the $v$ in Problem 4.3, so the maximum $v_x$ is still achieved at $\cos\theta = 2/3$. But the angle in eq. (7.63) is larger than this. (However, see Problem 2 for a setup involving rotations where the max $v_x$ *is* relevant.) ♣

2. **Sliding ladder**

The important point to realize in this problem is that the ladder loses contact with the wall before it hits the ground. Let's find where this loss of contact occurs.

Let $r = \ell/2$, for convenience. While the ladder is in contact with the wall, its CM moves in a circle of radius $r$. This follows from the fact that the median to the hypotenuse of a right triangle has half the length of the hypotenuse. Let $\theta$ be the

[8] In short, the ball can be considered to be instantaneously rotating around the contact point, so the parallel-axis theorem leads to the factor of $(1+\eta)$ in the rotational kinetic energy around this point.

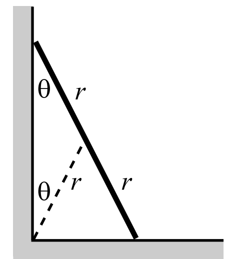



**Figure 7.57**

angle between the wall and the radius from the corner to the CM; see Fig. 7.57. This is also the angle between the ladder and the wall.

We'll solve this problem by assuming that the CM always moves in a circle, and then determining the position at which the horizontal CM speed starts to decrease, that is, the point at which the normal force from the wall would have to become negative. Since the normal force of course can't be negative, this is the point where the ladder loses contact with the wall.

By conservation of energy, the kinetic energy of the ladder equals the loss in potential energy, which is $mgr(1-\cos\theta)$. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$, because the CM travels in a circle of radius $r$. The rotational energy is $I\dot{\theta}^2/2$. The same $\dot{\theta}$ applies here as in the CM translational motion, because $\theta$ is the angle between the ladder and the vertical, and thus is the angle of rotation of the ladder.

Letting $I \equiv \eta m r^2$ to be general ($\eta = 1/3$ for our ladder), the conservation of energy statement is $(1+\eta)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, which is $v = r\dot{\theta}$, equals

$$v = \sqrt{\frac{2gr(1-\cos\theta)}{1+\eta}}\,. \tag{7.65}$$

The horizontal component of this is

$$v_x = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{7.66}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the horizontal speed is maximum when $\cos\theta = 2/3$. Therefore the ladder loses contact with the wall when

$$\cos\theta = \frac{2}{3} \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ. \tag{7.67}$$

Note that this is independent of $\eta$. This means that, for example, a dumbbell (two masses at the ends of a massless rod, with $\eta = 1$) will lose contact with the wall at the same angle.

Plugging this value of $\theta$ into eq. (7.66), and using $\eta = 1/3$, we obtain a final horizontal speed of

$$v_x = \frac{\sqrt{2gr}}{3} \equiv \frac{\sqrt{g\ell}}{3}\,. \tag{7.68}$$

Note that this is 1/3 of the $\sqrt{2gr}$ horizontal speed that the ladder would have if it were arranged (perhaps by having the top end slide down a curve) to eventually slide horizontally along the ground.

You are encouraged to compare various aspects of this problem with those in Problem 1 and Problem 4.3.

REMARK: The normal force from the wall is zero at the start and finish, so it must reach a maximum at some intermediate value of $\theta$. Let's find this $\theta$. Taking the derivative of $v_x$ in eq. (7.66) to find the CM's horizontal acceleration $a_x$, and then using $\dot{\theta} \propto \sqrt{1-\cos\theta}$ from eq. (7.65), we see that the force from the wall is proportional to

$$a_x \;\propto\; \frac{\dot{\theta}\sin\theta(3\cos\theta-2)}{\sqrt{1-\cos\theta}} \;\propto\; \sin\theta(3\cos\theta-2). \tag{7.69}$$

Taking the derivative of this, we find that the force from the wall is maximum when

$$\cos\theta = \frac{1+\sqrt{19}}{6} \quad \Longrightarrow \quad \theta \approx 26.7^\circ. \quad ♣ \tag{7.70}$$

3. **Leaning rectangle**

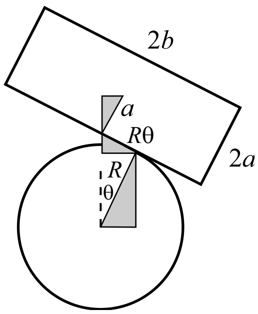


**Figure 7.58**

We must first find the position of the rectangle's CM when it has rotated through an angle $\theta$. Using Fig. 7.58, we can obtain this position (relative to the center of the cylinder) by adding up the distances along the three shaded triangles. Note that the contact point has moved a distance $R\theta$ along the rectangle. We find that the position of the CM is

$$(x, y) = R(\sin\theta, \cos\theta) + R\theta(-\cos\theta, \sin\theta) + a(\sin\theta, \cos\theta), \tag{7.71}$$

We'll now use the Lagrangian method to find the equation of motion and the frequency of small oscillations. Using eq. (7.71), you can show that the square of the speed of the CM is

$$v^2 = \dot{x}^2 + \dot{y}^2 = (a^2 + R^2\theta^2)\dot{\theta}^2. \tag{7.72}$$

Remark: The simplicity of this result suggests that there is a quicker way to obtain it. And indeed, the CM instantaneously rotates around the contact point with angular speed $\dot{\theta}$, and from Fig. 7.58 the distance to the contact point is $\sqrt{a^2 + R^2\theta^2}$. Therefore, the speed of the CM is $\omega r = \dot{\theta}\sqrt{a^2 + R^2\theta^2}$. ♣

The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}m(a^2 + R^2\theta^2)\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - mg\Big((R+a)\cos\theta + R\theta\sin\theta\Big). \tag{7.73}$$

The equation of motion is (as you can show)

$$(ma^2 + mR^2\theta^2 + I)\ddot{\theta} + mR^2\theta\dot{\theta}^2 = mga\sin\theta - mgR\theta\cos\theta. \tag{7.74}$$

Let us now consider small oscillations. Using the small-angle approximations, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$, and keeping terms only to first order in $\theta$, we obtain

$$(ma^2 + I)\ddot{\theta} + mg(R - a)\theta = 0. \tag{7.75}$$

The coefficient of $\theta$ is positive if $a < R$. Therefore, oscillatory motion occurs for $a < R$. Note that this condition is independent of $b$. The frequency of small oscillations is

$$\omega = \sqrt{\frac{mg(R-a)}{ma^2 + I}}\,. \tag{7.76}$$

Remarks: Let's look at some special cases. If $I = 0$ (that is, all of the rectangle's mass is located at the CM), we have $\omega = \sqrt{g(R-a)/a^2}$. If in addition $a \ll R$, then $\omega \approx \sqrt{gR/a^2}$. You can also derive these results by considering the CM to be a point mass sliding in a parabolic potential. If the rectangle is instead a uniform horizontal stick, so that $a \ll R$, $a \ll b$, and $I \approx mb^2/3$, then we have $\omega \approx \sqrt{3gR/b^2}$. If the rectangle is a vertical stick (satisfying $a < R$), so that $b \ll a$ and $I \approx ma^2/3$, then we have $\omega \approx \sqrt{3g(R-a)/4a^2}$. If in addition $a \ll R$, then $\omega \approx \sqrt{3gR/4a^2}$.

Without doing much work, there are two other ways we can determine the condition under which there is oscillatory motion. The first is to look at the height of the CM. Using small-angle approximations in eq. (7.71), the height of the CM is $y \approx (R + a) + (R - a)\theta^2/2$.

Therefore, if $a < R$, the potential energy increases with $\theta$, so the rectangle wants to decrease its $\theta$ and fall back down to the middle. But if $a > R$, the potential energy decreases with $\theta$, so the rectangle wants to increase its $\theta$ and fall off the cylinder.

The second way is to look at the horizontal positions of the CM and the contact point. Small-angle approximations in eq. (7.71) show that the former equals $a\theta$ and the latter equals $R\theta$. Therefore, if $a < R$ then the CM is to the left of the contact point, so the torque from gravity (relative to the contact point) makes $\theta$ decrease, and the motion is stable. But if $a > R$ then the torque from gravity makes $\theta$ increase, and the motion is unstable. ♣

4. **Mass in a tube**

The Lagrangain is

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{3}M\ell^2\right)\dot{\theta}^2 + \left(\frac{1}{2}mx^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2\right) + mgx\sin\theta + Mg\left(\frac{\ell}{2}\right)\sin\theta. \qquad (7.77)$$

The Euler-Lagrange equations are then

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} \quad &\Longrightarrow \quad m\ddot{x} = mx\dot{\theta}^2 + mg\sin\theta, \qquad (7.78)\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial \theta} \quad &\Longrightarrow \quad \frac{d}{dt}\left(\frac{1}{3}M\ell^2\dot{\theta} + mx^2\dot{\theta}\right) = \left(mgx + \frac{Mg\ell}{2}\right)\cos\theta\\
&\Longrightarrow \quad \left(\frac{1}{3}M\ell^2 + mx^2\right)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = \left(mgx + \frac{Mg\ell}{2}\right)\cos\theta.
\end{aligned}$$

In term of $\eta \equiv x/\ell$, these equations become

$$\begin{aligned}
\ddot{\eta} &= \eta\dot{\theta}^2 + \tilde{g}\sin\theta\\
(1+3r\eta^2)\ddot{\theta} &= \left(3r\tilde{g}\eta + \frac{3\tilde{g}}{2}\right)\cos\theta - 6r\eta\dot{\eta}\dot{\theta}, \qquad (7.79)
\end{aligned}$$

where $r \equiv m/M$ and $\tilde{g} \equiv g/\ell$. Below is a Maple program that numerically finds the value of $\eta$ when $\theta$ equals $\pi/2$, in the case where $r = 1$. As mentioned in Problem 2 in Appendix B, this value of $\eta$ does not depend on $g$ or $\ell$, and hence not $\tilde{g}$. In the program, we'll denote $\tilde{g}$ by `g`, which we'll give the arbitrary value of 10. We'll use `q` for $\theta$, and `n` for $\eta$. Also, we denote $\dot{\theta}$ by `q1` and $\ddot{\theta}$ by `q2`, etc. Even if you don't know Maple, this program should still be understandable. See Appendix D for more discussion on solving differential equations numerically.

```
n:=0:       # initial n value
n1:=0:       # initial n speed
q:=0:       # initial angle
q1:=0:       # initial angular speed
e:=.0001:      # small time interval
g:=10:       # value of g/l
r:=1:       # value of m/M
while q<1.57079 do     # do this process until the angle is pi/2
n2:=n*q1^2+g*sin(q):      # the first E-L equation
q2:=((3*r*g*n+3*g/2)*cos(q)-6*r*n*n1*q1)/(1+3*r*n^2):
        # the second E-L equation
n:=n+e*n1:      # how n changes
n1:=n1+e*n2:      # how n1 changes
```

```
q:=q+e*q1:      # how q changes
q1:=q1+e*q2:      # how q1 changes
end do:      # stop the process
n;     # print the value of n (eta)
```

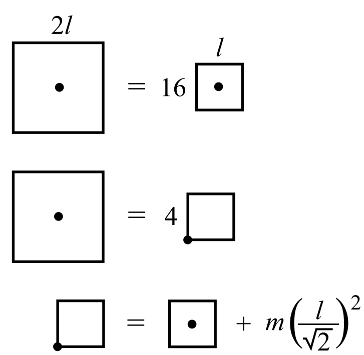
The resulting value for $\eta$ is 0.378. If you actually run this program on Maple with different values of g, you will find that the result for n doesn't depend on g, as we stated above. A few results for $\eta$ for various values of $r$ are, in $(r, \eta)$ notation: $(0, .349)$, $(1, .378)$, $(2, .410)$, $(10, .872)$, $(20, 3.290)$. It turns out that $r \approx 11.25$ yields $\eta \approx 1$. That is, the mass $m$ gets to the end of the tube right when the tube becomes vertical.

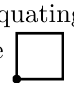
For $\eta$ values larger than 1, we could imagine attaching a massless tubular extension on the end of the given tube. It turns out that $\eta \to \infty$ as $r \to \infty$. In this case, the mass $m$ essentially drops straight down, causing the tube to quickly swing down to a nearly vertical position. But $m$ ends up slightly to one side, and then takes a very long time to move over to become directly below the pivot.

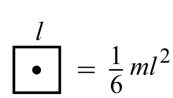
5. **Slick calculations of $I$**

(a) We claim that the $I$ for a square of side $2\ell$ is 16 times the $I$ for a square of side $\ell$, assuming that the axes pass through any two corresponding points. This is true because $dm$ goes like the area, which is proportional to length squared, so the corresponding $dm$'s differ by a factor of 4. And then there are the two powers of $r$ in the integrand. Therefore, when changing variables from one square to the other, there are four powers of 2 in the integral $\int r^2\,dm = \int r^2 \rho\,dx\,dy$.

As is Section 7.3.2, we can express the relevant relations in terms of pictures:

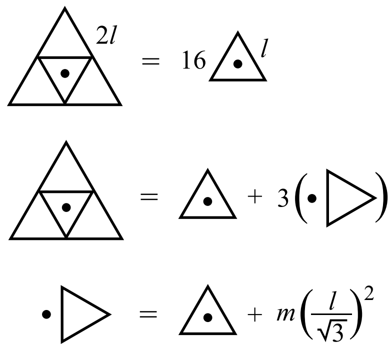


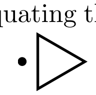

The first line comes from the scaling argument, the second comes from the fact that moments of inertia simply add, and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

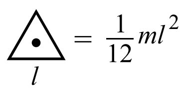



This agrees with the result of Example 12 in Section 7.3.1, with $a = b = \ell$.

(b) This is again a two-dimensional object, so the $I$ for a triangle of side $2\ell$ is 16 times the $I$ for a triangle of side $\ell$, assuming that the axes pass through any two corresponding points. With pictures, we have:




The first line comes from the scaling argument, the second comes from the fact that moments of inertia simply add, and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

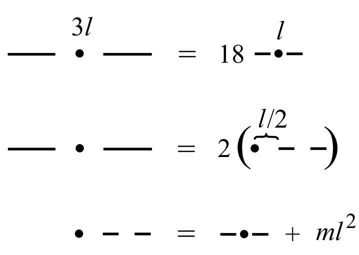



This agrees with the result of Example 11 in Section 7.3.1, with $N = 3$. The "radius" $R$ used in that example equals $\ell/\sqrt{3}$ in the present notation.

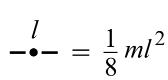
6. **Slick calculations of $I$ for fractal objects**

   (a) The scaling argument here is a little trickier than that in Section 7.3.2. Our object is self-similar to an object 3 times as big, so let's increase the length by a factor of 3 and see what happens to $I$. In the integral $\int x^2\,dm$, the $x$'s pick up a factor of 3, so this gives a factor of 9. But what happens to the $dm$? Well, tripling the size of our object increases its mass by a factor of 2, because the new object is simply made up of two of the smaller ones, plus some empty space in the middle. So the $dm$ picks up a factor of 2. Therefore, the $I$ for an object of length $3\ell$ is 18 times the $I$ for an object of length $\ell$, assuming that the axes pass through any two corresponding points.

   With pictures, we have (the following symbols denote our fractal object):




The first line comes from the scaling argument, the second comes from the fact that moments of inertia simply add, and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

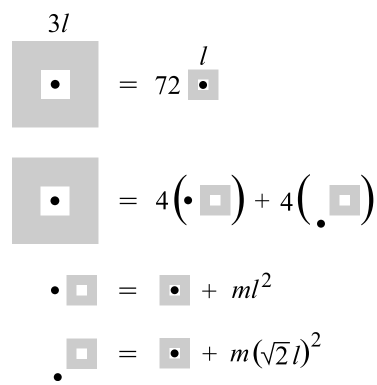


This is larger than the $I$ for a uniform stick, namely $m\ell^2/12$, because the mass here is generally farther away from the center.

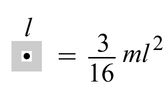
Remark: When we increase the length of our object by a factor of 3 here, the factor of 2 in the $dm$ is larger than the factor of 1 relevant to a zero-dimensional object, but smaller than the factor of 3 relevant to a one-dimensional object. So in some sense our object has a dimension between 0 and 1. It is reasonable to define the dimension, $d$, of an object as the number for which $r^d$ is the increase in "volume" when the dimensions are increased by a factor of $r$. In this problem, we have $3^d = 2$, so $d = \log_3 2 \approx 0.63$. ♣

(b) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 3 and see what happens to $I$. In the integral $\int x^2\,dm$, the $x$'s pick up a factor of 3, so this gives a factor of 9. But what happens to the $dm$? Tripling the size of our object increases its mass by a factor of 8, because the new object is made up of eight of the smaller ones, plus an empty square in the middle. So the $dm$ picks up a factor of 8. Therefore, the $I$ for an object of side $3\ell$ is 72 times the $I$ for an object of side $\ell$, assuming that the axes pass through any two corresponding points.

With pictures, we have (the following symbols denote our fractal object):




The first line comes from the scaling argument, the second comes from the fact that moments of inertia simply add, and the third and fourth come from the parallel-axis theorem. Equating the right-hand sides of the first two, and then using the third and fourth to eliminate  and  gives



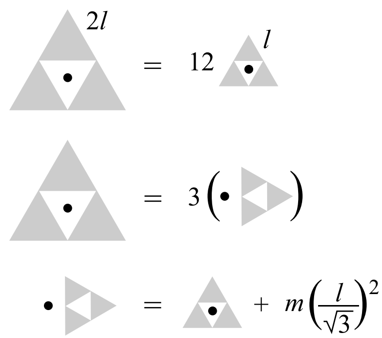

This is larger than the $I$ for the uniform square in Problem 5, namely $m\ell^2/6$, because the mass here is generally farther away from the center.

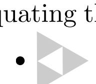
Note: Increasing the size of our object by a factor of 3 increases the "volume" by a factor of 8. So the dimension is given by $3^d = 8 \implies d = \log_3 8 \approx 1.89$.

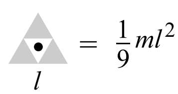
(c) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 2 and see what happens to $I$. In the integral $\int x^2\,dm$, the $x$'s pick up a factor of 2, so this gives a factor of 4. But what happens to the

$dm$? Doubling the size of our object increases its mass by a factor of 3, because the new object is simply made up of three of the smaller ones, plus an empty triangle in the middle. So the $dm$ picks up a factor of 3. Therefore, the $I$ for an object of side $2\ell$ is 12 times the $I$ for an object of side $\ell$, assuming that the axes pass through any two corresponding points.

With pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second comes from the fact that moments of inertia simply add, and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives



This is larger than the $I$ for the uniform triangle in Problem 5, namely $m\ell^2/12$, because the mass here is generally farther away from the center.

NOTE: Note: Increasing the size of our object by a factor of 2 increases the "volume" by a factor of 3. So the dimension is given by $2^d = 3 \implies d = \log_2 3 \approx 1.58$.

7. **Minimum** $I$

The shape should be a cylinder with the $z$-axis as its symmetry axis. A quick proof (by contradiction) is as follows.

Assume that the optimal blob is not a cylinder, and consider the surface of the blob. If the blob is not a cylinder, then there exist two points on the surface, $P_1$ and $P_2$, that are located at different distances, $r_1$ and $r_2$, from the $z$-axis. Assume $r_1 < r_2$ (see Fig. 7.59). If we move a small piece of the blob from $P_2$ to $P_1$, then we decrease the moment of inertia, $\int r^2 \rho \, dV$. Therefore, the proposed non-cylindrical blob cannot be the one with the smallest $I$.

In order to avoid this contradiction, all points on the surface must be equidistant from the $z$-axis. The only blob with this property is a cylinder.

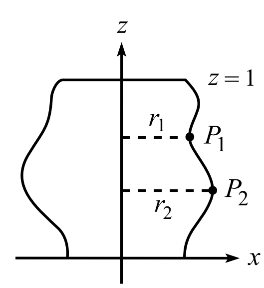



**Figure 7.59**

8. **Zero torque from internal forces**

Let $\mathbf{F}_{ij}^{\text{int}}$ be the force that the $i$th particle feels from the $j$th particle (see Fig. 7.60). Then

$$\mathbf{F}_i^{\text{int}} = \sum_j \mathbf{F}_{ij}^{\text{int}}. \tag{7.80}$$

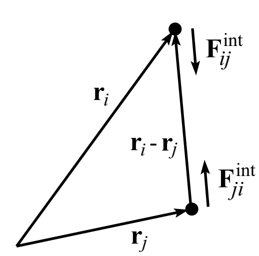



**Figure 7.60**

The total internal torque, relative to the chosen origin, is therefore

$$\boldsymbol{\tau}^{\text{int}} \equiv \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.81}$$

But if we interchange the indices (which were labeled arbitrarily), we have

$$\boldsymbol{\tau}^{\text{int}} = \sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} = -\sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ij}^{\text{int}}, \tag{7.82}$$

where we have used Newton's third law, $\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}$. Adding the two previous equations gives

$$2\boldsymbol{\tau}^{\text{int}} = \sum_i \sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.83}$$

But $\mathbf{F}_{ij}^{\text{int}}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$, by assumption. Therefore, each cross product in the sum equals zero.

The above sums might make this solution look a bit involved. But the idea is simply that the torques cancel in pairs. This is clear from Fig. 7.60, because the two forces shown are equal and opposite, and they have the same lever arm relative to the origin.

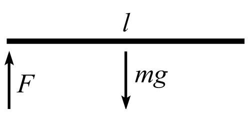


**Figure 7.61**

9. **Removing a support**

(a) **First Solution:** Let the desired force on the left support be $F$, and let the downward acceleration of the stick's CM be $a$. Then the $F = ma$ and $\tau = I\alpha$ (relative to the fixed support; see Fig. 7.61) equations, along with the circular-motion relation between $a$ and $\alpha$, are

$$\begin{aligned} mg - F &= ma, \\ mg\frac{\ell}{2} &= \left(\frac{m\ell^2}{3}\right)\alpha, \\ a &= \frac{\ell}{2}\alpha. \end{aligned} \tag{7.84}$$

The second equation gives $\alpha = 3g/2\ell$. The third equation then gives $a = 3g/4$. And the first equation then gives $F = mg/4$. Note that the right end of the stick accelerates at $2a = 3g/2$, which is larger than $g$.

**Second Solution:** Looking at torques around the CM, we have

$$F\frac{\ell}{2} = \left(\frac{m\ell^2}{12}\right)\alpha. \tag{7.85}$$

And looking at torques around the fixed support, we have

$$mg\frac{\ell}{2} = \left(\frac{m\ell^2}{3}\right)\alpha. \tag{7.86}$$

Dividing the first of these equations by the second gives $F = mg/4$.

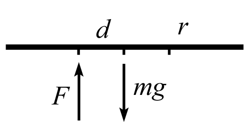


**Figure 7.62**

(b) **First Solution:** As in the first solution above, we have (using the parallel-axis theorem; see Fig. 7.62)

$$\begin{aligned} mg - F &= ma, \\ mgd &= (\eta mr^2 + md^2)\alpha \\ a &= d\alpha. \end{aligned} \tag{7.87}$$

Solving for $F$ gives $F = mg/(1 + d^2/\eta r^2)$. For $d = r$ and $\eta = 1/3$, we obtain the answer in part (a).

**Second Solution:** As in the second solution above, looking at torques around the CM, we have

$$Fd = (\eta mr^2)\alpha. \tag{7.88}$$

And looking at torques around the fixed support, we have

$$mgd = (\eta mr^2 + md^2)\alpha. \tag{7.89}$$

Dividing the first of these equations by the second gives $F = mg/(1 + d^2/\eta r^2)$.

Some limits: For the special case $d = r$, we have the following: If $\eta = 0$ then $F = 0$; if $\eta = 1$ then $F = mg/2$; and if $\eta = \infty$ (we could put masses at the ends of massless extensions of the stick) then $F = mg$; these all make intuitive sense. In the limit $d = 0$, we have $F = mg$. And in the limit $d = \infty$, we have $F = 0$. Technically, we should be writing $d \ll \sqrt{\eta}\, r$ or $d \gg \sqrt{\eta}\, r$ here.

10. **Oscillating ball**

Let the angle from the bottom of the cylinder to the ball be $\theta$ (see Fig. 7.63), and let $F_f$ be the friction force. Then the tangential $F = ma$ equation is

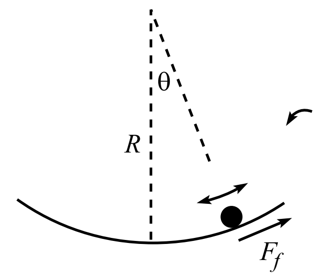



**Figure 7.63**

$$F_f - mg\sin\theta = ma, \tag{7.90}$$

where we have chosen rightward to be the positive direction for $a$ and $F_f$. Also, the $\tau = I\alpha$ equation (relative to the CM) is

$$-rF_f = \frac{2}{5}mr^2\alpha, \tag{7.91}$$

where we have chosen clockwise to be the positive direction for $\alpha$. Using $r\alpha = a$, the torque equation becomes $F_f = -(2/5)ma$. Plugging this into eq. (7.90), and using $\sin\theta \approx \theta$, we obtain $mg\theta + (7/5)ma = 0$. Under the assumption $r \ll R$, we have $a \approx R\ddot{\theta}$, so we arrive at

$$\ddot{\theta} + \left(\frac{5g}{7R}\right)\theta = 0. \tag{7.92}$$

This is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5g}{7R}}\,. \tag{7.93}$$

This answer is slightly smaller than the $\sqrt{g/R}$ answer for the case where the ball slides. The rolling ball effectively has a larger inertial mass, but the same gravitational mass.

You can also solve this problem by using the contact point as the origin around which $\tau$ and $L$ are calculated.

Remarks: If we omit the $r \ll R$ assumption, you can show that $r\alpha = a$ still holds, but $a = R\ddot{\theta}$ is replaced by $a = (R - r)\ddot{\theta}$. Therefore, the exact result for the frequency is $\omega = \sqrt{5g/7(R - r)}$. This goes to infinity as $r \to R$.

In general, if the ball has a moment of inertia equal to $\eta mr^2$, you can show that the frequency of small oscillations equals $\sqrt{g/(1 + \eta)R}$. ♣

11. **Oscillating cylinders**

The moments of inertia of the cylinders are simply $I_1 = M_1R_1^2$ and $I_2 = M_2R_2^2$. Let $F$ be the force between the two cylinders. And let $\theta_1$ and $\theta_2$ be the angles of rotation of the cylinders (with counterclockwise positive), relative to the position where the small cylinder is at the bottom of the big cylinder. Then the torque equations are

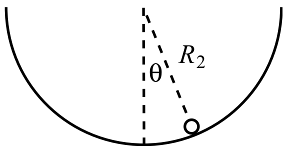



**Figure 7.64**

$$\begin{aligned} FR_1 &= M_1R_1^2\ddot{\theta}_1, \\ FR_2 &= -M_2R_2^2\ddot{\theta}_2. \end{aligned} \quad (7.94)$$

We are not so much concerned with $\theta_1$ and $\theta_2$ as we are with the angular position that $M_1$ makes with the vertical. Call this angle $\theta$ (see Fig. 7.64). In the approximation $R_1 \ll R_2$, the non-slipping condition says that $R_2\theta \approx R_2\theta_2 - R_1\theta_1$. Eqs. (7.94) then give

$$F\left(\frac{1}{M_1} + \frac{1}{M_2}\right) = -R_2\ddot{\theta}. \quad (7.95)$$

The force equation on $M_1$ is

$$F - M_1 g \sin\theta = M_1(R_2\ddot{\theta}). \quad (7.96)$$

Substituting the $F$ from (7.95) into this gives (with $\sin\theta \approx \theta$)

$$\left(M_1 + \frac{1}{\frac{1}{M_1} + \frac{1}{M_2}}\right)\ddot{\theta} + \left(\frac{M_1 g}{R_2}\right)\theta = 0. \quad (7.97)$$

The frequency of small oscillations is therefore

$$\omega = \sqrt{\frac{g}{R_2}}\sqrt{\frac{M_1 + M_2}{M_1 + 2M_2}}. \quad (7.98)$$

REMARKS: In the limit $M_2 \ll M_1$, we obtain $\omega \approx \sqrt{g/R_2}$. There is essentially no friction force between the cylinders; only a normal force. So the small cylinder essentially acts like a pendulum of length $R_2$. In the limit $M_1 \ll M_2$, we obtain $\omega \approx \sqrt{g/2R_2}$. The large cylinder is essentially fixed, so we simply have the setup mentioned in the remark in the solution to Problem 10, with $\eta = 1$. ♣

12. **A triangle of circles**

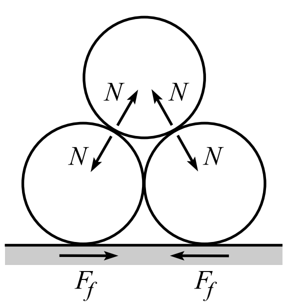



**Figure 7.65**

(a) Let $N$ be the normal force between the cylinders, and let $F$ be the friction force from the ground (see Fig. 7.65). Let $a_x$ be the initial horizontal acceleration of the right bottom cylinder (so $\alpha = a_x/R$ is its angular acceleration), and let $a_y$ be the initial vertical acceleration of the top cylinder (with downward taken to be positive).

If we consider the torque around the center of one of the bottom cylinders, then the only relevant force is $F$, because $N$, gravity, and the normal force from the ground all point through the center. The equations expressing $F_x = Ma_x$ on the bottom right cylinder, $F_y = Ma_y$ on the top cylinder, and $\tau = I\alpha$ on the bottom right cylinder are, respectively,

$$\begin{aligned} N\cos 60^\circ - F &= Ma_x, \\ Mg - 2N\sin 60^\circ &= Ma_y, \\ FR &= (\eta MR^2)(a_x/R). \end{aligned} \quad (7.99)$$

We have four unknowns, $N$, $F$, $a_x$, and $a_y$. So we need one more equation. Fortunately, $a_x$ and $a_y$ are related. The contact surface between the top and bottom cylinders lies (initially) at an angle of 30° with the horizontal. Therefore, if the bottom cylinders move a distance $d$ to the side, then the top cylinder moves a distance $d \tan 30°$ downward. Hence,

$$a_x = \sqrt{3} a_y. \tag{7.100}$$

We now have four equations in four unknowns. Solving for $a_y$, by your method of choice, gives

$$a_y = \frac{g}{7 + 6\eta}. \tag{7.101}$$

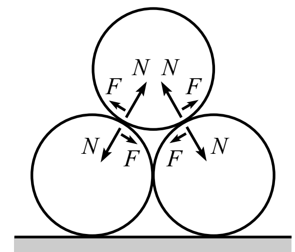



**Figure 7.66**

(b) Let $N$ be the normal force between the cylinders, and let $F$ be the friction force between the cylinders (see Fig. 7.66). Let $a_x$ be the initial horizontal acceleration of the right bottom cylinder, and let $a_y$ be the initial vertical acceleration of the top cylinder (with downward taken to be positive). Let $\alpha$ be the angular acceleration of the right bottom cylinder (with counterclockwise taken to be positive). Note that $\alpha$ is *not* equal to $a_x/R$, because the bottom cylinders slip on the ground.

If we consider the torque around the center of one of the bottom cylinders, then the only relevant force is $F$. And from the same reasoning as in part (a), we have $a_x = \sqrt{3} a_y$. Therefore, the four equations analogous to eqs. (7.99) and (7.100) are

$$\begin{aligned} N\cos 60° - F\sin 60° &= Ma_x, \\ Mg - 2N\sin 60° - 2F\cos 60° &= Ma_y, \\ FR &= (\eta MR^2)\alpha, \\ a_x &= \sqrt{3} a_y. \end{aligned} \tag{7.102}$$

We have five unknowns, $N$, $F$, $a_x$, $a_y$, and $\alpha$. So we need one more equation. The tricky part is relating $\alpha$ to $a_x$. One way to do this is to ignore the $y$ motion of the top cylinder and imagine the bottom right cylinder to be rotating up and around the top cylinder, which is held fixed. If the bottom cylinder moves an infinitesimal distance $d$ to the right, then its center moves a distance $d/\cos 30°$ up and to the right. So the angle through which the bottom cylinder rotates is $(d/\cos 30°)/R = (2/\sqrt{3})(d/R)$. Bringing back in the vertical motion of the cylinders does not change this result. Therefore,

$$\alpha = \frac{2}{\sqrt{3}} \frac{a_x}{R}. \tag{7.103}$$

We now have five equations and five unknowns. Solving for $a_y$, by your method of choice, gives

$$a_y = \frac{g}{7 + 8\eta}. \tag{7.104}$$

REMARKS: If $\eta = 0$, that is, if all the mass is at the center of the cylinders,[9] then the results in both parts (a) and (b) reduce to $g/7$. If $\eta \neq 0$, then the result in part (b) is smaller than that in part (a). This isn't so obvious, but the basic reason is that the bottom cylinders in part (b) take up more energy because they have to rotate slightly faster, because $\alpha = (2/\sqrt{3})(a_x/R)$ instead of $\alpha = a_x/R$. ♣

[9] The $\eta = 0$ case is also equivalent to all the surfaces simply being frictionless, because then nothing rotates.

13. **Falling stick**

(a) Let's calculate $\tau$ and $L$ relative to the pivot point. The torque is due to gravity, which effectively acts on the CM and has magnitude $mgb$. The moment of inertia of the stick around the horizontal axis through the pivot (and perpendicular to the massless stick) is simply $mb^2$. So when the stick starts to fall, the $\tau = dL/dt$ equation is $mgb = (mb^2)\alpha$. Therefore, the initial acceleration of the CM, namely $b\alpha$, is

$$b\alpha = g, \tag{7.105}$$

which is independent of $\ell$ and $b$. This answer makes sense. The stick initially falls straight down, and the pivot provides no force because it doesn't know right away that the stick is moving.

(b) The only change from part (a) is the moment of inertia of the stick around the horizontal axis through the pivot (and perpendicular to the massless stick). From the parallel-axis theorem, this moment is $mb^2 + m\ell^2/12$. So when the stick starts to fall, the $\tau = dL/dt$ equation is $mgb = (mb^2 + m\ell^2/12)\alpha$. Therefore, the initial acceleration of the CM is

$$b\alpha = \frac{g}{1 + (\ell^2/12b^2)} \,. \tag{7.106}$$

For $\ell \ll b$, this goes to $g$, as it should. And for $\ell \gg b$, it goes to zero, as it should. In this case, a tiny movement of the CM corresponds to a very large movement of the points far out along the stick. Therefore, by conservation of energy, the CM must be moving very slowly.

14. **Lengthening the string**

Consider the angular momentum $\mathbf{L}$ relative to the support point $P$. The forces on the mass are the tension in the string and gravity. The former provides no torque around $P$, and the latter provides no torque in the $z$-direction. Therefore, $L_z$ is constant. If we let $\omega_\ell$ be the frequency of the circular motion when the string has length $\ell$, then we can say that

$$L_z = mr^2\omega_\ell \tag{7.107}$$

is constant.

The frequency $\omega_\ell$ can be obtained by using $F = ma$ for the circular motion. The tension in the string is $mg/\cos\theta$ (to make the forces in the $y$-direction cancel), so the horizontal radial force is $mg\tan\theta$. Therefore,

$$mg\tan\theta = mr\omega_\ell^2 = m(\ell\sin\theta)\omega_\ell^2 \qquad \Longrightarrow \qquad \omega_\ell = \sqrt{\frac{g}{\ell\cos\theta}} = \sqrt{\frac{g}{h}} \,. \tag{7.108}$$

Plugging this into eq. (7.107), we see that the constant value of $L_z$ is

$$L_z = mr^2\sqrt{\frac{g}{h}} \,. \tag{7.109}$$

Let's now look at the two cases.

(a) For $\theta \approx 0$, we have $h \approx \ell$, so eq. (7.109) says that $r^2/\sqrt{\ell}$ is constant. Therefore,

$$r \propto \ell^{1/4}, \tag{7.110}$$

which means that $r$ grows very slowly with $\ell$.

(b) For $\theta \approx \pi/2$, we have $r \approx \ell$, so eq. (7.109) says that $\ell^2/\sqrt{h}$ is constant. Therefore,

$$h \propto \ell^4, \qquad (7.111)$$

which means that $h$ grows very quickly with $\ell$.

Note that eq. (7.109) says that $h \propto r^4$ for any value of $\theta$. So if you slowly lengthen the string so that $r$ doubles, then $h$ increases by a factor of 16.

15. **Falling Chimney**

Before we start dealing with the forces in the rods, let's first determine $\ddot{\theta}$ as a function of $\theta$ (the angle through which the chimney has fallen). Let $\ell$ be the height of the chimney. Then the moment of inertia around the pivot point on the ground is $m\ell^2/3$ (if we ignore the width), and the torque (around the pivot point) due to gravity is $\tau = mg(\ell/2)\sin\theta$. Therefore, $\tau = dL/dt$ gives $mg(\ell/2)\sin\theta = (1/3)m\ell^2\ddot{\theta}$, or

$$\ddot{\theta} = \frac{3g\sin\theta}{2\ell}\,. \qquad (7.112)$$

Let's now determine the forces in the rods. Our strategy will be to imagine that the chimney consists of a chimney of height $h$, with another chimney of height $\ell - h$ placed on top of it. We'll find the forces in the rods connecting these two "sub-chimneys," and then we'll maximize one of these forces ($T_2$, defined below) as a function of $h$.

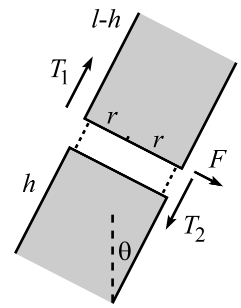


**Figure 7.67**

The forces on the top piece are gravity and also the forces from the two rods at each end of the bottom board. Let's break these latter forces up into transverse and longitudinal forces along the chimney. Let $T_1$ and $T_2$ be the two longitudinal components, and let $F$ be the sum of the transverse components, as shown in Fig. 7.67. We have picked the positive directions for $T_1$ and $T_2$ so that positive $T_1$ corresponds to a compression in the left rod, and positive $T_2$ corresponds to a tension in the right rod (which is what the forces will turn out to be, as we'll see). It turns out that if the width (which we'll call $2r$) is much less than the height, then $T_2 \gg F$ (as we will see below), so the tension in the right rod is essentially equal to $T_2$. We will therefore be concerned with maximizing $T_2$.

In writing down the force and torque equations for the top piece, we have three equations (the radial and tangential $F = ma$ equations, and $\tau = dL/dt$ around the CM), and three unknowns ($F$, $T_1$, and $T_2$). If we define the fraction $f \equiv h/\ell$, then the top piece has length $(1-f)\ell$ and mass $(1-f)m$, and its CM travels in a circle of radius $(1+f)\ell/2$, Therefore, our three force and torque equations are, respectively,

$$\begin{aligned} T_2 - T_1 + (1-f)mg\cos\theta &= (1-f)m\left(\frac{(1+f)\ell}{2}\right)\dot{\theta}^2, \\ F + (1-f)mg\sin\theta &= (1-f)m\left(\frac{(1+f)\ell}{2}\right)\ddot{\theta}, \\ (T_1+T_2)r - F\frac{(1-f)\ell}{2} &= (1-f)m\left(\frac{(1-f)^2\ell^2}{12}\right)\ddot{\theta}. \end{aligned} \qquad (7.113)$$

At this point, we could plow forward and solve this system of three equations in three unknowns. But things simplify greatly in the limit where $r \ll \ell$. The third equation says that $T_1 + T_2$ is of order $1/r$, and the first equation says that $T_2 - T_1$ is of order 1. These imply that $T_1 \approx T_2$, to leading order in $1/r$. Therefore, we may set $T_1 + T_2 \approx 2T_2$ in the third equation. Using this approximation, along with the value of $\ddot{\theta}$ from eq. (7.112), the second and third equations become

$$\begin{aligned} F + (1-f)mg\sin\theta &= \frac{3}{4}(1-f^2)mg\sin\theta, \\ 2rT_2 - F\frac{(1-f)\ell}{2} &= \frac{1}{8}(1-f)^3 mg\ell\sin\theta. \end{aligned} \tag{7.114}$$

This first of these equations gives

$$F = \frac{mg\sin\theta}{4}(-1+4f-3f^2), \tag{7.115}$$

and then the second gives

$$T_2 \approx \frac{mg\ell\sin\theta}{8r} f(1-f)^2. \tag{7.116}$$

As stated above, this is much greater than $F$ (because $\ell/r \gg 1$), so the tension in the right rod is essentially equal to $T_2$. Taking the derivative of $T_2$ with respect to $f$, we see that it is maximum at

$$f \equiv \frac{h}{\ell} = \frac{1}{3}\,. \tag{7.117}$$

Therefore, the chimney is most likely to break at a point one-third of the way up (assuming that the width is much less than the height). Interestingly, $f = 1/3$ makes the force $F$ in eq. (7.115) exactly equal to zero.

16. **Ball hitting stick**

Let $V$, $v$, and $\omega$ be the speed of the ball, the speed of the stick's CM, and the angular speed of the stick, respectively, after the collision. Then conservation of momentum, angular momentum (around the fixed point that coincides with the initial center of the stick), and energy give

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + \eta m\ell^2\omega, \\ MV_0^2 &= MV^2 + mv^2 + \eta m\ell^2\omega^2. \end{aligned} \tag{7.118}$$

We must solve these three equations for $V$, $v$, and $\omega$. The first two equations quickly give $vd = \eta\ell^2\omega$. Solving for $V$ in the first equation and plugging the result into the third, and then eliminating $\omega$ through $vd = \eta\ell^2\omega$ gives

$$v = \frac{2V_0}{1 + \frac{m}{M} + \frac{d^2}{\eta\ell^2}} \qquad \Longrightarrow \qquad \omega = V_0 \frac{2\frac{d}{\eta\ell^2}V_0}{1 + \frac{m}{M} + \frac{d^2}{\eta\ell^2}}\,. \tag{7.119}$$

Having found $v$, the first equation above gives $V$ as

$$V = V_0 \frac{1 - \frac{m}{M} + \frac{d^2}{\eta\ell^2}}{1 + \frac{m}{M} + \frac{d^2}{\eta\ell^2}}\,. \tag{7.120}$$

You are encouraged to check various limits of these answers.

Remark: Another solution to eqs. (7.118) is of course $V = V_0$, $v = 0$, and $\omega = 0$. The initial conditions certainly satisfy conservation of $p$, $L$, and $E$ with the initial conditions. A fine tautology, indeed. Nowhere in eqs. (7.118) does it say that the ball actually hits the stick. ♣

17. **A ball and stick theorem**

As in the solution to Problem 16, we have

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + I\omega, \\ MV_0^2 &= MV^2 + mv^2 + I\omega^2. \end{aligned} \tag{7.121}$$

The speed of the contact point on the stick right after the collision equals the speed of the CM plus the rotational speed relative to the CM. In other words, it equals $v + \omega d$. The desired relative speed is therefore $(v + \omega d) - V$. We can determine the value of this relative speed by solving the above three equations for $V$, $v$, and $\omega$. Or equivalently, we can just use the results of Problem 16. There is, however, a much more appealing method, which is as follows.

The first two equations quickly give $mvd = I\omega$. The last equation may then be written as, using $I\omega^2 = (I\omega)\omega = (mvd)\omega$,

$$M(V_0 - V)(V_0 + V) = mv(v + \omega d). \tag{7.122}$$

If we now write the first equation as

$$M(V_0 - V) = mv, \tag{7.123}$$

we can divide eq. (7.122) by eq. (7.123) to obtain $V_0 + V = v + \omega d$, or

$$V_0 = (v + \omega d) - V, \tag{7.124}$$

as we wanted to show. In terms of velocities, the correct statement is that the final relative velocity is the negative of the initial relative velocity. In other words, $V_0 - 0 = -\big(V - (v + \omega d)\big)$.

18. **Sliding to rolling**

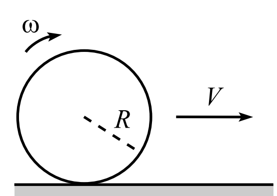



**Figure 7.68**

(a) Define all linear quantities to be positive to the right, and all angular quantities to be positive clockwise, as shown in Fig. 7.68. Then, for example, the friction force $F_f$ is negative. The friction force slows down the translational motion and speeds up the rotational motion, according to

$$\begin{aligned} F_f &= ma, \\ -F_f R &= I\alpha, \end{aligned} \tag{7.125}$$

where we have calculated the torque relative to the CM. Eliminating $F_f$, and using $I = \eta m R^2$, gives $a = -\eta R\alpha$. Integrating this over time, up to the time when the ball stops slipping, gives

$$\Delta V = -\eta R \Delta\omega. \tag{7.126}$$

Note that we could have obtained this by simply using the impulse equation, eq. (7.58). Using $\Delta V = V_f - V_0$, and $\Delta\omega = \omega_f - \omega_0 = \omega_f$, and also $\omega_f = V_f/R$ (the non-slipping condition), eq. (7.126) gives

$$V_f = \frac{V_0}{1+\eta}, \tag{7.127}$$

independent of the nature of $F_f$. $F_f$ can depend on position, time, speed, or anything else. The relation $a = -\eta R\alpha$, and hence also eq. (7.126), will still be true at all times.

REMARK: We can also calculate $\tau$ and $L$ relative to a dot painted on the ground that is the contact point at a given instant. There is zero torque relative to this point. To find $L$, we must add the $L$ of the CM and the $L$ relative to the CM. Therefore, $\tau = dL/dt$ gives $0 = (d/dt)(mRv + \eta mR^2\omega)$, and so $a = -\eta R\alpha$, as above. ♣

Using eq. (7.127), and also the relation $\omega_f = V_f/R$, the loss in kinetic energy is

$$\begin{aligned} \Delta KE &= \frac{1}{2}mV_0^2 - \left(\frac{1}{2}mV_f^2 + \frac{1}{2}I\omega_f^2\right) \\ &= \frac{1}{2}mV_0^2\left(1 - \frac{1}{(1+\eta)^2} - \frac{\eta}{(1+\eta)^2}\right) \\ &= \frac{1}{2}mV_0^2\left(\frac{\eta}{1+\eta}\right). \end{aligned} \tag{7.128}$$

For $\eta \to 0$, no energy is lost, which makes sense. And for $\eta \to \infty$, all the energy is lost, which also makes sense. This case is essentially like a sliding block which can't rotate.

(b) Let's first find $t$. The friction force is $F_f = -\mu mg$, so $F = ma$ gives $-\mu g = a$. Therefore, $\Delta V = at = -\mu gt$. But eq. (7.127) says that $\Delta V \equiv V_f - V_0 = -V_0\eta/(1+\eta)$. Therefore,

$$t = \frac{\eta}{(1+\eta)}\frac{V_0}{\mu g}. \tag{7.129}$$

For $\eta \to 0$, we have $t \to 0$, which makes sense. And for $\eta \to \infty$, we have $t \to V_0/(\mu g)$ which is exactly the time a sliding block would take to stop.

Let's now find $d$. We have $d = V_0 t + (1/2)at^2$. Using $a = -\mu g$, and plugging in the $t$ from eq. (7.129), we obtain

$$d = \frac{\eta(2+\eta)}{(1+\eta)^2}\frac{V_0^2}{2\mu g}. \tag{7.130}$$

The two extreme cases for $\eta$ check here.

To calculate the work done by friction, we might be tempted to write down the product $F_f d$, with $F_f = -\mu mg$ and $d$ given in eq. (7.130). But the result doesn't look much like the loss in kinetic energy calculated in eq. (7.128). What's wrong with this reasoning? The error is that the friction force does not act over a distance $d$. To find the distance over which $F_f$ acts, we must find how far the surface of the ball moves relative to the ground.

The speed of a dot on the ball that is instantaneously the contact point is $V_{\rm rel} = V(t) - R\omega(t) = (V_0 + at) - R\alpha t$. Using $\alpha = -a/\eta R$ and $a = -\mu g$, this becomes

$$V_{\rm rel} = V_0 - \frac{1+\eta}{\eta}\mu gt. \tag{7.131}$$

Integrating this from $t = 0$ to the $t$ given in eq. (7.129) gives

$$d_{\rm rel} = \int V_{\rm rel}\,dt = \frac{V_0^2\eta}{2\mu g(1+\eta)}. \tag{7.132}$$

The work done by friction is $F_f d_{\rm rel} = -\mu mg d_{\rm rel}$, which does indeed give the loss in kinetic energy given in eq. (7.128).

19. **The superball**

Since we are told that $|v_y|$ is unchanged by the bounce, we can ignore it when applying conservation of energy. And since the vertical impulse from the floor provides no torque around the ball's CM, we can completely ignore the $y$ motion in this problem.

The horizontal impulse from the floor is responsible for changing both $v_x$ and $\omega$. With positive directions defined as in the statement of the problem, eq. (7.58) gives

$$\begin{aligned} \Delta L &= R\Delta p \\ \Longrightarrow \quad I(\omega' - \omega) &= Rm(v_x' - v_x). \end{aligned} \tag{7.133}$$

But conservation of energy gives

$$\begin{aligned} \frac{1}{2}m{v_x'}^2 + \frac{1}{2}I\omega'^2 &= \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 \\ \Longrightarrow \quad I(\omega'^2 - \omega^2) &= m(v_x^2 - {v_x'}^2). \end{aligned} \tag{7.134}$$

Dividing this equation by eq. (7.133) gives[10]

$$R(\omega' + \omega) = -(v_x' + v_x). \tag{7.135}$$

We can now combine this equation with eq. (7.133), which can be rewritten as, using $I = (2/5)mR^2$,

$$\frac{2}{5}R(\omega' - \omega) = v_x' - v_x. \tag{7.136}$$

Given $v_x$ and $\omega$, the previous two equations are two linear equations in the two unknowns, $v_x'$ and $\omega'$. Solving for $v_x'$ and $\omega'$, and then writing the result in matrix notation, gives

$$\begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} v_x \\ R\omega \end{pmatrix}, \tag{7.137}$$

as desired. As an exercise, you can use this result to show that the relative velocity of the ball's contact point and the ground simply changes sign during the bounce.

20. **Many bounces**

Eq. (7.59) gives the result after one bounce, so the result after two bounces is

$$\begin{aligned} \begin{pmatrix} v_x'' \\ R\omega'' \end{pmatrix} &= \begin{pmatrix} 3/7 & -4/7 \\ -10/7 & -3/7 \end{pmatrix}\begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} \\ &= \begin{pmatrix} 3/7 & -4/7 \\ -10/7 & -3/7 \end{pmatrix}^2\begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\ &= \begin{pmatrix} v_x \\ R\omega \end{pmatrix}. \end{aligned} \tag{7.138}$$

The square of the matrix turns out to be the identity. Therefore, after two bounces, both $v_x$ and $\omega$ return to their original values. The ball then repeats the motion of

[10] We have divided out the trivial $\omega' = \omega$ and $v_x' = v_x$ solution, which corresponds to slipping motion on a frictionless plane. The nontrivial solution we will find shortly is the non-slipping one. Basically, to conserve energy, there must be no work done by friction. And since work is force times distance, this means that either the plane is frictionless, or that there is no relative motion between ball's contact point and the plane. The latter case is the one we are concerned with here.

the previous two bounces (and so on, after every two bounces). The only difference between successive pairs of bounces is that the ball may shift horizontally. You are strongly encouraged to experimentally verify this interesting periodic behavior.

21. **Rolling over a bump**

We will use the fact that the angular momentum of the ball with respect to the corner of the step (call this point $P$) is unchanged by the collision. This is true because any forces exerted at point $P$ provide zero torque around $P$.[11] This fact will allow us to find the energy of the ball right after the collision, which we will then require to be greater than $mgh$.

Breaking the initial $L$ into the contribution relative to the CM, plus the contribution from the ball treated like a point mass located at the CM, we see that the initial angular momentum is $L = (2/5)mR^2\omega_0 + mV_0(R-h)$, where $\omega_0$ is the initial angular speed. But the non-slipping condition tells us that $\omega_0 = V_0/R$. Therefore, $L$ may be written as

$$L = \frac{2}{5}mRV_0 + mV_0(R-h) = mV_0\left(\frac{7R}{5} - h\right). \tag{7.139}$$

Let $\omega'$ be the angular speed of the ball around point $P$ immediately after the collision. The parallel-axis theorem says that the moment of inertia around $P$ is equal to $(2/5)mR^2 + mR^2 = (7/5)mR^2$. Conservation of $L$ (around point $P$) during the collision then gives

$$mV_0\left(\frac{7R}{5} - h\right) = \frac{7}{5}mR^2\omega' \qquad \Longrightarrow \qquad \omega' = \frac{V_0}{R}\left(1 - \frac{5h}{7R}\right). \tag{7.140}$$

The energy of the ball right after the collision is therefore

$$E = \frac{1}{2}\left(\frac{7}{5}mR^2\right)\omega'^2 = \frac{1}{2}\left(\frac{7}{5}mR^2\right)\frac{V_0^2}{R^2}\left(1 - \frac{5h}{7R}\right)^2 = \frac{7}{10}mV_0^2\left(1 - \frac{5h}{7R}\right)^2. \tag{7.141}$$

The ball will climb up over the step if $E \geq mgh$, which gives

$$V_0 \geq \sqrt{\frac{10gh}{7}}\left(1 - \frac{5h}{7R}\right)^{-1}. \tag{7.142}$$

REMARKS: Note that is possible for the ball to rise up over the step even if $h > R$, provided that the ball sticks to the corner, without slipping. (If $h > R$, the step would have to be "hollowed out" so that the ball doesn't collide with the side of the step.) But note that $V_0 \to \infty$ as $h \to 7R/5$. For $h \geq 7R/5$, it is impossible for the ball to make it up over the step, no matter how large $V_0$ is. The ball will get pushed down into the ground, instead of rising up, if $h > 7R/5$.

For an object with a general moment of inertia $I = \eta mR^2$ (so $\eta = 2/5$ in our problem), you can show that the minimum initial speed is

$$V_0 \geq \sqrt{\frac{2gh}{1+\eta}}\left(1 - \frac{h}{(1+\eta)R}\right)^{-1}. \tag{7.143}$$

This decreases as $\eta$ increases. It is smallest when the "ball" is a wheel with all the mass on its rim (so that $\eta = 1$), in which case it is possible for the wheel to climb up over the step even if $h$ is close to $2R$. ♣

[11] The torque from gravity will be relevant once the ball rises up off the ground. But during the (instantaneous) collision, $L$ will not change.

22. **Lots of sticks**

Consider the collision between two sticks. Let $V$ be the speed of the contact point on the heavy one. Since this stick is essentially infinitely heavy, we may consider it to be an infinitely heavy ball, moving at speed $V$. The rotational degree of freedom of the heavy stick is irrelevant, as far as the light stick is concerned.

We may therefore invoke the result of Problem 17 to say that the relative speed of the contact points is the same before and after the collision. This implies that the contact point on the light stick picks up a speed of $2V$, because the heavy stick is essentially unaffected by the collision and keeps moving at speed $V$.

Let us now find the speed of the other end of the light stick. This stick receives an impulse from the heavy stick, so we can apply eq. (7.58) to the light stick to obtain

$$\eta m r^2 \omega = r(m v_{\text{CM}}) \qquad \Longrightarrow \qquad r\omega = \frac{v_{\text{CM}}}{\eta}\,. \tag{7.144}$$

The speed of the struck end is $v_{\text{str}} = r\omega + v_{\text{CM}}$, because the rotational speed adds to the CM motion. The speed of the other end is $v_{\text{oth}} = r\omega - v_{\text{CM}}$, because the rotational speed subtracts from the CM motion.[12] The ratio of these speeds is

$$\frac{v_{\text{oth}}}{v_{\text{str}}} = \frac{\frac{v_{\text{CM}}}{\eta} - v_{\text{CM}}}{\frac{v_{\text{CM}}}{\eta} + v_{\text{CM}}} = \frac{1-\eta}{1+\eta}\,. \tag{7.145}$$

In the problem at hand, we have $v_{\text{str}} = 2V$. Therefore,

$$v_{\text{oth}} = V\left(\frac{2(1-\eta)}{1+\eta}\right). \tag{7.146}$$

The same analysis holds for all the other collisions. Therefore, the bottom ends of the sticks move with speeds that form a geometric progression with ratio $2(1-\eta)/(1+\eta)$. If this ratio is less than 1 (that is, if $\eta > 1/3$), then the speeds go to zero as $n \to \infty$. If it is greater than 1 (that is, if $\eta < 1/3$), then the speeds go to infinity as $n \to \infty$. If it equals 1 (that is, if $\eta = 1/3$), then the speeds remain equal to $V$ and are thus independent of $n$, as we wanted to show. A uniform stick has $\eta = 1/3$ relative to its center (which is usually written in the form $I = m\ell^2/12$, where $\ell = 2r$).

[12] Since $\eta \leq 1$ for any real stick, we have $r\omega = v_{\text{CM}}/\eta \geq v_{\text{CM}}$. Therefore, $r\omega - v_{\text{CM}}$ is greater than or equal to zero.

# Chapter 8

# Angular Momentum, Part II (General $\hat{\text{L}}$)



In the Chapter 7, we discussed situations where the direction of the vector **L** remains constant, and only its magnitude changes. In this chapter, we will look at the more complicated situations where the direction of **L** is allowed to change. The vector nature of **L** will prove to be vital, and we will arrive at all sorts of strange results for spinning tops and such things.

This chapter is rather long, alas. The first three sections consist of general theory, and then in Section 8.4 we start solving some actual problems.

## 8.1 Preliminaries concerning rotations

### 8.1.1 The form of general motion

Before getting started, we should make sure we're all on the same page concerning a few important things about rotations. Because rotations generally involve three dimensions, they can often be hard to visualize. A rough drawing on a piece of paper might not do the trick. For this reason, this topic is one of the more difficult ones in this book.

The next few pages consist of some definitions and helpful theorems. This first theorem describes the form of general motion. You might consider it obvious, but let's prove it anyway.

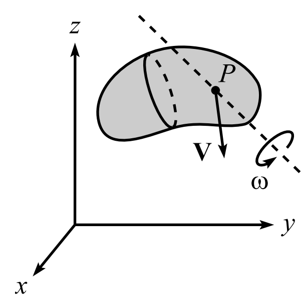



**Figure 8.1**

**Theorem 8.1** *Consider a rigid body undergoing arbitrary motion. Pick any point $P$ in the body. Then at any instant (see Fig. 8.1), the motion of the body may be written as the sum of the translational motion of $P$, plus a rotation around some axis, $\boldsymbol{\omega}$, through $P$ (the axis $\boldsymbol{\omega}$ may change with time).*[1]

**Proof:** The motion of the body may be written as the sum of the translational motion of $P$, plus some other motion relative to $P$ (this is true because relative

[1] In other words, what we mean here is that a person at rest with respect to a frame whose origin is $P$, and whose axes are parallel to the fixed-frame axes, will see the body undergoing a rotation around some axis through $P$.

coordinates are additive quantities). We must show that this latter motion is simply a rotation. This seems quite plausible, and it holds because the body is rigid; that is, all points keep the same relative distances. (If the body weren't rigid, then this theorem wouldn't be true.)

To be rigorous, consider a sphere fixed in the body, centered at $P$. The motion of the body is completely determined by the motion of the points on this sphere, so we need only examine what happens to the sphere. And because we are looking at motion relative to $P$, we have reduced the problem to the following: In what manner can a rigid sphere transform into itself? We claim that *any such transformation requires that two points end up where they started.*[2]

If this claim is true, then we are done, because for an infinitesimal transformation, a given point moves in only one direction (since there is no time to do any bending). So a point that ends up where it started must have always been fixed. Therefore, the diameter joining the two fixed points remains stationary (because distances are preserved), and we are left with a rotation around this axis.

This claim is quite believable, but nevertheless tricky to prove. I can't resist making you think about it, so I've left it as a problem (Problem 1). Try to solve it on your own. ■

We will invoke this theorem repeatedly in this chapter (often without bothering to say so). Note that it is required that $P$ be a point in the body, since we used the fact that $P$ keeps the same distances from other points in the body.

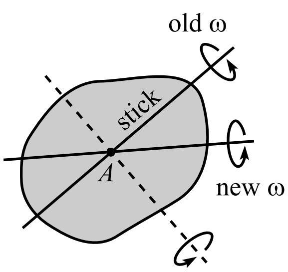



**Figure 8.2**

Remark: A situation where our theorem is not so obvious is the following. Consider an object rotating around a fixed axis, the stick shown in Fig. 8.2. In this case, $\boldsymbol{\omega}$ simply points along the stick. But now imagine grabbing the stick and rotating it around some other axis (the dotted line shown). It is not immediately obvious that the resulting motion is (instantaneously) a rotation around some new axis through $A$. But indeed it is. (We'll be quantitative about this in the "Rotating Sphere" example near the end of this section.) ♣

### 8.1.2 The angular velocity vector

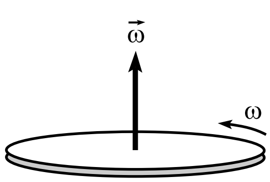



**Figure 8.3**

It is extremely useful to introduce the angular velocity vector, $\boldsymbol{\omega}$, which is defined to point along the axis of rotation, with a magnitude equal to the angular speed. The choice of the two possible directions is given by the right-hand rule. (Curl your right-hand fingers in the direction of the spin, and your thumb will point in the direction of $\boldsymbol{\omega}$.) For example, a spinning record has $\boldsymbol{\omega}$ perpendicular to the record, through its center (as shown in Fig. 8.3), with magnitude equal to the angular speed, $\omega$.

Remark: You could, of course, break the mold and use the left-hand rule, as long as you use it consistently. The direction of $\vec{\omega}$ would be opposite, but that doesn't matter, because $\vec{\omega}$ isn't really physical. Any physical result (for example, the velocity of a particle,

[2] This claim is actually true for *any* transformation of a rigid sphere into itself, but for the present purposes we are concerned only with infinitesimal transformations (because we are only looking at what happens at a given instant in time).

or the force on it) will come out the same, independent of which hand you (consistently) use.

> When studying vectors in school,
> You'll use your right hand as a tool.
> But look in a mirror,
> And then you'll see clearer,
> You can just use the left-handed rule. ♣

The points on the axis of rotation are the ones that (instantaneously) do not move. Of, course, the direction of $\boldsymbol{\omega}$ may change over time, so the points that were formerly on $\boldsymbol{\omega}$ may now be moving.

REMARK: The fact that we can specify a rotation by specifying a vector $\boldsymbol{\omega}$ is a peculiarity to three dimensions. If we lived in one dimension, then there would be no such thing as a rotation. If we lived in two dimensions, then all rotations would take place in that plane, so we could label a rotation by simply giving its speed, $\omega$. In three dimensions, rotations take place in $\binom{3}{2} = 3$ independent planes. And we choose to label these, for convenience, by the directions orthogonal to these planes, and by the angular speed in each plane. If we lived in four dimensions, then rotations could take place in $\binom{4}{2} = 6$ planes, so we would have to label a rotation by giving 6 planes and 6 angular speeds. Note that a vector (which has four components in four dimensions) would not do the trick here. ♣

In addition to specifying the points that are instantaneously motionless, $\boldsymbol{\omega}$ also easily produces the velocity of any point in the rotating object. Consider the case where the axis of rotation passes through the origin (which we will generally assume to be the case in this chapter, unless otherwise stated). Then we have the following theorem.

**Theorem 8.2** *Given an object rotating with angular velocity* $\boldsymbol{\omega}$*, the velocity of any point in the object is given by (with* $\mathbf{r}$ *being the position of the point)*

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}. \tag{8.1}$$

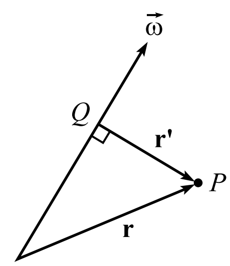


**Figure 8.4**

**Proof:** Drop a perpendicular from the point in question (call it $P$) to the axis $\boldsymbol{\omega}$ (call the point there $Q$). Let $\mathbf{r}'$ be the vector from $Q$ to $P$ (see Fig. 8.4). From the properties of the cross product, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is orthogonal to $\boldsymbol{\omega}$, $\mathbf{r}$, and also $\mathbf{r}'$ (since $\mathbf{r}'$ is a linear combination of $\boldsymbol{\omega}$ and $\mathbf{r}$). Therefore, the direction of $\mathbf{v}$ is correct (it lies in a plane perpendicular to $\boldsymbol{\omega}$, and is also perpendicular to $\mathbf{r}'$, so it describes circular motion around the axis $\boldsymbol{\omega}$; also, by the right-hand rule, it points in the proper orientation around $\boldsymbol{\omega}$). And since

$$|\mathbf{v}| = |\boldsymbol{\omega}||\mathbf{r}| \sin\theta = \omega r', \tag{8.2}$$

which is the speed of the circular motion around $\boldsymbol{\omega}$, we see that $\mathbf{v}$ has the correct magnitude. So $\mathbf{v}$ is indeed the correct velocity vector. ■

Note that if we have the special case where $P$ lies along $\omega$, then $\mathbf{r}$ is parallel to $\boldsymbol{\omega}$, and so the cross product gives a zero result for $\mathbf{v}$, as it should.

Eq. (8.1) is extremely useful and will be applied repeatedly in this chapter. Even if it's hard to visualize what's going on with a given rotation, all you have to do to find the speed of any given point is calculate the cross product $\boldsymbol{\omega} \times \mathbf{r}$.

Conversely, if the speed of every point in a moving body is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, then the body is undergoing a rotation with angular velocity $\boldsymbol{\omega}$ (because all points on the axis $\boldsymbol{\omega}$ are motionless, and all other points move with the proper speed for this rotation).

A very nice thing about angular velocities is that they simply add. Stated more precisely, we have the following theorem.

**Theorem 8.3** *Let coordinate systems $S_1$, $S_2$, and $S_3$ have the same origin. Let $S_1$ rotate with angular velocity $\boldsymbol{\omega}_{1,2}$ with respect to $S_2$. Let $S_2$ rotate with angular velocity $\boldsymbol{\omega}_{2,3}$ with respect to $S_3$. Then $S_1$ rotates (instantaneously) with angular velocity*

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} \tag{8.3}$$

*with respect to $S_3$.*

**Proof:** If $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the same direction, then the theorem is clear; the angular speeds just add. If, however, they don't point in the same direction, then things are a little harder to visualize. But we can prove the theorem by simply making abundant use of the definition of $\boldsymbol{\omega}$.

Pick a point $P_1$ at rest in $S_1$. Let $\mathbf{r}$ be the vector from the origin to $P_1$. The velocity of $P_1$ (relative to a very close point $P_2$ at rest in $S_2$) due to the rotation about $\boldsymbol{\omega}_{1,2}$ is $\mathbf{V}_{P_1P_2} = \boldsymbol{\omega}_{1,2} \times \mathbf{r}$. The velocity of $P_2$ (relative to a very close point $P_3$ at rest in $S_3$) due to the rotation about $\boldsymbol{\omega}_{2,3}$ is $\mathbf{V}_{P_2P_3} = \boldsymbol{\omega}_{2,3} \times \mathbf{r}$ (because $P_2$ is also located essentially at position $\mathbf{r}$). Therefore, the velocity of $P_1$ (relative to $P_3$) is $\mathbf{V}_{P_1P_2} + \mathbf{V}_{P_2P_3} = (\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}) \times \mathbf{r}$. This holds for any point $P_1$ at rest in $S_1$. So the frame $S_1$ rotates with angular velocity $(\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3})$ with respect to $S_3$. ■

Note that if $\boldsymbol{\omega}_{1,2}$ is constant in $S_2$, then the vector $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ will change with respect to $S_3$ as time goes by (because $\boldsymbol{\omega}_{1,2}$, which is fixed in $S_2$, is changing with respect to $S_3$). But at any instant, $\boldsymbol{\omega}_{1,3}$ may be obtained by simply adding the present values of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$. Consider the following example.

---

**Example (Rotating sphere):** A sphere rotates with angular speed $\omega_3$ around a stick that initially points in the $\hat{\mathbf{z}}$ direction. You grab the stick and rotate it around the $\hat{\mathbf{y}}$-axis with angular speed $\omega_2$. What is the angular velocity of the sphere, with respect to the lab frame, as time goes by?

**Solution:** In the language of Theorem 8.3, the sphere defines the $S_1$ frame; the stick and the $\hat{\mathbf{y}}$-axis define the $S_2$ frame; and the lab frame is the $S_3$ frame. The instant after you grab the stick, we are given that $\boldsymbol{\omega}_{1,2} = \omega_3\hat{\mathbf{z}}$, and $\boldsymbol{\omega}_{2,3} = \omega_2\hat{\mathbf{y}}$. Therefore, the angular velocity of the sphere with respect to the lab frame is $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} = \omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$. This is shown is Fig. 8.5. As time goes by, the stick (and hence $\boldsymbol{\omega}_{1,2}$) rotates around the $\mathbf{y}$ axis, so $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ traces out a cone around the $\mathbf{y}$ axis, as shown.

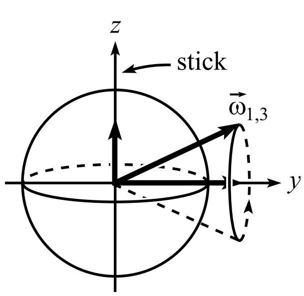



**Figure 8.5**

REMARK: Note the different behavior of $\vec{\omega}_{1,3}$ for a slightly different statement of the problem: Let the sphere initially rotate with angular velocity $\omega_2\hat{\mathbf{y}}$. Grab the axis (which points in the $\hat{\mathbf{y}}$ direction) and rotate it with angular velocity $\omega_3\hat{\mathbf{z}}$. For this situation, $\vec{\omega}_{1,3}$ initially points in the same direction as in the above statement of the problem (it is initially equal to $\omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$), but as time goes by, it is the $\omega_2\hat{\mathbf{y}}$ vector that will change, so $\vec{\omega}_{1,3} = \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$ traces out a cone around the **z** axis, as shown in Fig. 8.6. ♣

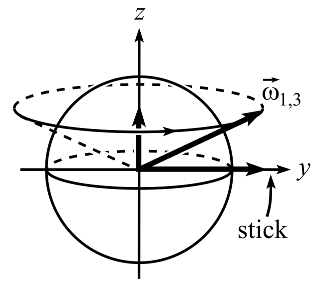



**Figure 8.6**

An important point concerning rotations in that they are defined with respect to a *coordinate system*. It makes no sense to ask how fast an object is rotating with respect to a certain point, or even a certain axis. Consider, for example, an object rotating with angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$, with respect to the lab frame. Saying only, "The object has angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$," is not sufficient, because someone standing in the frame of the object would measure $\boldsymbol{\omega} = 0$, and would therefore be very confused by your statement.

Throughout this chapter, we'll try to remember to state the coordinate system with respect to which $\boldsymbol{\omega}$ is measured. But if we forget, the default frame is the lab frame.

If you want to strain some brain cells thinking about $\boldsymbol{\omega}$ vectors, you are encouraged to solve Problem 3, and then also to look at the three given solutions.

This section was a bit abstract, so don't worry too much about it at the moment. The best strategy is probably to read on, and then come back for a second pass after digesting a few more sections. At any rate, we'll be discussing many other aspects of $\boldsymbol{\omega}$ in Section 8.7.2.

## 8.2 The inertia tensor

Given an object undergoing general motion, the *inertia tensor* is what relates the angular momentum, **L**, to the angular velocity, $\boldsymbol{\omega}$. This tensor[3] depends on the geometry of the object, as we will see. In finding the **L** due to general motion, we will (in the same spirit as in Section 7.1) first look at the special case of rotation around an axis through the origin. Then we will look at the most general possible motion.

### 8.2.1 Rotation about an axis through the origin

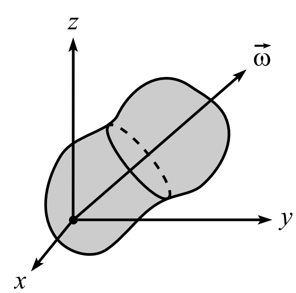



**Figure 8.7**

The three-dimensional object in Fig. 8.7 rotates with angular velocity $\boldsymbol{\omega}$. Consider a little piece of the body, with mass $dm$ and position $\mathbf{r}$. The velocity of this piece is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. So the angular momentum (relative to the origin) of this piece is equal to $\mathbf{r} \times \mathbf{p} = (dm)\mathbf{r} \times \mathbf{v} = (dm)\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. The angular momentum of the entire body is therefore

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm, \qquad (8.4)$$

where the integration runs over the volume of the body.

[3] "Tensor" is just a fancy name for "matrix" here.

In the case where the rigid body is made up of a collection of point masses, $m_i$, the angular momentum is simply

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \tag{8.5}$$

This double cross-product looks a bit intimidating, but it's actually not so bad. First, we have

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{r} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} \\ &= (\omega_2 z - \omega_3 y)\hat{\mathbf{x}} + (\omega_3 x - \omega_1 z)\hat{\mathbf{y}} + (\omega_1 y - \omega_2 x)\hat{\mathbf{z}}. \end{aligned} \tag{8.6}$$

Therefore,

$$\begin{aligned} \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & z \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} \\ &= \Big(\omega_1(y^2 + z^2) - \omega_2 xy - \omega_3 zx\Big)\hat{\mathbf{x}} \\ &\quad + \Big(\omega_2(z^2 + x^2) - \omega_3 yz - \omega_1 xy\Big)\hat{\mathbf{y}} \\ &\quad + \Big(\omega_3(x^2 + y^2) - \omega_1 zx - \omega_2 yz\Big)\hat{\mathbf{z}}. \end{aligned} \tag{8.7}$$

The angular momentum in eq. (8.4) may therefore be written in the nice, concise, matrix form,

$$\begin{aligned} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv \mathbf{I}\boldsymbol{\omega} \end{aligned} \tag{8.8}$$

For sake of clarity, we have not bothered to write the $dm$ part of each integral. The matrix **I** is called the *inertia tensor*. If the word "tensor" scares you, just ignore it. **I** is simply a matrix. It acts on one vector (the angular velocity) to yield another vector (the angular momentum).

REMARKS:

1. **I** is a rather formidable-looking object. Therefore, you will undoubtedly be very pleased to hear that you will rarely have to use it. It's nice to know that it's there if you do need it, but the concept of *principal axes* in Section 8.3 provides a much better way of solving problems, which avoids the use of the inertia tensor.

2. **I** is a symmetric matrix. (This fact will be important in Section 8.3.) There are therefore only six independent entries, instead of nine.

3. In the case where the rigid body is made up of a collection of point masses, $m_i$, the entries in the matrix are just sums. For example, the upper left entry is $\sum m_i(y_i^2+z_i^2)$.
4. **I** depends only on the geometry of the object, and not on $\boldsymbol{\omega}$.
5. To construct an **I**, you not only need to specify the origin, you also need to specify the $x$,$y$,$z$ axes of your coordinate system. (These basis vectors must be orthogonal, because the cross-product calculation above is valid only for an orthonormal basis.) If someone else comes along and chooses a different orthonormal basis (but the same origin), then her **I** will have different *entries*, as will her $\boldsymbol{\omega}$, as will her **L**. But her $\boldsymbol{\omega}$ and **L** will be exactly the same *vectors* as your $\boldsymbol{\omega}$ and **L**. They will only appear different because they are written in a different coordinate system. (A vector is what it is, independent of how you choose to look at it. If you each point your arm in the direction of what you calculate **L** to be, then you will both be pointing in the same direction.) ♣

All this is fine and dandy. Given any rigid body, we can calculate **I** (relative to a given origin, using a given set of axes). And given $\boldsymbol{\omega}$, we can then apply **I** to it to find **L** (relative to the origin). But what do these entries in **I** really mean? How do we interpret them? Note, for example, that the $L_3$ in eq. (8.8) contains terms involving $\omega_1$ and $\omega_2$. But $\omega_1$ and $\omega_2$ have to do with rotations around the $x$ and $y$ axes, so what in the world are they doing in $L_3$? Consider the following examples.

**Example 1 (Point-mass in $x$-$y$ plane):** Consider a point-mass $m$ traveling in a circle (centered at the origin) in the $x$-$y$ plane, with frequency $\omega_3$. Let the radius of the circle be $r$ (see Fig. 8.8).

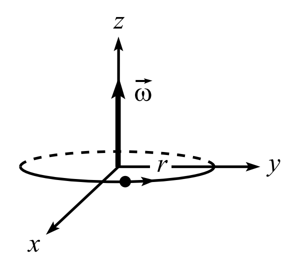


**Figure 8.8**

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = 0$ in eq. (8.8) (with a discrete sum of only one object, instead of the integrals), the angular momentum with respect to the origin is

$$\mathbf{L} = (0, 0, mr^2\omega_3). \tag{8.9}$$

The $z$-component is $mrv$, as it should be. And the $x$- and $y$-components are 0, as they should be. This case where $\omega_1 = \omega_2 = 0$ and $z = 0$ is simply the case we studied in the Chapter 7.

**Example 2 (Point-mass in space):** Consider a point-mass $m$ traveling in a circle of radius $r$, with frequency $\omega_3$. But now let the circle be centered at the point $(0, 0, z_0)$, with the plane of the circle parallel to the $x$-$y$ plane (see Fig. 8.9).

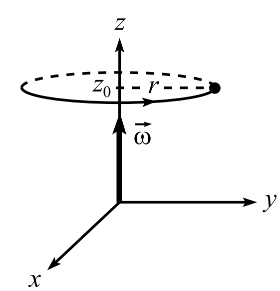


**Figure 8.9**

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), the angular momentum with respect to the origin is

$$\mathbf{L} = m\omega_3(-xz_0, -yz_0, r^2). \tag{8.10}$$

The $z$-component is $mrv$, as it should be. But, surprisingly, we have nonzero $L_1$ and $L_2$, even though our mass is simply rotating around the $z$-axis. What's going on?

Consider the instant when the mass is in the $x$-$z$ plane. The velocity of the mass is then in the $\hat{\mathbf{y}}$ direction. Therefore, the particle most certainly has angular momentum around the $x$-axis, as well as the $z$-axis. (Someone looking at a split-second movie of the particle at this point could not tell whether the mass was rotating around the

$x$-axis, the $z$-axis, or undergoing some complicated motion. But the past and future motion is irrelevant; at any instant in time, as far as the angular momentum goes, we are concerned only with what is happening at that instant.)

At this instant, the angular momentum around the $x$-axis is $-mz_0v$ (since $z_0$ is the distance from the $x$-axis; and the minus sign comes from the right-hand rule). Using $v = \omega_3 x$, we have $L_1 = -mxz_0\omega_3$, in agreement with eq. (8.10).

At this instant, $L_2$ is zero, since the velocity is parallel to the $y$-axis. This agrees with eq. (8.10), since $y = 0$. And you can check that eq. (8.10) is indeed correct when the mass is at a general point $(x, y, z_0)$.

For a point mass, $\mathbf{L}$ is much more easily obtained by simply calculating $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (you should use this to check the results of this example). But for more complicated objects, the tensor $\mathbf{I}$ must be used.

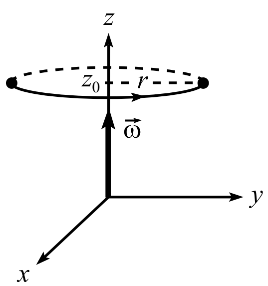


**Figure 8.10**

**Example 3 (Two point-masses):** Add another point-mass $m$ to the previous example. Let it travel in the same circle, at the diametrically opposite point (see Fig. 8.10).

Using $\omega = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), you can show that the angular momentum with respect to the origin is

$$\mathbf{L} = 2m\omega_3(0, 0, r^2). \tag{8.11}$$

The $z$-component is $2mrv$, as it should be. And $L_1$ and $L_2$ are zero, unlike in the previous example, because these components of the $\mathbf{L}$'s of the two particles cancel. This occurs because of the symmetry of the masses around the $z$-axis, which causes the $I_{zx}$ and $I_{zy}$ entries in the inertia tensor to vanish (because they are each the sum of two terms, with opposite $x$ values, or opposite $y$ values).

Let's now look at the kinetic energy of our object (which is rotating about an axis passing through the origin). To find this, we need to add up the kinetic energies of all the little pieces. A little piece has energy $(dm)\, v^2/2 = dm\, |\boldsymbol{\omega} \times \mathbf{r}|^2/2$. So, using eq. (8.6), the total kinetic energy is

$$T = \frac{1}{2}\int \Big( (\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2 \Big)\, dm. \tag{8.12}$$

Multiplying this out, we see (after a little work) that we may write $T$ as

$$\begin{aligned} T &= \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \end{aligned} \tag{8.13}$$

If $\boldsymbol{\omega} = \omega_3 \hat{\mathbf{z}}$, then this reduces to the $T = I_{33}\omega_3^2/2$ result in eq. (7.8) in Chapter 7 (with a slight change in notation).

### 8.2.2 General motion

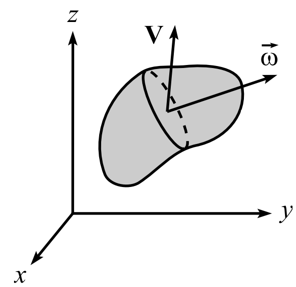


**Figure 8.11**

How do we deal with general motion in space? For the motion in Fig. 8.11, the various pieces of mass are not traveling in circles about the origin, so we cannot write $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, as we did prior to eq. (8.4).

To determine $\mathbf{L}$ (relative to the origin), and also the kinetic energy $T$, we will invoke Theorem 8.1. In applying this theorem, we may choose any point in the body to be the point $P$ in the theorem. However, only in the case that $P$ is the object's CM can we extract anything useful. The theorem then says that the motion of the body is the sum of the motion of the CM plus a rotation about the CM. So, let the CM move with velocity $\mathbf{V}$, and let the body instantaneously rotate with angular velocity $\boldsymbol{\omega}'$ around the CM. (That is, with respect to the frame whose origin is the CM, and whose axes are parallel to the fixed-frame axes.)

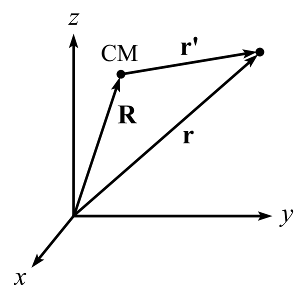


**Figure 8.12**

Let the CM coordinates be $\mathbf{R} = (X, Y, Z)$, and let the coordinates relative to the CM be $\mathbf{r}' = (x', y', z')$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 8.12). Let the velocity relative to the CM be $\mathbf{v}'$ (so $\mathbf{v}' = \boldsymbol{\omega}' \times \mathbf{r}'$). Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$.

Let's look at $L$ first. The angular momentum is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times \Big(\mathbf{V} + (\boldsymbol{\omega}' \times \mathbf{r}')\Big)\, dm \\
&= \int (\mathbf{R} \times \mathbf{V})\, dm + \int \mathbf{r}' \times (\boldsymbol{\omega}' \times \mathbf{r}')\, dm \\
&= M(\mathbf{R} \times \mathbf{V}) + \mathbf{L}_{\text{CM}}.
\end{aligned} \tag{8.14}$$

The cross terms vanish because the integrands are linear in $\mathbf{r}'$ (and so the integrals, which involve $\int \mathbf{r}'\, dm$, are zero by definition of the CM). $\mathbf{L}_{\text{CM}}$ is the angular momentum relative to the CM.[4]

As in the pancake case Section 7.1.2, we see that the angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the CM and finding the angular momentum of this point mass (relative to the origin), and by then adding on the angular momentum of the body, relative to the CM. Note that these two parts of the angular momentum need not point in the same direction (as they did in the pancake case).

Now let's look at $T$. The kinetic energy is

$$\begin{aligned}
T &= \int \frac{1}{2} v^2\, dm \\
&= \int \frac{1}{2} |\mathbf{V} + \mathbf{v}'|^2\, dm \\
&= \int \frac{1}{2} V^2\, dm + \int \frac{1}{2} v'^2\, dm \\
&= \frac{1}{2} M V^2 + \int \frac{1}{2} |\boldsymbol{\omega}' \times \mathbf{r}'|^2\, dm
\end{aligned}$$

[4]By this, we mean the angular momentum as measured in the coordinate system whose origin is the CM, and whose axes are parallel to the fixed-frame axes.

$$\equiv \frac{1}{2}MV^2 + \frac{1}{2}\boldsymbol{\omega}' \cdot \mathbf{L}_{\text{CM}}, \tag{8.15}$$

where the last line follows from the steps leading to eq. (8.13). The cross term $\int \mathbf{V} \cdot \mathbf{v}'\,dm = \int \mathbf{V} \cdot (\boldsymbol{\omega}' \times \mathbf{r}')\,dm$ vanishes because the integrand is linear in $\mathbf{r}'$ (and thus yields a zero integral, by definition of the CM).

As in the pancake case Section 7.1.2, we see that the kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to motion relative to the CM.

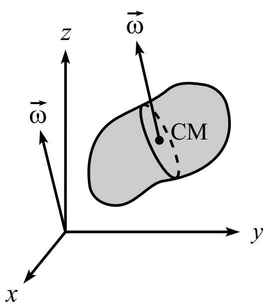



**Figure 8.13**

### 8.2.3 The parallel-axis theorem

Consider the special case where the CM rotates around the origin with the same angular velocity at which the body rotates around the CM (see Fig. 8.13). That is, $\mathbf{V} = \boldsymbol{\omega}' \times \mathbf{R}$, (This may be achieved, for example, by having a rod stick out of the body and pivoting one end of the rod at the origin.) This means that we have the nice situation where all points in the body travel in fixed circles around the axis of rotation (because $\mathbf{v} = \mathbf{V} + \mathbf{v}' = \boldsymbol{\omega}' \times \mathbf{R} + \boldsymbol{\omega}' \times \mathbf{r}' = \boldsymbol{\omega}' \times \mathbf{r}$). Dropping the prime on $\boldsymbol{\omega}$, eq. (8.14) becomes

$$\mathbf{L} = M\mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{R}) + \int \mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')\,dm \tag{8.16}$$

Expanding the double cross-products as in the steps leading to eq. (8.8), we may write this as

$$\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= M \begin{pmatrix} Y^2+Z^2 & -XY & -ZX \\ -XY & Z^2+X^2 & -YZ \\ -ZX & -YZ & X^2+Y^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&\quad + \begin{pmatrix} \int(y'^2+z'^2) & -\int x'y' & -\int z'x' \\ -\int x'y' & \int(z'^2+x'^2) & -\int y'z' \\ -\int z'x' & -\int y'z' & \int(x'^2+y'^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&\equiv (\mathbf{I}_{\text{R}} + \mathbf{I}_{\text{CM}})\boldsymbol{\omega}. 
\end{aligned} \tag{8.17}$$

This is the generalized parallel-axis theorem. It says that once you've calculated $\mathbf{I}_{\text{CM}}$ for an axis through the CM, then if you want to calculate $\mathbf{I}$ around any parallel axis, you simply have to add on the $\mathbf{I}_{\text{R}}$ matrix (obtained by treating the object like a point-mass at the CM). So you have to compute six numbers (there are only six, instead of nine, because the matrix is symmetric) instead of just the one $MR^2$ in the parallel-axis theorem in Chapter 7, given in eq. (7.12).

Likewise, if $\mathbf{V} = \boldsymbol{\omega}' \times \mathbf{R}$, then eq. (8.15) gives (dropping the prime on $\boldsymbol{\omega}$) a kinetic energy of

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_{\text{R}} + \mathbf{I}_{\text{CM}})\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \tag{8.18}$$

## 8.3 Principal axes

The cumbersome expressions in the previous section may seem a bit unsettling, but it turns out that you will rarely have to invoke them. The strategy for avoiding all the previous mess is to use the *principal axes* of a body, which we will define below.

In general, the inertia tensor $\mathbf{I}$ in eq. (8.8) has nine nonzero entries (six independent ones). In addition to depending on the origin chosen, this inertia tensor depends on the set of orthonormal basis vectors chosen for the coordinate system. (The $x$,$y$,$z$ variables in the integrals in $\mathbf{I}$ depend on the coordinate system with respect to which they are measured, of course.)

Given a blob of material, and given an arbitrary origin,[5] any orthonormal set of basis vectors is usable, but there is one special set that makes all our calculations very nice. These special basis vectors are called the *principal axes.* They can be defined in various equivalent ways.

- The principal axes are the orthonormal basis vectors for which $\mathbf{I}$ is diagonal, that is, for which[6]

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \qquad (8.19)$$

  $I_1$, $I_2$, and $I_3$ are called the *principal moments.*

  For many objects, it is quite obvious what the principal axes are. For example, consider a uniform rectangle in the $x$-$y$ plane, and let the CM be the origin (and let the sides be parallel to the coordinate axes). Then the principal axes are clearly the $x$, $y$, and $z$ axes, because all the off-diagonal elements of the inertia tensor in eq. (8.8) vanish, by symmetry. For example $I_{xy} \equiv -\int xy\, dm$ equals zero, because for every point $(x, y)$ in the rectangle, there is a corresponding point $(-x, y)$. So the contributions to $\int xy\, dm$ cancel in pairs. Also, the integrals involving $z$ are identically zero, because $z = 0$.

- The principal axes are the orthonormal set $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, $\hat{\boldsymbol{\omega}}_3$ with the property that

$$\mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_2 = I_2\hat{\boldsymbol{\omega}}_2, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_3 = I_3\hat{\boldsymbol{\omega}}_3. \qquad (8.20)$$

  (That is, they are the $\boldsymbol{\omega}$'s for which $\mathbf{L}$ points in the same direction as $\boldsymbol{\omega}$.) These three statements are equivalent to eq. (8.19), because the vectors $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, and $\hat{\boldsymbol{\omega}}_3$ are simply $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in the frame in which they are the basis vectors.

- The principal axes are the axes around which the object can rotate with constant speed, without the need for any torque. (So in some sense, the object is

[5] The CM is often chosen to be the origin, but it need not be. There are principal axes associated with any origin.

[6] Technically, we should be writing $I_{11}$ instead of $I_1$, etc., in this matrix, because we're talking about elements of a matrix. (The one-index object $I_1$ looks like a component of a vector.) But the two-index notation gets cumbersome, so we'll be sloppy and just use $I_1$, etc.

"happy" to spin around a principal axis.) This is equivalent to the previous definition for the following reason. Assume the object rotates around an axis $\hat{\boldsymbol{\omega}}_1$, for which $\mathbf{L} = \mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1$, as in eq. (8.20). Then, since $\hat{\boldsymbol{\omega}}_1$ is assumed to be fixed, we see that $\mathbf{L}$ is also fixed. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt = \mathbf{0}$.

The lack of need for any torque, for rotation around a principal axis $\hat{\boldsymbol{\omega}}$, means that if the object is pivoted at the origin, and if the origin is the only place where any force is applied, then the object can undergo rotation with constant angular velocity $\boldsymbol{\omega}$. If you try to set up this scenario with a non-principal axis, it won't work.

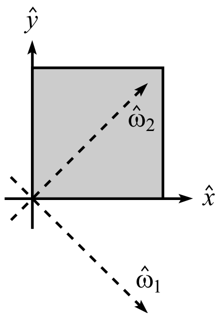


**Figure 8.14**

**Example (Square with origin at corner):** Consider the uniform square in Fig. 8.14. In Appendix G, we show that the principal axes are the dotted lines shown (and also the $z$-axis perpendicular to the page). But there is no need to use the techniques of the appendix to see this, because in this basis it is clear that the integral $\int x_1x_2$ is zero, by symmetry. (And $x_3 \equiv z$ is identically zero, which makes the other off-diagonal terms in $\mathbf{I}$ also equal to zero.)

Furthermore, it is intuitively clear that the square will be happy to rotate around any one of these axes indefinitely. During such a rotation, the pivot will certainly be supplying a *force* (if the axis is $\hat{\boldsymbol{\omega}}_1$ or $\hat{\mathbf{z}}$), to provide the centripetal acceleration for the circular motion of the CM. But it will not be applying a *torque* relative to the origin (because the $\mathbf{r}$ in $\mathbf{r} \times \mathbf{F}$ is $\mathbf{0}$). This is good, because for a rotation around one of these principal axes, $d\mathbf{L}/dt = \mathbf{0}$, and there is no need for any torque.

It is fairly clear that it is impossible to make the square rotate around, say, the $x$-axis, assuming that its only contact with the world is through a free pivot at the origin. The square simply doesn't want to remain in that circular motion. There are various ways to demonstrate this rigorously. One is to show that $\mathbf{L}$ (relative to the origin) will not point along the $x$-axis, so it will therefore precess around the $x$-axis along with the square, tracing out the surface of a cone. This means that $\mathbf{L}$ is changing. But there is no torque available (relative to the origin) to provide for this change in $\mathbf{L}$. Hence, such a rotation cannot exist.

Note also that the integral $\int xy$ is not equal to zero (every point gives a positive contribution). So the inertia tensor is not diagonal in the $x$-$y$ basis, which means that $\hat{x}$ and $\hat{y}$ are not principal axes.

At the moment, it is not at all obvious that an orthonormal set of principal axes exists for an arbitrary object. This is the task of Theorem 8.4 below. But assuming that principal axes do exist, the $\mathbf{L}$ and $T$ in eqs. (8.8) and (8.13) take on the particularly nice forms,

$$\begin{aligned} \mathbf{L} &= (I_1\omega_1, I_2\omega_2, I_3\omega_3), \\ T &= \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \end{aligned} \tag{8.21}$$

in the basis of the principal axes. (The numbers $\omega_1$, $\omega_2$, and $\omega_3$ are the components of a general vector $\boldsymbol{\omega}$ written in the principal-axis basis; that is, $\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{\omega}}_1 + \omega_2\hat{\boldsymbol{\omega}}_2 +$

$\omega_3\hat{\boldsymbol{\omega}}_3$.) This is a vast simplification over the general formulas in eqs. (8.8) and (8.13). We will therefore invariably work with principal axes in the remainder of this chapter.

REMARK: Note that the directions of the principal axes (relative to the body) depend only on the geometry of the body. They may therefore be considered to be painted onto the object. Hence, they will generally move around in space as the body rotates. (For example, in the special case where the object is rotating happily around a principal axis, then that axis will stay fixed, and the other two principal axes will rotate around it in space.) In equations like $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, the components $\omega_i$ and $I_i\omega_i$ are measured along the *instantaneous* principal axes $\hat{\boldsymbol{\omega}}_i$. Since these axes change with time, the components $\omega_i$ and $I_i\omega_i$ will generally change with time (except in the case where we have a nice rotation around a principal axis). ♣

Let us now prove that a set of principal axes does indeed exist, for any object, and any origin. Actually, we'll just state the theorem here. The proof involves a rather slick and useful technique, but it's slightly off the main line of thought, so we'll relegate it to Appendix F. Take a look at the proof if you wish, but if you want to simply accept the fact that the principal axes exist, that's fine.

**Theorem 8.4** *Given a real symmetric* $3\times 3$ *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* $I_k$*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k\hat{\boldsymbol{\omega}}_k. \tag{8.22}$$

**Proof:** See Appendix F. ∎

Since the inertia tensor in eq. (8.8) is indeed symmetric, for any body and any origin, this theorem says that we can always find three orthogonal basis-vectors for which $\mathbf{I}$ is a diagonal matrix. That is, principal axes always exist. Invariably, it is best to work in a coordinate system that has this basis. (As mentioned above, the CM is generally chosen to be the origin, but this is not necessary. There are principal axes associated with any origin.)

Problem 5 gives another way to show the existence of principal axes in the special case of a pancake object.

For an object with a fair amount of symmetry, the principal axes are usually the obvious choices and can be written down by simply looking at the object (examples are given below). If, however, you are given an unsymmetrical body, then the only way to determine the principal axes is to pick an arbitrary basis, then find $\mathbf{I}$ in this basis, then go through a diagonalization procedure. This diagonalization procedure basically consists of the steps at the beginning of the proof of Theorem 8.4 (given in Appendix F), with the addition of one more step to get the actual vectors, so we'll relegate it to Appendix G. You need not worry much about this method. Virtually every problem we encounter will involve an object with sufficient symmetry to enable you to simply write down the principal axes.

Let's now prove two very useful (and very similar) theorems, and then we'll give some examples.

**Theorem 8.5** *If two principal moments are equal ($I_1 = I_2 \equiv I$), then any axis (through the chosen origin) in the plane of the corresponding principal axes is a principal axis (and its moment is also $I$).*

*Similarly, if all three principal moments are equal ($I_1 = I_2 = I_3 \equiv I$), then any axis (through the chosen origin) in space is a principal axis (and its moment is also $I$).*

**Proof:** This first part was already proved at the end of the proof in Appendix F, but we'll do it again here. Let $I_1 = I_2 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$. Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$. Therefore, any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Similarly, let $I_1 = I_2 = I_3 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$, and $\mathbf{I}\mathbf{u}_3 = I\mathbf{u}_3$. Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3) = I(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3)$. Therefore, any linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (that is, any vector in space) is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Basically, if $I_1 = I_2 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the space spanned by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. And if $I_1 = I_2 = I_3 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the entire space. ■

If two or three moments are equal, so that there is freedom in choosing the principal axes, then it is possible to pick a non-orthogonal group of them. We will, however, always choose ones that are orthogonal. So when we say "a set of principal axes", we mean an orthonormal set.

**Theorem 8.6** *If a pancake object is symmetric under a rotation through an angle $\theta \neq 180°$ in the $x$-$y$ plane (for example, a hexagon), then every axis in the $x$-$y$ plane (with the origin chosen to be the center of the symmetry rotation) is a principal axis.*

**Proof:** Let $\hat{\boldsymbol{\omega}}_0$ be a principal axis in the plane, and let $\hat{\boldsymbol{\omega}}_\theta$ be the axis obtained by rotating $\hat{\boldsymbol{\omega}}_0$ through the angle $\theta$. Then $\hat{\boldsymbol{\omega}}_\theta$ is also a principal axis with the same principal moment (due to the symmetry of the object). Therefore, $\mathbf{I}\hat{\boldsymbol{\omega}}_0 = I\hat{\boldsymbol{\omega}}_0$, and $\mathbf{I}\hat{\boldsymbol{\omega}}_\theta = I\hat{\boldsymbol{\omega}}_\theta$.

Now, any vector $\boldsymbol{\omega}$ in the $x$-$y$ plane can be written as a linear combination of $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$, provided that $\theta \neq 180°$ (this is where we use that assumption). That is, $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$ span the $x$-$y$ plane. Therefore, any vector $\boldsymbol{\omega}$ may be written as $\boldsymbol{\omega} = a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta$, and so

$$\mathbf{I}\boldsymbol{\omega} = \mathbf{I}(a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta) = aI\hat{\boldsymbol{\omega}}_0 + bI\hat{\boldsymbol{\omega}}_\theta = I\boldsymbol{\omega}. \tag{8.23}$$

Hence, $\boldsymbol{\omega}$ is also a principal axis. (Problem 6 gives another proof of this theorem.) ■

Let's now give some examples. We'll state the principal axes for the following objects (relative to the origin). Your exercise is to show that these are correct. Usually, a quick symmetry argument shows that

$$\mathbf{I} \equiv \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \tag{8.24}$$

is diagonal. In all of these examples (see Fig. 8.15), the origin for the principal axes is the origin of the given coordinate system (which is not necessarily the CM). In describing the axes, they thus all pass through the origin, in addition to having the other properties stated.

**Example 1:** Point mass at the origin.
*principal axes:* any axes.

**Example 2:** Point mass at the point $(x_0, y_0, z_0)$.
*principal axes:* axis through point, any axes perpendicular to this.

**Example 3:** Rectangle centered at the origin, as shown.
*principal axes:* $z$-axis, axes parallel to sides.

**Example 4:** Cylinder with axis as $z$-axis.
*principal axes:* $z$-axis, any axes in $x$-$y$ plane.

**Example 5:** Square with one corner at origin, as shown.
*principal axes:* $z$ axis, axis through CM, axis perp to this.

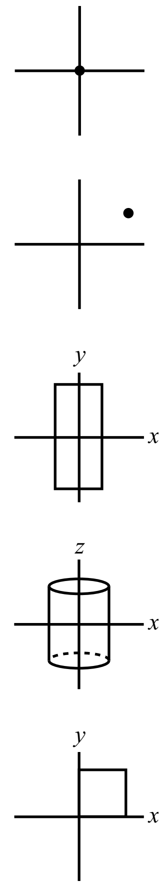


**Figure 8.15**

## 8.4 Two basic types of problems

The previous three sections introduced many new, and somewhat abstract, concepts. We will now (finally) get our hands dirty and solve some actual problems. The concept of principal axes, in particular, gives us the ability to solve many kinds of problems. Two types, however, come up again and again. There are variations on these, of course, but they may be generally stated as follows.

- Strike a rigid object with an impulsive (that is, quick) blow. What is the motion of the object immediately after the blow?
- An object rotates around a fixed axis. A given torque is applied. What is the frequency of the rotation? (Or conversely, given the frequency, what is the required torque?)

Let's work through an example for each of these problems. In both cases, the solution involves a few standard steps, so we'll write them out explicitly.

### 8.4.1 Motion after an impulsive blow

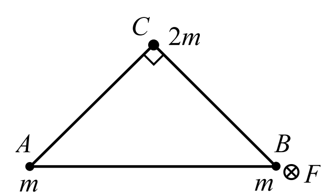


**Figure 8.16**

**Problem:** Consider the rigid object in Fig. 8.16. Three masses are connected by three massless rods, in the shape of an isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $2m$, and the other two masses are $m$. Label them $A$, $B$, $C$, as shown. Assume that the object is floating freely in space. (Alternatively, let the object hang from a long thread attached to mass $C$.)

Mass $B$ is struck with a quick blow, directed into the page. Let the imparted impulse have magnitude $\int F\,dt = P$. (See Section 7.6 for a discussion of impulse and angular impulse.) What are the velocities of the three masses immediately after the blow?

**Solution:** The strategy of the solution will be to find the angular momentum of the system (relative to the CM) using the angular impulse, then calculate the principal moments and find the angular velocity vector (which will give the velocities relative to the CM), and then add on the CM motion.

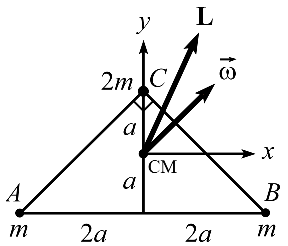


**Figure 8.17**

The altitude from the right angle to the hypotenuse has length $2a$, and the CM is easily seen to be located at its midpoint (see Fig. 8.17). Picking the CM as our origin, and letting the plane of the paper be the $x$-$y$ plane, the positions of the three masses are $\mathbf{r}_A = (-2a, -a, 0)$, $\mathbf{r}_B = (2a, -a, 0)$, and $\mathbf{r}_C = (0, a, 0)$. There are now five standard steps that we must perform.

- **Find L:** The positive $z$-axis is directed out of the page, so the impulse vector is $\mathbf{P} \equiv \int \mathbf{F}\,dt = (0, 0, -P)$. Therefore, the angular momentum of the system (relative to the CM) is

$$\begin{aligned}\mathbf{L} = \int \boldsymbol{\tau}\,dt &= \int (\mathbf{r}_B \times \mathbf{F})\,dt = \mathbf{r}_B \times \int \mathbf{F}\,dt \\ &= (2a, -a, 0) \times (0, 0, -P) = aP(1, 2, 0), \end{aligned} \tag{8.25}$$

as shown in Fig. 8.17. We have used the fact that $\mathbf{r}_B$ is essentially constant during the blow (because the blow is assumed to happen very quickly) in taking $\mathbf{r}_B$ outside the integral in the above equation.

- **Calculate the principal moments:** The principal axes are clearly the $x$, $y$, and $z$ axes. The moments (relative to the CM) are

$$\begin{aligned} I_x &= ma^2 + ma^2 + (2m)a^2 = 4ma^2, \\ I_y &= m(2a)^2 + m(2a)^2 + (2m)0^2 = 8ma^2, \\ I_z &= I_x + I_y = 12ma^2. \end{aligned} \tag{8.26}$$

We have used the perpendicular-axis theorem, eq. (7.17), to obtain $I_z$. But $I_z$ will not be needed to solve the problem.

- **Find $\boldsymbol{\omega}$:** We now have two expressions for the angular momentum of the system. One expression is in terms of the given impulse, eq. (8.25). The other is in terms of the moments and the angular velocity components, eq. (8.21). Therefore,

$$\begin{aligned} (I_x\omega_x, I_y\omega_y, I_z\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (4ma^2\omega_x, 8ma^2\omega_y, 12ma^2\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (\omega_x, \omega_y, \omega_z) &= \frac{P}{4ma}(1, 1, 0), \end{aligned} \tag{8.27}$$

as shown in Fig. 8.17.

- **Calculate speeds relative to CM:** Right after the blow, the object rotates around the CM with the angular velocity found above. The speeds relative to the CM are therefore $\mathbf{u}_i = \boldsymbol{\omega} \times \mathbf{r}_i$. That is,

$$\begin{aligned} \mathbf{u}_A &= \boldsymbol{\omega} \times \mathbf{r}_A = \frac{P}{4ma}(1,1,0) \times (-2a,-a,0) = (0,0,P/4m), \\ \mathbf{u}_B &= \boldsymbol{\omega} \times \mathbf{r}_B = \frac{P}{4ma}(1,1,0) \times (2a,-a,0) = (0,0,-3P/4m), \\ \mathbf{u}_C &= \boldsymbol{\omega} \times \mathbf{r}_C = \frac{P}{4ma}(1,1,0) \times (0,a,0) = (0,0,P/4m). \end{aligned} \tag{8.28}$$

- **Add on speed of CM:** The impulse (that is, the change in linear momentum) supplied to the whole system is $\mathbf{P} = (0,0,-P)$. The total mass of the system is $M = 4m$. Therefore, the velocity of the CM is

$$V_{\text{CM}} = \frac{\mathbf{P}}{M} = (0,0,-P/4m). \tag{8.29}$$

The total velocities of the masses are therefore

$$\begin{aligned} \mathbf{v}_A &= \mathbf{u}_A + V_{\text{CM}} = (0,0,0), \\ \mathbf{v}_B &= \mathbf{u}_B + V_{\text{CM}} = (0,0,-P/m), \\ \mathbf{v}_C &= \mathbf{u}_C + V_{\text{CM}} = (0,0,0). \end{aligned} \tag{8.30}$$

Remarks:

1. We see that masses $A$ and $C$ are instantaneously at rest immediately after the blow, and mass $B$ acquires all of the imparted impulse. In retrospect, this is quite clear. Basically, it is possible for both $A$ and $C$ to remain at rest while $B$ moves a tiny bit, so this is what happens. (If $B$ moves into the page by a small distance $\epsilon$, then $A$ and $C$ won't know that $B$ has moved, since their distances to $B$ will change only by a distance of order $\epsilon^2$.) If we changed the problem and added a mass $D$ at, say, the midpoint of the hypotenuse, then this would not be the case; it would not be possible for $A$, $C$, and $D$ to remain at rest while $B$ moved a tiny bit. So there would be some other motion, in addition to $B$'s.
2. As time goes on, the system will undergo a rather complicated motion. What will happen is that the CM will move with constant velocity, and the masses will rotate around it in a messy (but understandable) manner. Since there are no torques acting on the system (after the initial blow), we know that $\mathbf{L}$ will forever remain constant. It turns out that $\boldsymbol{\omega}$ will move around $\mathbf{L}$, and the body will rotate around this changing $\boldsymbol{\omega}$. These matters are the subject of Section 8.6. (Although in that discussion, we restrict ourselves to symmetric tops; that is, ones with two equal moments.) But these issues aside, it's good to know that we can, without too much difficulty, determine what's going on immediately after the blow.
3. The body in the above problem was assumed to be floating freely in space. If we instead have an object that is pivoted at a given (fixed) point, then we simply want to use the pivot as our origin, and there is no need to perform the last step of adding on the velocity of the origin (which was the CM, above), since this velocity is now zero. Equivalently, just consider the pivot to be an infinite mass, which is therefore the location of the (motionless) CM. ♣

### 8.4.2 Frequency of motion due to a torque

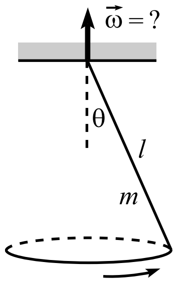



**Figure 8.18**

**Problem:** Consider a stick of length $\ell$, mass $m$, and uniform mass density. The stick is pivoted at its top end and swings around the vertical axis. Assume conditions have been set up so that the stick always makes an angle $\theta$ with the vertical, as shown in Fig. 8.18. What is the frequency, $\omega$, of this motion?

**Solution:** The strategy of the solution will be to find the principal moments and then the angular momentum of the system (in terms of $\omega$), then find the rate of change of **L**, and then calculate the torque and equate it with $d\mathbf{L}/dt$. We will choose the pivot to be the origin.[7] Again, there are five standard steps that we must perform.

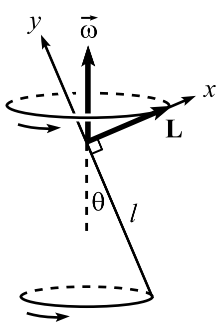



**Figure 8.19**

- **Calculate the principal moments:** The principal axes are clearly the axis along the stick, along with any two orthogonal axes perpendicular to the stick. So let the $x$- and $y$-axes be as shown in Fig. 8.19, and let the $z$-axis point out of the page. The moments (relative to the pivot) are $I_x = m\ell^2/3$, $I_y = 0$, and $I_z = m\ell^2/3$. ($I_z$ won't be needed in this solution.)

- **Find L:** The angular velocity vector points vertically,[8] so in the basis of the principal axes, the angular velocity vector is $\boldsymbol{\omega} = (\omega\sin\theta, \omega\cos\theta, 0)$, where $\omega$ is yet to be determined. The angular momentum of the system (relative to the pivot) is thus

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = (m\ell^2\omega\sin\theta/3, 0, 0). \tag{8.31}$$

- **Find $d\mathbf{L}/dt$:** The vector **L** therefore points upwards to the right, along the $x$-axis (at the instant shown in Fig. 8.19), with magnitude $L = m\ell^2\omega\sin\theta/3$. As the stick rotates around the vertical axis, **L** traces out the surface of a cone. That is, the tip of **L** traces out a horizontal circle. The radius of this circle is the horizontal component of **L**, which is $L\cos\theta$. The speed of the tip (that is, the magnitude of $d\mathbf{L}/dt$) is therefore $(L\cos\theta)\omega$, because **L** rotates around the vertical axis with the same frequency as the stick. So, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta, \tag{8.32}$$

and it points into the page.

REMARK: In more complicated problems (where $I_y \neq 0$), **L** will point in some messy direction (not along a principal axis), and the length of the horizontal component (that is, the radius of the circle **L** traces out) won't be immediately obvious. In this case, you can either explicitly calculate the horizontal component (see the Gyroscope example in Section 8.7.5), or you can simply do things the formal (and easier) way by

[7] This is a better choice than the CM, because this way we won't have to worry about any messy forces acting at the pivot, when computing the torque.

[8] However, see the third Remark, following this solution.

finding the rate of change of $\mathbf{L}$ via the expression $d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L}$ (which holds for all the same reasons that $\mathbf{v} \equiv d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$ holds). In the present problem, we obtain

$$d\mathbf{L}/dt = (\omega \sin\theta, \omega\cos\theta, 0) \times (m\ell^2\omega\sin\theta/3, 0, 0) = (0, 0, -m\ell^2\omega^2 \sin\theta\cos\theta/3), \tag{8.33}$$

which agrees with eq. (8.32). And the direction is correct, since the negative $z$-axis points into the page. Note that we calculated this cross-product in the principal-axis basis. Although these axes are changing in time, they present a perfectly good set of basis vectors at any instant. ♣

- **Calculate the torque:** The torque (relative to the pivot) is due to gravity, which effectively acts on the CM of the stick. So $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ has magnitude

$$\tau = rF\sin\theta = (\ell/2)(mg)\sin\theta, \tag{8.34}$$

and it points into the page.

- **Equate $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$:** The vectors $d\mathbf{L}/dt$ and $\boldsymbol{\tau}$ both point into the page (they had better point in the same direction). Equating their magnitudes gives

$$\frac{m\ell^2\omega^2\sin\theta\cos\theta}{3} = \frac{mg\ell\sin\theta}{2}$$
$$\Longrightarrow \qquad \omega = \sqrt{\frac{3g}{2\ell\cos\theta}}\,. \tag{8.35}$$

REMARKS:

1. This frequency is slightly larger than the frequency obtained if we instead have a mass at the end of a massless stick of length $\ell$. From Problem 12, the frequency in that case is $\sqrt{g/\ell\cos\theta}$. So, in some sense, a uniform stick of length $\ell$ behaves like a mass at the end of a massless stick of length $2\ell/3$, as far as these rotations are concerned.
2. As $\theta \to \pi/2$, the frequency $\omega$ goes to $\infty$, which makes sense. And as $\theta \to 0$, $\omega$ approaches $\sqrt{3g/2\ell}$, which isn't so obvious.
3. As explained in Problem 2, the instantaneous $\boldsymbol{\omega}$ is not uniquely defined in some situations. At the instant shown in Fig. 8.18, the stick is moving directly into the page. So let's say someone else wants to think of the stick as (instantaneously) rotating around the axis $\boldsymbol{\omega}'$ perpendicular to the stick (the $x$-axis, from above), instead of the vertical axis, as shown in Fig. 8.20. What is the angular speed $\omega'$?

   Well, if $\omega$ is the angular speed of the stick around the vertical axis, then we may view the tip of the stick as instantaneously moving in a circle of radius $\ell\sin\theta$ around the vertical axis $\boldsymbol{\omega}$. So $\omega(\ell\sin\theta)$ is the speed of the tip of the stick. But we may also view the tip of the stick as instantaneously moving in a circle of radius $\ell$ around $\boldsymbol{\omega}'$. The speed of the tip is still $\omega(\ell\sin\theta)$, so the angular speed about this axis is given by $\omega'\ell = \omega(\ell\sin\theta)$. Hence $\omega' = \omega\sin\theta$, which is simply the $x$-component of $\boldsymbol{\omega}$ that we found above, right before eq. (8.31). The moment of inertia around $\boldsymbol{\omega}'$ is $m\ell^2/3$, so the angular momentum has magnitude $(m\ell^2/3)(\omega\sin\theta)$, in agreement with eq. (8.31). And the direction is along the $x$-axis, as it should be.

   Note that although $\boldsymbol{\omega}$ is not uniquely defined at any instant, $\mathbf{L} \equiv \int(\mathbf{r}\times\mathbf{p})\,dm$ certainly is.[9] Choosing $\boldsymbol{\omega}$ to point vertically, as we did in the above solution, is in some sense the natural choice, because this $\boldsymbol{\omega}$ does not change with time. ♣

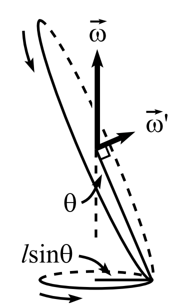


**Figure 8.20**

[9]The non-uniqueness of $\vec{\omega}$ arises from the fact that $I_y = 0$ here. If all the moments are nonzero, then $(L_x, L_y, L_z) = (I_x\omega_x, I_y\omega_y, I_z\omega_z)$ uniquely determines $\vec{\omega}$, for a given $\mathbf{L}$.

## 8.5 Euler's equations

Consider a rigid body instantaneously rotating around an axis $\boldsymbol{\omega}$. ($\boldsymbol{\omega}$ may change as time goes on, but all we care about for now is what it is at a given instant.) The angular momentum, $\mathbf{L}$, is given by eq. (8.8) as

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}, \tag{8.36}$$

where $\mathbf{I}$ is the inertial tensor, calculated with respect to a given set of axes (and $\boldsymbol{\omega}$ is written in the same basis, of course).

As usual, things are much nicer if we use the principal axes (relative to the chosen origin) as the basis vectors of our coordinate system. Since these axes are fixed with respect to the rotating object, they will of course rotate with respect to the fixed reference frame. In this basis, $\mathbf{L}$ takes the nice form,

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3), \tag{8.37}$$

where $\omega_1$, $\omega_2$, and $\omega_3$ are the components of $\boldsymbol{\omega}$ along the principal axes. In other words, if you take the vector $\mathbf{L}$ in space and project it onto the instantaneous principal axes, then you get these components.

On one hand, writing $\mathbf{L}$ in terms of the rotating principal axes allows us to write it in the nice form of (8.37). But on the other hand, writing $\mathbf{L}$ in this way makes it nontrivial to determine how it changes in time (since the principal axes themselves are changing). The benefits outweigh the detriments, however, so we will invariably use the principal axes as our basis vectors.

The goal of this section is to find an expression for $d\mathbf{L}/dt$, and to then equate this with the torque. The result will be Euler's equations, eqs. (8.43).

### Derivation of Euler's equations

If we write $\mathbf{L}$ in terms of the body frame, then we see that $\mathbf{L}$ can change (relative to the lab frame) due to two effects. $\mathbf{L}$ can change because its coordinates in the body frame may change, and $\mathbf{L}$ can also change because of the rotation of the body frame.

To be precise, let $\mathbf{L}_0$ be the vector $\mathbf{L}$ at a given instant. At this instant, imagine painting the vector $\mathbf{L}_0$ onto the body frame (so that $\mathbf{L}_0$ will then rotate with the body frame). The rate of change of $\mathbf{L}$ with respect to the lab frame may be written in the (identically true) way,

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{L} - \mathbf{L}_0)}{dt} + \frac{d\mathbf{L}_0}{dt}\,. \tag{8.38}$$

The second term here is simply the rate of change of a body-fixed vector, which we know is $\boldsymbol{\omega} \times \mathbf{L}_0$ (which equals $\boldsymbol{\omega} \times \mathbf{L}$ at this instant). The first term is the rate of change of $\mathbf{L}$ with respect to the body frame, which we will denote by $\delta\mathbf{L}/\delta t$. So we end up with

$$\frac{d\mathbf{L}}{dt} = \frac{\delta\mathbf{L}}{\delta t} + \boldsymbol{\omega} \times \mathbf{L}. \tag{8.39}$$

This is actually a general statement, true for any vector in any rotating frame.[10] There is nothing particular to $\mathbf{L}$ that we used in the above derivation. Also, there was no need to restrict ourselves to principal axes.

In words, what we've shown is that the total change equals the change relative to the rotating frame, plus the change of the rotating frame relative to the fixed frame. Simply addition of changes.

Let us now be specific and choose our body-axes to be the principal-axes. This will put eq. (8.39) in a very usable form. Using eq. (8.37), we have

$$\frac{d\mathbf{L}}{dt} = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.40}$$

This equation equates two vectors. As is true for any vector, these (equal) vectors have an existence that is independent of what coordinate system we choose to describe them with (eq. (8.39) makes no reference to a coordinate system). But since we've chosen an explicit frame on the right-hand side of eq. (8.40), we should choose the same frame for the left-hand side; we can then equate the components on the left with the components on the right. Projecting $d\mathbf{L}/dt$ onto the instantaneous principal axes, we have

$$\left(\left(\frac{d\mathbf{L}}{dt}\right)_1, \left(\frac{d\mathbf{L}}{dt}\right)_2, \left(\frac{d\mathbf{L}}{dt}\right)_3\right) = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.41}$$

REMARK: The left-hand side looks nastier than it really is. At the risk of belaboring the point, consider the following (this is a remark that has to be read very slowly): We could have written the left-hand side as $(d/dt)(L_1, L_2, L_3)$, but this might cause confusion as to whether the $L_i$ refer to the components with respect to the rotating axes, or the components with respect to the fixed set of axes that coincide with the rotating principal axes at this instant. That is, do we project $\mathbf{L}$ onto the principal axes, and then take the derivative; or do we take the derivative and then project? The latter is what we mean in eq. (8.41). (The former is $\delta\mathbf{L}/\delta t$, by definition.) The way we've written the left-hand side of eq. (8.41), it's clear that we're taking the derivative first. We are, after all, simply projecting eq. (8.39) onto the principal axes. ♣

The time derivatives on the right-hand side of eq. (8.41) are $\delta(I_1\omega_1)/\delta t = I_1\dot{\omega}_1$ (because $I_1$ is constant), etc. Performing the cross product and equating the corresponding components on each side yields the three equations,

$$\begin{aligned}
\left(\frac{d\mathbf{L}}{dt}\right)_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2, \\
\left(\frac{d\mathbf{L}}{dt}\right)_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3, \\
\left(\frac{d\mathbf{L}}{dt}\right)_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1.
\end{aligned} \tag{8.42}$$

[10] We will prove eq. (8.39) in another more mathematical way in Chapter 9.

If we have chosen the origin of our rotating frame to be either a fixed point or the CM (which we will always do), then the results of Section 7.4 tell us that we may equate $d\mathbf{L}/dt$ with the torque, $\boldsymbol{\tau}$. We therefore have

$$\begin{aligned}\tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2,\\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3,\\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1.\end{aligned} \tag{8.43}$$

These are *Euler's equations*. You need only remember one of them, because the other two can be obtained by cyclic permutation of the indices.

REMARKS:

1. We repeat that the left- and right-hand sides of eq. (8.43) are components that are measured with respect to the instantaneous principal axes. Let's say we do a problem, for example, where at all times $\tau_1 = \tau_2 = 0$, and $\tau_3$ equals some nonzero number. This doesn't mean, of course, that $\boldsymbol{\tau}$ is a constant vector. On the contrary, $\boldsymbol{\tau}$ always points along the $\hat{\mathbf{x}}_3$ vector in the rotating frame, but this vector is changing in the fixed frame (unless $\hat{\mathbf{x}}_3$ points along $\boldsymbol{\omega}$).

   The two types of terms on the right-hand sides of eqs. (8.42) are the two types of changes that $\mathbf{L}$ can undergo. $\mathbf{L}$ can change because its components with respect to the rotating frame change, and $\mathbf{L}$ can also change because the body is rotating around $\boldsymbol{\omega}$.

2. Section 8.6.1 on the free symmetric top (viewed from the body frame) provides a good example of the use of Euler's equations. Another interesting application is the famed "tennis racket theorem" (Problem 14).

3. It should be noted that you never *have* to use Euler's equations. You can simply start from scratch and use eq. (8.39) each time you solve a problem. The point is that we've done the calculation of $d\mathbf{L}/dt$ once and for all, so you can just invoke the result in eqs. (8.43). ♣

## 8.6 Free symmetric top

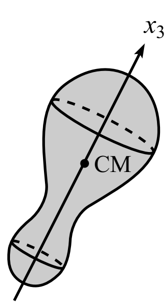


**Figure 8.21**

The free symmetric top is the classic example of an application of the Euler equations. Consider an object which has two of its principal moments equal (with the CM as the origin). Let the object be in outer space, far from any external forces.[11] We will choose our object to have cylindrical symmetry around some axis (see Fig. 8.21), although this is not necessary (a square cross-section, for example, would yield two equal moments). The principal axes are then the symmetry axis and any two orthogonal axes in the cross-section plane through the CM. Let the symmetry axis be chosen as the $\hat{\mathbf{x}}_3$ axis. Then our moments are $I_1 = I_2 \equiv I$, and $I_3$.

### 8.6.1 View from body frame

Plugging $I_1 = I_2 \equiv I$ into Euler's equations, eqs. (8.43), with the $\tau_i$ equal to zero (since there are no torques, because the top is "free"), gives

$$0 = I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2,$$

[11]Equivalently, the object is thrown up in the air, and we are traveling along on the CM.

$$0 = I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3,$$
$$0 = I_3\dot{\omega}_3. \tag{8.44}$$

The last equation says that $\omega_3$ is a constant. If we then define

$$\Omega \equiv \left(\frac{I_3 - I}{I}\right)\omega_3, \tag{8.45}$$

the first two equations become

$$\dot{\omega}_1 + \Omega\omega_2 = 0, \qquad \text{and} \qquad \dot{\omega}_2 - \Omega\omega_1 = 0. \tag{8.46}$$

Taking the derivative of the first of these, and then using the second one to eliminate $\dot{\omega}_2$, gives

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0, \tag{8.47}$$

and likewise for $\omega_2$. This is a nice simple-harmonic equation. The solutions for $\omega_1(t)$ and (by using eq. (8.46)) $\omega_2(t)$ are

$$\omega_1(t) = A\cos(\Omega t + \phi), \qquad \omega_2(t) = A\sin(\Omega t + \phi). \tag{8.48}$$

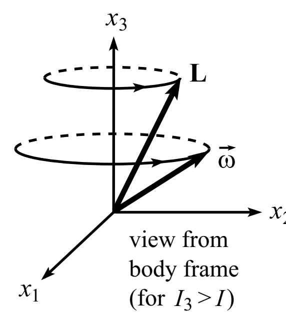


**Figure 8.22**

Therefore, $\omega_1(t)$ and $\omega_2(t)$ are the components of a circle in the body frame. Hence, the $\boldsymbol{\omega}$ vector traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency $\Omega$, as viewed by someone standing on the body. The angular momentum is

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \Big(IA\cos(\Omega t + \phi), IA\sin(\Omega t + \phi), I_3\omega_3\Big), \tag{8.49}$$

so $\mathbf{L}$ also traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency $\Omega$, as viewed by someone standing on the body.

The frequency, $\Omega$, in eq. (8.45) depends on the value of $\omega_3$ and on the geometry of the object. But the amplitude, $A$, of the $\boldsymbol{\omega}$ cone is determined by the initial values of $\omega_1$ and $\omega_2$.

Note that $\Omega$ may be negative (if $I > I_3$). In this case, $\boldsymbol{\omega}$ traces out its cone in the opposite direction compared to the $\Omega > 0$ case.

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**Example (The earth):** Let's consider the earth to be our object. Then $\omega_3 \approx 2\pi/(1 \text{ day})$.[12] The bulge at the equator (caused by the spinning of the earth) makes $I_3$ slightly larger than $I$, and it turns out that $(I_3 - I)/I \approx 1/300$. Therefore, eq. (8.45) gives $\Omega \approx (1/300)\, 2\pi/(1 \text{ day})$. So the $\boldsymbol{\omega}$ vector should precess around its cone once every 300 days, as viewed by someone on the earth. The true value is more like 400 days. The difference has to do with various things, including the non-rigidity of the earth. But at least we got an answer in the right ballpark.

How do you determine the direction of $\boldsymbol{\omega}$? Simply make an extended-time photograph exposure at night. The stars will form arcs of circles. At the center of all these circles is a point that doesn't move. This is the direction of $\boldsymbol{\omega}$.

[12]This isn't quite correct, since the earth rotates 366 times for every 365 days (due to the motion around the sun), but it's close enough for the purposes here.

How big is the $\boldsymbol{\omega}$ cone, for the earth? Equivalently, what is the value of $A$ in eq. (8.48)? Observation has shown that $\boldsymbol{\omega}$ pierces the earth at a point on the order of 10 m from the north pole. Hence, $A/\omega_3 \approx (10\,\text{m})/R_E$. The half-angle of the $\boldsymbol{\omega}$ cone is therefore found to be only on the order of $10^{-4}$ degrees. So if you use an extended-time photograph exposure one night to see which point in the sky stands still, and then if you do the same thing 200 nights later, you probably won't be able to tell that they're really two different points.

### 8.6.2 View from fixed frame

Now let's see what our symmetric top looks like from a fixed frame. In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned} \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.50}$$

Eliminating the $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2)$ term from these equations gives (in terms of the $\Omega$ defined in eq. (8.45))

$$\mathbf{L} = I(\boldsymbol{\omega} + \Omega\hat{\mathbf{x}}_3), \qquad \text{or} \qquad \boldsymbol{\omega} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3, \tag{8.51}$$

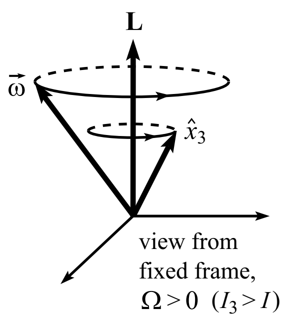


**Figure 8.23**

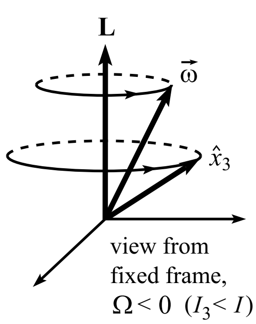


**Figure 8.24**

where $L = |\mathbf{L}|$, and $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction. The linear relationship between $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$, implies that these three vectors lie in a plane. Since there are no torques on the system, $\mathbf{L}$ remains constant. Therefore, $\boldsymbol{\omega}$ and $\hat{\mathbf{x}}_3$ precess (as we will see below) around $\mathbf{L}$, with the three vectors always coplanar. See Fig. 8.23 for the case $I_3 > I$ (an *oblate* top, such as a coin), and Fig. 8.24 for the case $I_3 < I$ (a *prolate* top, such as a carrot).

What is the frequency of this precession, as viewed from the fixed frame? The rate of change of $\hat{\mathbf{x}}_3$ is $\boldsymbol{\omega} \times \hat{\mathbf{x}}_3$ (because $\hat{\mathbf{x}}_3$ is fixed in the body frame, so its change comes only from rotation around $\boldsymbol{\omega}$). Therefore, eq. (8.51) gives

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \left(\frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3\right) \times \hat{\mathbf{x}}_3 = \left(\frac{L}{I}\hat{\mathbf{L}}\right) \times \hat{\mathbf{x}}_3. \tag{8.52}$$

But this is simply the expression for the rate of change of a vector rotating around the fixed vector $\tilde{\boldsymbol{\omega}} \equiv (L/I)\hat{\mathbf{L}}$. The frequency of this rotation is $|\tilde{\boldsymbol{\omega}}| = L/I$. Therefore, $\hat{\mathbf{x}}_3$ precesses around the fixed vector $\mathbf{L}$ with frequency

$$\tilde{\omega} = \frac{L}{I}\,, \tag{8.53}$$

in the fixed frame (and therefore $\boldsymbol{\omega}$ does also, since it is coplanar with $\hat{\mathbf{x}}_3$ and $\mathbf{L}$).

Remarks:

1. We just said that $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with frequency $L/I$. What, then, is wrong with the following reasoning: "Just as the rate of change of $\hat{\mathbf{x}}_3$ equals $\boldsymbol{\omega}\times\hat{\mathbf{x}}_3$, the rate of change of $\boldsymbol{\omega}$ should equal $\boldsymbol{\omega}\times\boldsymbol{\omega}$, which is zero. Hence, $\boldsymbol{\omega}$ should remain constant." The error is that the vector $\boldsymbol{\omega}$ is not fixed in the body frame. A vector $\mathbf{A}$ must be fixed in the body frame in order for its rate of change to be given by $\boldsymbol{\omega}\times\mathbf{A}$.

2. We found in eqs. (8.49) and (8.45) that a person standing on the rotating body sees $\mathbf{L}$ (and $\boldsymbol{\omega}$) precess with frequency $\Omega \equiv \omega_3(I_3 - I)/I$ around $\hat{\mathbf{x}}_3$. But we found in eq. (8.53) that a person standing in the fixed frame sees $\hat{\mathbf{x}}_3$ (and $\boldsymbol{\omega}$) precess with frequency $L/I$ around $\mathbf{L}$. Are these two facts compatible? Should we have obtained the same frequency from either point of view? (Answers: yes, no).

   These two frequencies are indeed consistent, as can be seen from the following reasoning. Consider the plane (call it $S$) containing the three vectors $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$. We know from eq. (8.49) that $S$ rotates with frequency $\Omega\hat{\mathbf{x}}_3$ with respect to the body. Therefore, the body rotates with frequency $-\Omega\hat{\mathbf{x}}_3$ with respect to $S$. And from eq. (8.53), $S$ rotates with frequency $(L/I)\hat{\mathbf{L}}$ with respect to the fixed frame. Therefore, the total angular velocity of the body with respect to the fixed frame (using the frame $S$ as an intermediate step) is

$$\boldsymbol{\omega}_{\text{total}} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3. \tag{8.54}$$

   But from eq. (8.51), this is simply $\boldsymbol{\omega}$, as it should be. So the two frequencies in eqs. (8.45) and (8.53) are indeed consistent.

   For the earth, $\Omega \equiv \omega_3(I_3 - I)/I$ and $L/I$ are much different. $L/I$ is roughly equal to $L/I_3$, which is essentially equal to $\omega_3$. $\Omega$, on the other hand is about $(1/300)\omega_3$. Basically, an external observer sees $\boldsymbol{\omega}$ precess around its cone at roughly the rate at which the earth spins. But it's not exactly the same rate, and this difference is what causes the earth-based observer to see $\boldsymbol{\omega}$ precess with a nonzero $\Omega$. ♣

## 8.7 Heavy symmetric top

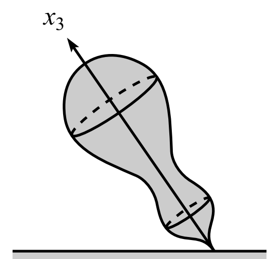


**Figure 8.25**

Consider now a heavy symmetrical top; that is, one that spins on a table, under the influence of gravity (see Fig. 8.25). Assume that the tip of the top is fixed on the table by a free pivot. We will solve for the motion of the top in two different ways. The first will use $\boldsymbol{\tau} = d\mathbf{L}/dt$. The second will use the Lagrangian method.

### 8.7.1 Euler angles

For both of these methods, it is very convenient to use the *Euler angles*, $\theta$,$\phi$,$\psi$, which are shown in Fig. 8.26 and are defined as follows.

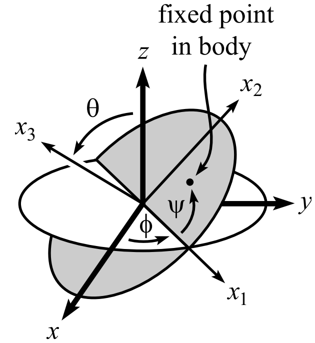


**Figure 8.26**

- $\theta$: Let $\hat{\mathbf{x}}_3$ be the symmetry axis of the top. Define $\theta$ to be the angle that $\hat{\mathbf{x}}_3$ makes with the vertical axis $\hat{\mathbf{z}}$ of the fixed frame.

- $\phi$: Draw the plane orthogonal to $\hat{\mathbf{x}}_3$. Let $\hat{\mathbf{x}}_1$ be the intersection of this plane with the horizontal $x$-$y$ plane. Define $\phi$ to be the angle $\hat{\mathbf{x}}_1$ makes with the $\hat{\mathbf{x}}$ axis of the fixed frame.

- $\psi$**:** Let $\hat{\mathbf{x}}_2$ be orthogonal to $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_1$, as shown. Let frame $S$ be the frame whose axes are $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$. Define $\psi$ to be the angle of rotation of the body around the $\hat{\mathbf{x}}_3$ axis in frame $S$. (That is, $\dot{\psi}\hat{\mathbf{x}}_3$ is the angular velocity of the body with respect to $S$.) Note that the angular velocity of frame $S$ with respect to the fixed frame is $\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1$.

The angular velocity of the body with respect to the fixed frame is equal to the angular velocity of the body with respect to frame $S$, plus the angular velocity of frame $S$ with respect to the fixed frame. In other words, it is

$$\boldsymbol{\omega} = \dot{\psi}\hat{\mathbf{x}}_3 + (\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1). \tag{8.55}$$

Note that the vector $\hat{\mathbf{z}}$ is not orthogonal to $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$. It is often more convenient to rewrite $\boldsymbol{\omega}$ entirely in terms of the orthogonal $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$ basis vectors. Since $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x}}_3 + \sin\theta\hat{\mathbf{x}}_2$, eq. (8.55) gives

$$\boldsymbol{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{x}}_3 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_2 + \dot{\theta}\hat{\mathbf{x}}_1. \tag{8.56}$$

This form of $\boldsymbol{\omega}$ is often more useful, because $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are principal axes of the body. (We are assuming that we are working with a symmetrical top, with $I_1 = I_2 \equiv I$. Hence, any axes in the $\hat{\mathbf{x}}_1$-$\hat{\mathbf{x}}_2$ plane are principal axes.) Although $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are not fixed in the object, they are still good principal axes at any instant.

### 8.7.2 Digression on the components of $\vec{\omega}$

The previous expressions for $\boldsymbol{\omega}$ look rather formidable, but there is a very helpful diagram we can draw (see Fig. 8.27) which makes it easier to see what is going on. Let's talk a bit about this before returning to the original problem of the spinning top. The diagram is rather pithy, so we'll go through it nice and slowly.

In the following discussion, we will simplify things by setting $\dot{\theta} = 0$. All the interesting features of $\boldsymbol{\omega}$ remain. The $\dot{\theta}\hat{\mathbf{x}}_1$ component of $\boldsymbol{\omega}$ in eqs. (8.55) and (8.56) simply arises from the easily-visualizable rising and falling of the top. We will therefore concentrate here on the more complicated issues, namely the components of $\boldsymbol{\omega}$ in the plane of $\hat{\mathbf{x}}_3$, $\hat{\mathbf{z}}$, and $\hat{\mathbf{x}}_2$.

With $\dot{\theta} = 0$, Fig. 8.27 shows the vector $\boldsymbol{\omega}$ in the $\hat{\mathbf{x}}_3$-$\hat{\mathbf{z}}$-$\hat{\mathbf{x}}_2$ plane (the way we've drawn it, $\hat{\mathbf{x}}_1$ points into the page, in contrast with Fig. 8.26). This is an extremely useful diagram, and we will refer to it many times in the problems for this chapter. There are numerous comments to be made on it, so let's just list them out.

1. If someone asks you to "decompose" $\boldsymbol{\omega}$ into pieces along $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}_3$, what would you do? Would you draw the lines perpendicular to these axes to obtain the lengths shown (which we will label as $\omega_z$ and $\omega_3$), or would you draw the lines parallel to these axes to obtain the lengths shown (which we will label as $\Omega$ and $\omega'$)? There is no "correct" answer to this question. The four quantities, $\omega_z$, $\omega_3$, $\Omega$, $\omega'$ simply represent different things. We will interpret each of these below, along with $\omega_2$ (the projection of $\boldsymbol{\omega}$ along $\hat{\mathbf{x}}_2$). It turns out that $\Omega$ and $\omega'$ are the frequencies that your eye can see the easiest, while $\omega_2$ and $\omega_3$ are what you want to use when you're doing calculations involving the angular momentum. (And as far as I can see, $\omega_z$ is not of much use.)

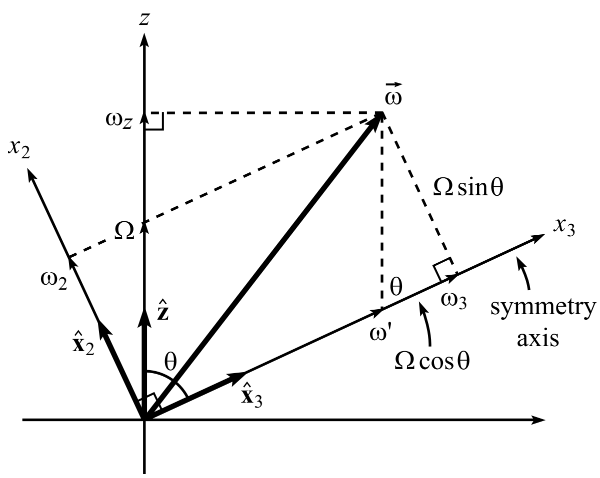


**Figure 8.27**

2. Note that it is true that
$$\boldsymbol{\omega} = \omega' \hat{\mathbf{x}}_3 + \Omega \hat{\mathbf{z}}, \tag{8.57}$$
but it is *not* true that $\boldsymbol{\omega} = \omega_z \hat{\mathbf{z}} + \omega_3 \hat{\mathbf{x}}_3$. Another true statement is
$$\boldsymbol{\omega} = \omega_3 \hat{\mathbf{x}}_3 + \omega_2 \hat{\mathbf{x}}_2. \tag{8.58}$$

3. In terms of the Euler angles, we see (by comparing eq. (8.57) with eq. (8.55), with $\dot{\theta} = 0$) that
$$\begin{aligned} \omega' &= \dot{\psi}, \\ \Omega &= \dot{\phi}. \end{aligned} \tag{8.59}$$
And we also have (by comparing eq. (8.58) with eq. (8.56), with $\dot{\theta} = 0$)
$$\begin{aligned} \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta = \omega' + \Omega\cos\theta, \\ \omega_2 &= \dot{\phi}\sin\theta = \Omega\sin\theta. \end{aligned} \tag{8.60}$$
These are also clear from Fig. 8.27.

   There is therefore technically no need to introduce the new $\omega_2$, $\omega_3$, $\Omega$, $\omega'$ definitions in Fig. 8.27, since the Euler angles are quite sufficient. But we will be referring to this figure many times, and it is a little easier to refer to these omega's than to the various combinations of Euler angles.

4. $\Omega$ is the easiest of these frequencies to visualize. It is simply the frequency of precession of the top around the vertical $\hat{\mathbf{z}}$ axis.[13] In other words, the

[13] Although we're using the same letter, this $\Omega$ doesn't have anything to do with the $\Omega$ defined in eq. (8.45), except for the fact that they both represent a precession frequency.

symmetry axis $\hat{\mathbf{x}}_3$ traces out a cone around the $\hat{\mathbf{z}}$ axis with frequency $\Omega$. (Note that this precession frequency is *not* $\omega_z$.) Let's prove this.

The vector $\boldsymbol{\omega}$ is the vector which gives the speed of any point (at position $\mathbf{r}$) fixed in the top as $\boldsymbol{\omega} \times \mathbf{r}$. Therefore, since the vector $\hat{\mathbf{x}}_3$ is fixed in the top, we may write

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{x}}_3 = (\omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3 = (\Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3. \tag{8.61}$$

But this is precisely the expression for the rate of change of a vector rotating around the $\hat{\mathbf{z}}$ axis, with frequency $\Omega$. (This was exactly the same type of proof as the one leading to eq. (8.52).)

Remark: In the derivation of eq. (8.61), we've basically just stripped off a certain part of $\boldsymbol{\omega}$ that points along the $\hat{\mathbf{x}}_3$ axis, because a rotation around $\hat{\mathbf{x}}_3$ contributes nothing to the motion of $\hat{\mathbf{x}}_3$. Note, however, that there is in fact an infinite number of ways to strip off a piece along $\hat{\mathbf{x}}_3$. For example, we can also break $\boldsymbol{\omega}$ up as, say, $\boldsymbol{\omega} = \omega_3\hat{\mathbf{x}}_3 + \omega_2\hat{\mathbf{x}}_2$. We then obtain $d\hat{\mathbf{x}}_3/dt = (\omega_2\hat{\mathbf{x}}_2) \times \hat{\mathbf{x}}_3$, which means that $\hat{\mathbf{x}}_3$ is instantaneously rotating around $\hat{\mathbf{x}}_2$ with frequency $\omega_2$. Although this is true, it is not as useful as the result in eq. (8.61), because the $\hat{\mathbf{x}}_2$ axis changes with time. The point here is that the instantaneous angular velocity vector around which the symmetry axis rotates is not well-defined (Problem 2 discusses this issue).[14] But the $\hat{\mathbf{z}}$-axis is the only one of these angular velocity vectors that is fixed. When we look at the top, we therefore see it precessing around the $\hat{\mathbf{z}}$-axis. ♣

5. $\omega'$ is also easy to visualize. Imagine that you are at rest in a frame that rotates around the $\hat{\mathbf{z}}$-axis with frequency $\Omega$. Then you will see the symmetry axis of the top remain perfectly still, and the only motion you will see is the top spinning around this axis with frequency $\omega'$. (This is true because $\boldsymbol{\omega} = \omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}$, and the rotation of your frame causes you to not see the $\Omega\hat{\mathbf{z}}$ part.) If you paint a dot somewhere on the top, then the dot will trace out a fixed tilted circle, and the dot will return to, say, its maximum height at frequency $\omega'$.

   Note that someone in the lab frame will see the dot undergo a rather complicated motion, but she must observe the same frequency at which the dot returns to its highest point. Hence, $\omega'$ is something quite physical in the lab frame, also.

6. $\omega_3$ is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_3$, because $L_3 = I_3\omega_3$. It is not quite as easy to visualize as $\Omega$ and $\omega'$, but it is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_2$ axis with frequency $\omega_2$. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_2\hat{\mathbf{x}}_2$ part.) This rotation is a little harder to see, because the $\hat{\mathbf{x}}_2$ axis changes with time.

[14] The instantaneous angular velocity of the *whole body* is well defined, of course. But if you just look at the symmetry axis by itself, then there is an ambiguity (see footnote 9).

There is one physical scenario in which $\omega_3$ is the easily observed frequency. Imagine that the top is precessing around the $\hat{\mathbf{z}}$ axis at constant $\theta$, and imagine that the top has a frictionless rod protruding along its symmetry axis. If you grab the rod and stop the precession motion (so that the top is now spinning around its stationary symmetry axis), then this spinning will occur at frequency $\omega_3$. This is true because when you grab the rod, you apply a torque in only the (negative) $\hat{\mathbf{x}}_2$ direction. Therefore, you don't change $L_3$, and hence you don't change $\omega_3$.

7. $\omega_2$ is similar to $\omega_3$, of course. $\omega_2$ is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_2$, because $L_2 = I_2\omega_2$. It is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_3$ axis with frequency $\omega_3$. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_3\hat{\mathbf{x}}_3$ part.) Again, this rotation is a little harder to see, because the $\hat{\mathbf{x}}_3$ axis changes with time.

8. $\omega_z$ is not very useful (as far as I can see). The most important thing to note about it is that it is *not* the frequency of precession around the $\hat{\mathbf{z}}$-axis, even though it is the projection of $\boldsymbol{\omega}$ onto $\hat{\mathbf{z}}$. The frequency of the precession is $\Omega$, as we found above in eq. (8.61). A true, but somewhat useless, fact about $\omega_z$ is that if someone is at rest in a frame that rotates around the $\hat{\mathbf{z}}$ axis with frequency $\omega_z$, then she will see all points in the top instantaneously rotating around the $\hat{\mathbf{x}}$-axis with frequency $\omega_x$, where $\omega_x$ is the projection of $\boldsymbol{\omega}$ onto the horizontal $\hat{\mathbf{x}}$ axis. (This is true because $\boldsymbol{\omega} = \omega_x\hat{\mathbf{x}} + \omega_z\hat{\mathbf{z}}$, and the rotation of the frame causes you to not see the $\omega_z\hat{\mathbf{z}}$ part.)

### 8.7.3 Torque method

This method of solving the heavy top will be straightforward, although a little tedious. We include it here to (1) show that this problem can be done without resorting to Lagrangians, and to (2) get some practice using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

We will make use of the form of $\boldsymbol{\omega}$ given in eq. (8.56), because there it is broken up into the principal-axis components. For convenience, define $\dot\beta = \dot\psi + \dot\phi\cos\theta$, so that

$$\boldsymbol{\omega} = \dot\beta\hat{\mathbf{x}}_3 + \dot\phi\sin\theta\hat{\mathbf{x}}_2 + \dot\theta\hat{\mathbf{x}}_1. \tag{8.62}$$

Note that we've returned to the most general motion, where $\dot\theta$ is not necessarily zero.

We will choose the tip of the top as our origin, which is assumed to be fixed on the table.[15] Let the principal moments relative to this origin be $I_1 = I_2 \equiv I$, and $I_3$. The angular momentum of the body is then

$$\mathbf{L} = I_3\dot\beta\hat{\mathbf{x}}_3 + I\dot\phi\sin\theta\hat{\mathbf{x}}_2 + I\dot\theta\hat{\mathbf{x}}_1. \tag{8.63}$$

We must now calculate $d\mathbf{L}/dt$. What makes this nontrivial is the fact that the $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ unit vectors change with time (they change with $\theta$ and $\phi$). But let's

[15] We could use the CM as our origin, but then we would have to include the complicated forces acting at the pivot point, which is difficult.

forge ahead and take the derivative of eq. (8.63). Using the product rule (which works fine with the product of a scalar and a vector), we have

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\frac{d\dot{\beta}}{dt}\hat{\mathbf{x}}_3 + I\frac{d(\dot{\phi}\sin\theta)}{dt}\hat{\mathbf{x}}_2 + I\frac{d\dot{\theta}}{dt}\hat{\mathbf{x}}_1 \\ &\quad + I_3\dot{\beta}\frac{d\hat{\mathbf{x}}_3}{dt} + I\dot{\phi}\sin\theta\frac{d\hat{\mathbf{x}}_2}{dt} + I\dot{\theta}\frac{d\hat{\mathbf{x}}_1}{dt}\,. \qquad (8.64)\end{aligned}$$

Using a little geometry, you can show

$$\begin{aligned}\frac{d\hat{\mathbf{x}}_3}{dt} &= -\dot{\theta}\hat{\mathbf{x}}_2 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_2}{dt} &= \dot{\theta}\hat{\mathbf{x}}_3 - \dot{\phi}\cos\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_1}{dt} &= -\dot{\phi}\sin\theta\hat{\mathbf{x}}_3 + \dot{\phi}\cos\theta\hat{\mathbf{x}}_2. \qquad (8.65)\end{aligned}$$

As an exercise, prove these by making use of Fig. 8.26. In the first equation, for example, show that a change in $\theta$ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_2$ direction; and show that a change in $\phi$ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_1$ direction. Plugging eqs. (8.65) into eq. (8.64) gives

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\ddot{\beta}\hat{\mathbf{x}}_3 + \Big(I\ddot{\phi}\sin\theta + 2I\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\beta}\dot{\theta}\Big)\hat{\mathbf{x}}_2 \\ &\quad + \Big(I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\beta}\dot{\phi}\sin\theta\Big)\hat{\mathbf{x}}_1. \qquad (8.66)\end{aligned}$$

The torque on the top arises from gravity pulling down on the CM. $\boldsymbol{\tau}$ points in the $\hat{\mathbf{x}}_1$ direction and has magnitude $Mg\ell\sin\theta$, where $\ell$ is the distance from the pivot to CM. Equating $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$ gives

$$\ddot{\beta} = 0, \qquad (8.67)$$

for the $\hat{\mathbf{x}}_3$ component. Therefore, $\dot{\beta}$ is a constant, which we will call $\omega_3$ (an obvious label, in view of eq. (8.62)). The other two components of $\boldsymbol{\tau} = d\mathbf{L}/dt$ then give

$$\begin{aligned} I\ddot{\phi}\sin\theta + \dot{\theta}(2I\dot{\phi}\cos\theta - I_3\omega_3) &= 0, \\ (Mg\ell + I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi})\sin\theta &= I\ddot{\theta}. \qquad (8.68)\end{aligned}$$

We will wait to fiddle with these equations until we have derived them again using the Lagrangian method.

### 8.7.4 Lagrangian method

Eq. (8.13) gives the kinetic energy of the top as $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$. Eqs. (8.62) and (8.63) give (using $\dot{\psi} + \dot{\phi}\cos\theta$ instead of the shorthand $\dot{\beta}$)[16]

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{1}{2}I(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2). \qquad (8.69)$$

[16] It was ok to use $\beta$ in Subsection 8.7.3; we introduced it simply because it was quicker to write. But we can't use $\beta$ here, because it depends on the other coordinates, and the Lagrangian method requires the use of independent coordinates. (The variational proof back in Chapter 5 assumed this independence.)

The potential energy is

$$V = Mg\ell\cos\theta, \tag{8.70}$$

where $\ell$ is the distance from the pivot to CM. The Lagrangian is $\mathcal{L} = T - V$ (we'll use "$\mathcal{L}$" here to avoid confusion with the angular momentum, "$L$"), and so the equation of motion obtained from varying $\psi$ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi} = \frac{\partial\mathcal{L}}{\partial\psi} \Longrightarrow \frac{d}{dt}(\dot\psi + \dot\phi\cos\theta) = 0. \tag{8.71}$$

Therefore, $\dot\psi + \dot\phi\cos\theta$ is a constant. Call it $\omega_3$. The equations of motion obtained from varying $\phi$ and $\theta$ are then

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \quad &\Longrightarrow \quad \frac{d}{dt}\Big(I_3\omega_3\cos\theta + I\dot\phi\sin^2\theta\Big) = 0, \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} \quad &\Longrightarrow \quad I\ddot\theta = (Mg\ell + I\dot\phi^2\cos\theta - I_3\omega_3\dot\phi)\sin\theta.
\end{aligned} \tag{8.72}$$

These are equivalent to eqs. (8.68), as you can check. Note that there are two conserved quantities, arising from the facts that $\partial\mathcal{L}/\partial\psi$ and $\partial\mathcal{L}/\partial\phi$ equal zero. The conserved quantities are simply the angular momenta in the $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{z}}$ directions, respectively. (There is no torque in the plane spanned by these vectors, since the torque points in the $\hat{\mathbf{x}}_1$ direction.)

### 8.7.5 Gyroscope with $\dot\theta = 0$

A special case of eqs. (8.68) occurs when $\dot\theta = 0$. In this case, the first of eqs. (8.68) says that $\dot\phi$ is a constant. The CM of the top therefore undergoes uniform circular motion in a horizontal plane. Let $\Omega \equiv \dot\phi$ be the frequency of this motion (this is the same notation as in eq. (8.59)). Then the second of eqs. (8.68) says that

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mg\ell = 0. \tag{8.73}$$

This quadratic equation may be solved to yield two possible precessional frequencies for the top. (Yes, there are indeed two of them, provided that $\omega_3$ is greater than a certain minimum value.)

The previous pages in this "Heavy Symmetric Top" section have been a bit abstract. So let's now pause for a moment, take a breather, and rederive eq. (8.73) from scratch. That is, we'll assume $\dot\theta = 0$ from the start of the solution, and solve things by simply finding $\mathbf{L}$ and using $\boldsymbol{\tau} = d\mathbf{L}/dt$, in the spirit of Section 8.4.2.

The following Gyroscope example is the classic "top" problem. We'll warm up by solving it in an approximate way. Then we'll do it for real.

---

**Example (Gyroscope):** A symmetric top of mass $M$ has its CM a distance $\ell$ from its pivot. The moments of inertia relative to the pivot are $I_1 = I_2 \equiv I$ and $I_3$. The top spins around its symmetry axis with frequency $\omega_3$ (in the language of Section 8.7.2), and initial conditions have been set up so that the CM precesses in a circle around the vertical axis. The symmetry axis makes a constant angle $\theta$ with the vertical (see Fig. 8.28).

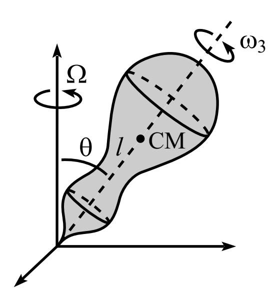

Figure 8.28

(a) Assuming that the angular momentum due to $\omega_3$ is much larger than any other angular momentum in the problem, find an approximate expression for the frequency, $\Omega$, of precession.

(b) Now do the problem exactly. That is, find $\Omega$ by considering all of the angular momentum.

**Solution:**

(a) The angular momentum (relative to the pivot) due to the spinning of the top has magnitude $L_3 = I_3\omega_3$, and it is directed along $\hat{\mathbf{x}}_3$. Let's label this angular momentum vector as $\mathbf{L}_3 \equiv L_3\hat{\mathbf{x}}_3$. As the top precesses, $\mathbf{L}_3$ traces out a cone around the vertical axis. So the tip of $\mathbf{L}_3$ moves in a circle of radius $L_3 \sin\theta$. The frequency of this circular motion is the frequency of precession, $\Omega$. So $d\mathbf{L}_3/dt$, which is the velocity of the tip, has magnitude

$$\Omega(L_3 \sin\theta) = \Omega I_3\omega_3 \sin\theta, \tag{8.74}$$

and is directed into the page.

The torque relative to the pivot point is due to gravity acting on the CM, so it has magnitude $Mg\ell \sin\theta$. It is directed into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = \frac{Mg\ell}{I_3\omega_3}\,. \tag{8.75}$$

Note that this is independent of $\theta$. And it is inversely proportional to $\omega_3$.

(b) The error in the above analysis is that we omitted the angular momentum arising from the $\hat{\mathbf{x}}_2$ (defined in Section 8.7.1) component of the angular velocity due to the precession of the top around the $\hat{\mathbf{z}}$-axis. This component has magnitude $\Omega \sin\theta$.[17] The angular momentum due to this angular velocity component has magnitude

$$L_2 = I\Omega \sin\theta, \tag{8.76}$$

and is directed along $\hat{\mathbf{x}}_2$. Let's label this as $\mathbf{L}_2 \equiv L_2\hat{\mathbf{x}}_2$. The total $\mathbf{L} = \mathbf{L}_2 + \mathbf{L}_3$ is shown in Fig. 8.29.

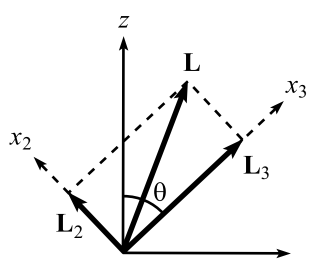

**Figure 8.29**

Only the horizontal component of $\mathbf{L}$ (call it $L_\perp$) changes. From the figure, $L_\perp$ is the difference in lengths of the horizontal components of $\mathbf{L}_3$ and $\mathbf{L}_2$. Therefore,

$$L_\perp = L_3 \sin\theta - L_2 \cos\theta = I_3\omega_3 \sin\theta - I\Omega \sin\theta \cos\theta. \tag{8.77}$$

The magnitude of the rate of change of $\mathbf{L}$ is simply $\Omega L_\perp = \Omega(I_3\omega_3 \sin\theta - I\Omega \sin\theta \cos\theta)$.[18] Equating this with the torque, $Mg\ell \sin\theta$, gives

$$I\Omega^2 \cos\theta - I_3\omega_3\Omega + Mg\ell = 0, \tag{8.78}$$

in agreement with eq. (8.73), as we wanted to show. The quadratic formula quickly gives the two solutions for $\Omega$, which may be written as

$$\Omega_\pm = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4MIg\ell\cos\theta}{I_3^2\omega_3^2}}\right). \tag{8.79}$$

[17]The angular velocity due to the precession is $\Omega\hat{\mathbf{z}}$. We may break this up into components along the orthogonal directions $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$. The $\Omega\cos\theta$ component along $\hat{\mathbf{x}}_3$ was absorbed into the definition of $\omega_3$ (see Fig. 8.27).

[18]This result can also be obtained in a more formal way. Since $\mathbf{L}$ precesses with angular velocity $\Omega\hat{\mathbf{z}}$, the rate of change of $\mathbf{L}$ is $d\mathbf{L}/dt = \Omega\hat{\mathbf{z}} \times \mathbf{L}$. This cross product is easily computed in the $x_2$-$x_3$ basis, and gives the same result.

REMARK: Note that if $\theta = \pi/2$, then eq. (8.78) is actually a linear equation, so there is only one solution for $\Omega$, which is the one in eq. (8.75). $\mathbf{L}_2$ points vertically, so it doesn't change. Only $\mathbf{L}_3$ contributes to $d\mathbf{L}/dt$. For this reason, a gyroscope is much easier to deal with when its symmetry axis is horizontal. ♣

The two solutions in eq. (8.79) are known as the *fast-precession* and *slow-precession* frequencies. For large $\omega_3$, you can show that the slow-precession frequency is

$$\Omega_- \approx \frac{Mg\ell}{I_3\omega_3}\,, \tag{8.80}$$

in agreement with the solution found in eq. (8.75).[19] This task, along with many other interesting features of this problem (including the interpretation of the fast-precession frequency, $\Omega_+$), is the subject of Problem 16, which you are strongly encouraged to do.

---

### 8.7.6 Nutation

Let us now solve eqs. (8.68) in a somewhat more general case, where $\theta$ is allowed to vary slightly. That is, we will consider a slight perturbation to the circular motion associated with eq. (8.73). We will assume $\omega_3$ is large here, and we will assume that the original circular motion corresponds to the slow precession, so that $\dot\phi$ is small. Under these assumptions, we will find that the top will bounce around slightly as it travels (roughly) in a circle. This bouncing is known as *nutation*.

Since $\dot\theta$ and $\dot\phi$ are small, we can (to a good approximation) ignore the quadratic terms in eqs. (8.68) and obtain

$$\begin{aligned} I\ddot\phi\sin\theta - \dot\theta I_3\omega_3 &= 0, \\ (Mg\ell - I_3\omega_3\dot\phi)\sin\theta &= I\ddot\theta. \end{aligned} \tag{8.81}$$

We must somehow solve these equations for $\theta(t)$ and $\phi(t)$. Taking the derivative of the first equation and dropping the quadratic term gives $\ddot\theta = (I\sin\theta/I_3\omega_3)\,d^2\dot\phi/dt^2$. Substituting this into the second equation gives

$$\frac{d^2\dot\phi}{dt^2} + \omega_n^2(\dot\phi - \Omega_s) = 0, \tag{8.82}$$

where

$$\omega_n \equiv \frac{I_3\omega_3}{I} \qquad \text{and} \qquad \Omega_s = \frac{Mg\ell}{I_3\omega_3} \tag{8.83}$$

are, respectively, the frequency of nutation (as we shall soon see), and the slow-precession frequency given in eq. (8.75). Shifting variables to $y \equiv \dot\phi - \Omega_s$ in eq. (8.82) gives us a nice harmonic-oscillator equation. Solving this and then shifting back to $\dot\phi$ yields

$$\dot\phi(t) = \Omega_s + A\cos(\omega_n t + \gamma), \tag{8.84}$$

[19] This is fairly clear. If $\omega_3$ is large enough compared to $\Omega$, then we can ignore the first term in eq. (8.78). That is, we can ignore the effects of $\mathbf{L}_2$, which is exactly what we did in the approximate solution in part (a).

where $A$ and $\gamma$ are determined from initial conditions. Integrating this gives

$$\phi(t) = \Omega_s t + \left(\frac{A}{\omega_n}\right)\sin(\omega_n t + \gamma), \tag{8.85}$$

plus an irrelevant constant.

Now let's solve for $\theta(t)$. Plugging $\phi(t)$ into the first of eqs. (8.81) gives

$$\dot{\theta}(t) = -\left(\frac{I\sin\theta}{I_3\omega_3}\right)A\omega_n\sin(\omega_n t + \gamma) = -A\sin\theta\sin(\omega_n t + \gamma). \tag{8.86}$$

Since $\theta(t)$ doesn't change much, we may set $\sin\theta \approx \sin\theta_0$, where $\theta_0$ is, say, the initial value of $\theta(t)$. (Any errors here are second-order effects in small quantities.) Integration then gives

$$\theta(t) = B + \left(\frac{A}{\omega_n}\sin\theta_0\right)\cos(\omega_n t + \gamma), \tag{8.87}$$

where $B$ is a constant of integration.

Eqs. (8.85) and (8.87) show that both $\phi$ (neglecting the uniform $\Omega_s t$ part) and $\theta$ oscillate with frequency $\omega_n$, and with amplitudes inversely proportional to $\omega_n$. Note that eq. (8.83) says $\omega_n$ grows with $\omega_3$.

---

**Example (Sideways kick):** Assume that uniform circular precession is initially taking place with $\theta = \theta_0$ and $\dot{\phi} = \Omega_s$. You then give the top a quick kick along the direction of motion, so that $\dot{\phi}$ is now equal to $\Omega_s + \Delta\Omega$ ($\Delta\Omega$ may be positive or negative). Find $\phi(t)$ and $\theta(t)$.

**Solution:** This is simply an exercise in initial conditions. We are given the initial values for $\dot{\phi}$, $\dot{\theta}$, and $\theta$. So we will need to solve for the unknowns $A$, $B$ and $\gamma$ in eqs. (8.84), (8.86), and (8.87). $\dot{\theta}$ is initially zero, so eq. (8.86) gives $\gamma = 0$. And $\dot{\phi}$ is initially $\Omega_s + \Delta\Omega$, so eq. (8.84) gives $A = \Delta\Omega$. Finally, $\theta$ is initially $\theta_0$, so eq. (8.87) gives $B = \theta_0 - (\Delta\Omega/\omega_n)\sin\theta_0$. Putting it all together, we have

$$\begin{aligned}
\phi(t) &= \Omega_s t + \left(\frac{\Delta\Omega}{\omega_n}\right)\sin\omega_n t, \\
\theta(t) &= \left(\theta_0 - \frac{\Delta\Omega}{\omega_n}\sin\theta_0\right) + \left(\frac{\Delta\Omega}{\omega_n}\sin\theta_0\right)\cos\omega_n t.
\end{aligned} \tag{8.88}$$

And for future reference (in the problems for this chapter), we'll also list the derivatives,

$$\begin{aligned}
\dot{\phi}(t) &= \Omega_s + \Delta\Omega\cos\omega_n t, \\
\dot{\theta}(t) &= -\Delta\Omega\sin\theta_0\sin\omega_n t.
\end{aligned} \tag{8.89}$$

REMARKS:

(a) With the initial conditions we have chosen, eq. (8.88) shows that $\theta$ always stays on one side of $\theta_0$. If $\Delta\Omega > 0$, then $\theta(t) \le \theta_0$ (that is, the top is always at a higher position, since $\theta$ is measured from the vertical). If $\Delta\Omega < 0$, then $\theta(t) \ge \theta_0$ (that is, the top is always at a lower position).

(b) The $\sin\theta_0$ coefficient of the $\cos\omega_n t$ term in eq. (8.88) implies that the amplitude of the $\theta$ oscillation is $\sin\theta_0$ times the amplitude of the $\phi$ oscillation. This is precisely the factor needed to make the CM travel in a circle around its average precessing position (because a change in $\theta$ causes a displacement of $\ell\, d\theta$, whereas a change in $\phi$ causes a displacement of $\ell \sin\theta_0\, d\phi$). ♣

## 8.8 Exercises

*Section 8.1: Preliminaries concerning rotations*

1. **Rolling wheel** $*$

   A wheel with spokes rolls on the ground. A stationary camera takes a picture of it, from the side. Do to the nonzero exposure time of the camera, the spokes will generally appear blurred. At what location (locations) in the picture does (do) a spoke (the spokes) *not* appear blurred?

*Section 8.2: The inertia tensor*

2. **Inertia tensor** $*$

   Calculate the $\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})$ double cross-product in eq. (8.7) by using the vector identity,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \tag{8.90}$$

*Section 8.3: Principal axes*

3. **Tennis racket theorem** $**$

   Problem 14 gives the statement of the "tennis racket theorem," and the solution there involves Euler's equations.

   Demonstrate the theorem here by using conservation of $L^2$ and conservation of rotational kinetic energy in the following way. Produce an equation which says that if $\omega_2$ and $\omega_3$ (or $\omega_1$ and $\omega_2$) start small, then they must remain small. And produce the analogous equation which says that if $\omega_1$ and $\omega_3$ start small, then they need *not* remain small.[20]

*Section 8.4: Two basic types of problems*

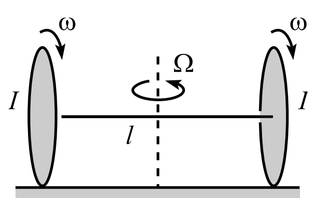


**Figure 8.30**

4. **Rotating axle** $**$

   Two wheels (with moment of inertia $I$) are connected by a massless axle of length $\ell$, as shown in Fig. 8.30. The system rests on a frictionless surface, and the wheels rotate with frequency $\omega$ around the axle. Additionally, the whole system rotates with frequency $\Omega$ around the vertical axis through the center of the axle. What is largest value of $\Omega$ for which both wheels stay on the ground?

*Section 8.7: Heavy symmetric top*

5. **Relation between $\Omega$ and $\omega'$** $**$

   Initial conditions have been set up so that a symmetric top undergoes precession, with its symmetry axis always making an angle $\theta$ with the vertical. The top has mass $M$, and the principal moments are $I_3$ and $I \equiv I_1 = I_2$. The CM

[20] It's another matter to show that they actually *won't* remain small. But don't bother with that here.

is a distance $\ell$ from the pivot. In the language of Fig. 8.27, show that $\omega'$ is related to $\Omega$ by

$$\omega' = \frac{Mg\ell}{I_3\Omega} + \Omega\cos\theta\left(\frac{I - I_3}{I_3}\right). \tag{8.91}$$

*Note*: You could just plug $\omega_3 = \omega' + \Omega\cos\theta$ (from eq. (8.60)) into eq. (8.73), and then solve for $\omega'$. But solve this problem from scratch, using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

6. **Sliding lollipop** $***$

   Consider a lollipop made of a solid sphere of mass $m$ and radius $r$, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground, which is frictionless (see Fig. 8.31). The sphere slides along the ground (keeping the same contact point on the sphere), with its center moving in a circle of radius $R$, with frequency $\Omega$.

   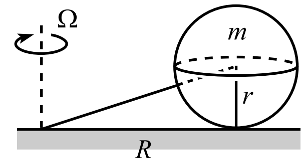
   

   **Figure 8.31**

   Show that the normal force between the ground and the sphere is $F_N = mg + mr\Omega^2$ (which is independent of $R$). Solve this by:

   (a) Using a simple $\mathbf{F} = m\mathbf{a}$ argument.[21]

   (b) Using a (more complicated) $\boldsymbol{\tau} = d\mathbf{L}/dt$ argument.

7. **Rolling wheel and axle** $***$

   A massless axle has one end attached to a wheel (which is a uniform disc of mass $m$ and radius $r$), with the other end pivoted on the ground (see Fig. 8.32). The wheel rolls on the ground without slipping, with the axle inclined at an angle $\theta$. The point of contact with the ground traces out a circle with frequency $\Omega$.

   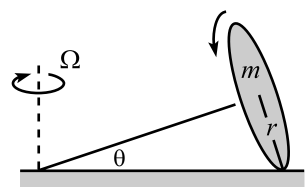
   

   **Figure 8.32**

   (a) Show that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown), with magnitude $\omega = \Omega/\tan\theta$.

   (b) Show that the normal force between the ground and the wheel is

$$N = mg\cos^2\theta + mr\Omega^2\left(\frac{3}{2}\cos^3\theta + \frac{1}{4}\cos\theta\sin^2\theta\right). \tag{8.92}$$

[21] This method happens to work here, due to the unusually nice nature of the sphere's motion. For more general motion (for example, in Problem 21, where the sphere is spinning), you must use $\vec{\tau} = d\mathbf{L}/dt$.

## 8.9 Problems

*Section 8.1: Preliminaries concerning rotations*

1. **Fixed points on a sphere** $**$

   Consider a transformation of a rigid sphere into itself. Show that two points on the sphere end up where they started.

2. **Many different $\vec{\omega}$'s** $*$

   Consider a particle at the point $(a, 0, 0)$, with velocity $(0, v, 0)$. This particle may be considered to be rotating around many different $\boldsymbol{\omega}$ vectors passing through the origin. (There is no one "correct" $\boldsymbol{\omega}$.) Find all the possible $\boldsymbol{\omega}$'s. (That is, find their directions and magnitudes.)

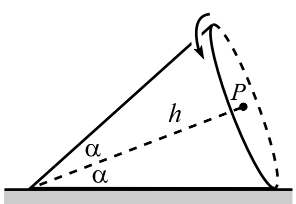


**Figure 8.33**

3. **Rolling cone** $**$

   A cone rolls without slipping on a table. The half-angle at the vertex is $\alpha$, and the axis of the cone has length $h$ (see Fig. 8.33). Let the speed of the center of the base (label this as point $P$) be $v$. What is the angular velocity of the cone with respect to the lab frame (at the instant shown)?

   There are many ways to do this problem, so you are encouraged to take a look at the three given solutions, even if you solve it.

*Section 8.2: The inertia tensor*

4. **Parallel-axis theorem**

   Let $(X, Y, Z)$ be the position of an object's CM, and let $(x', y', z')$ be the position relative to the CM. Prove the parallel-axis theorem, eq. (8.17), by setting $x = X + x'$, $y = Y + y'$, and $z = Z + z'$ in eq. (8.8).

*Section 8.3: Principal axes*

5. **Existence of principal axes for a pancake** $*$

   Given a pancake object in the $x$-$y$ plane, show that there exist principal axes by considering what happens to the integral $\int xy$ as the coordinate axes are rotated about the origin.

6. **Symmetries and principal axes for a pancake** $**$

   A rotation of the axes in the $x$-$y$ plane through an angle $\theta$ transforms the coordinates according to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8.93}$$

   Use this to show that if a pancake object in the $x$-$y$ plane has a symmetry under a rotation through $\theta \neq \pi$, then all axes (through the origin) in the plane are principal axes.

7. **Rotating square** $*$

Here's an exercise in geometry. Theorem 8.5 says that if the moments of inertia of two principal axes are equal, then any axis in the plane of these axes is a principal axis. This means that the object will rotate happily about any axis in this plane (no torque is needed). Demonstrate this explicitly for four masses $m$ in the shape of a square (which obviously has two moments equal), with the CM as the origin (see Fig. 8.34). Assume that the masses are connected with strings to the axis, as shown. Your task is to show that the tensions in the strings are such that there is no torque about the center of the square.

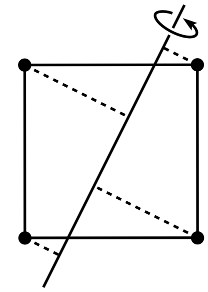

**Figure 8.34**

8. **A nice cylinder** $*$

What must the ratio of height to diameter of a cylinder be so that every axis is a principal axis (with the CM as the origin)?

*Section 8.4: Two basic types of problems*

9. **Rotating rectangle** $*$

A flat uniform rectangle with sides of length $a$ and $b$ sits it space (not rotating). You strike the corners at the ends of one diagonal, with equal and opposite forces (see Fig. 8.35). Show that the resulting initial $\boldsymbol{\omega}$ points along the other diagonal.

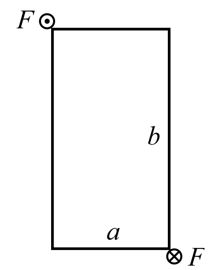


**Figure 8.35**

10. **Rotating stick** $**$

A stick of mass $m$ and length $\ell$ spins with frequency $\omega$ around an axis, as shown in Fig. 8.36. The stick makes an angle $\theta$ with the axis and is pivoted at its center. It is kept in this motion by two strings which are perpendicular to the axis. What is the tension in the strings?

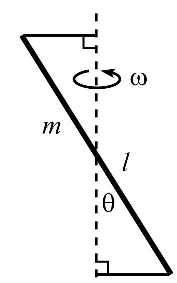


**Figure 8.36**

11. **Another rotating stick** $**$

A stick of mass $m$ and length $\ell$ is arranged to have its CM motionless and its top end slide in a circle on a frictionless rail (see Fig. 8.37). The stick makes an angle $\theta$ with the vertical. What is the frequency of this motion?

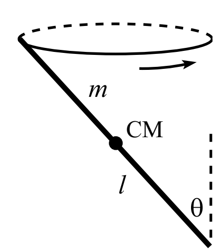


**Figure 8.37**

12. **Spherical pendulum** $**$

Consider a pendulum made of a massless rod of length $\ell$ and a point mass $m$. Assume conditions have been set up so that the mass moves in a horizontal circle. Let $\theta$ be the constant angle the rod makes with the vertical. Find the frequency, $\Omega$, of this circular motion in three different ways.

(a) Use $\mathbf{F} = m\mathbf{a}$. (The net force accounts for the centripetal acceleration.)[22]

(b) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the pendulum pivot as the origin.

(c) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the mass as the origin.

[22] This method works only if you have a point mass. With an extended object, you have to use one of the following methods involving torque.

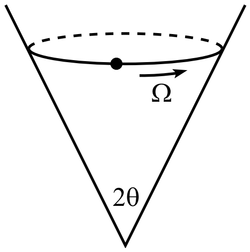


**Figure 8.38**

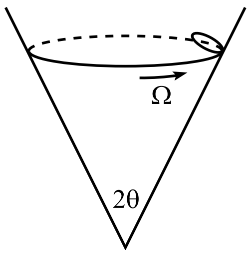


**Figure 8.39**

13. **Rolling in a cone** ∗∗

(a) A fixed cone stands on its tip, with its axis in the vertical direction. The half-angle at the vertex is $\theta$. A particle of negligible size slides on the inside frictionless surface of the cone (see Fig. 8.38).

Assume conditions have been set up so that the particle moves in a circle at height $h$ above the tip. What is the frequency, $\Omega$, of this circular motion?

(b) Assume now that the surface has friction, and a small ring of radius $r$ rolls without slipping on the surface. Assume conditions have been set up so that (1) the point of contact between the ring and the cone moves in a circle at height $h$ above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone (see Fig. 8.39).

What is the frequency, $\Omega$, of this circular motion? (You may work in the approximation where $r$ is much less than the radius of the circular motion, $h\tan\theta$.)

*Section 8.5: Euler's equations*

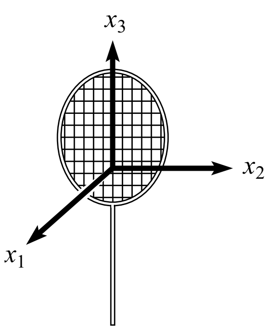


**Figure 8.40**

14. **Tennis racket theorem** ∗∗∗

If you try to spin a tennis racket (or a book, etc.) around any of its three principal axes, you will find that different things happen with the different axes. Assuming that the principal moments (relative to the CM) are labeled according to $I_1 > I_2 > I_3$ (see Fig. 8.40), you will find that the racket will spin nicely around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes, but it will wobble in a rather messy manner if you try to spin it around the $\hat{\mathbf{x}}_2$ axis.

Verify this claim experimentally with a book (preferably lightweight, and wrapped with a rubber band), or a tennis racket (if you happen to study with one on hand).

Verify this claim mathematically. The main point here is that you can't start the motion off with $\boldsymbol{\omega}$ pointing *exactly* along a principal axis. Therefore, what you want to show is that the motion around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes is *stable* (that is, small errors in the initial conditions remain small); whereas the motion around the $\hat{\mathbf{x}}_2$ axis is *unstable* (that is, small errors in the initial conditions get larger and larger, until the motion eventually doesn't resemble rotation around the $\hat{\mathbf{x}}_2$ axis).[23] Your task is to use Euler's equations to prove these statements about stability. (Exercise 3 gives another derivation of this result.)

[23] If you try for a long enough time, you will eventually be able to get the initial $\vec{\omega}$ pointing close enough to $\hat{\mathbf{x}}_2$ so that the book will remain rotating (almost) around $\hat{\mathbf{x}}_2$ for the entire time of its flight. There is, however, probably a better use for your time, as well as for the book...

*Section 8.6: Free symmetric top*

15. **Free-top angles** $*$

In Section 8.6.2, we showed that for a free symmetric top, the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the symmetry axis $\hat{\mathbf{x}}_3$ all lie in a plane. Let $\alpha$ be the angle between $\hat{\mathbf{x}}_3$ and $\mathbf{L}$, and let $\beta$ be the angle between $\hat{\mathbf{x}}_3$ and $\boldsymbol{\omega}$ (see Fig. 8.41). Find the relationship between $\alpha$ and $\beta$ in terms of the principal moments, $I$ and $I_3$.

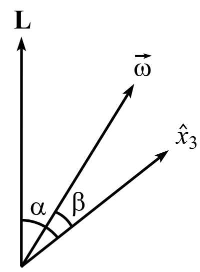

**Figure 8.41**

*Section 8.7: Heavy symmetric top*

16. **Gyroscope** $**$

This problem deals with the gyroscope example in Section 8.7.5, and uses the result for $\Omega$ in eq. (8.79).

(a) What is the minimum $\omega_3$ for which circular precession is possible?

(b) Let $\omega_3$ be very large, and find approximate expressions for $\Omega_\pm$. The phrase "very large" is rather meaningless, however. What mathematical statement should replace it?

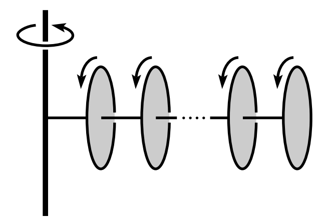
**Figure 8.42**

17. **Many gyroscopes** $**$

$N$ identical plates and massless sticks are arranged as shown in Fig. 8.42. Each plate is glued to the stick on its left. And each plate is attached by a free pivot to the stick on its right. (And the leftmost stick is attached by a free pivot to a pole.) You wish to set up a circular precession with the sticks always forming a straight horizontal line. What should the relative angular speeds of the plates be so that this is possible?

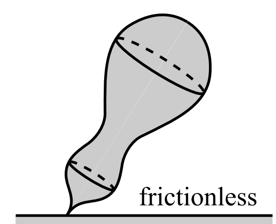

**Figure 8.43**

18. **Heavy top on slippery table** $*$

Solve the problem of a heavy symmetric top spinning on a frictionless table (see Fig. 8.43). You may do this by simply stating what modifications are needed in the derivation in Section 8.7.3 (or Section 8.7.4).

19. **Fixed highest point** $**$

Consider a top made of a uniform disc of radius $R$, connected to the origin by a massless stick (which is perpendicular to the disc) of length $\ell$. Paint a dot on the top at its highest point, and label this as point $P$ (see Fig. 8.44). You wish to set up uniform circular precession, with the stick making a constant angle $\theta$ with the vertical, and with $P$ always being the highest point on the top. What relation between $R$ and $\ell$ must be satisfied for this motion to be possible? What is the frequency of precession, $\Omega$?

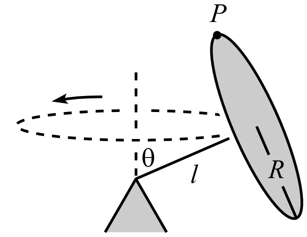

**Figure 8.44**

20. **Basketball on rim** ∗∗∗

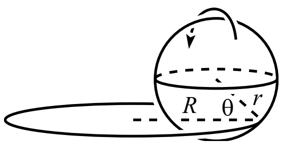


**Figure 8.45**

A basketball rolls without slipping around a basketball rim in such a way that the contact points trace out a great circle on the ball, and the CM moves around in a horizontal circle with frequency $\Omega$. The radii of the ball and rim are $r$ and $R$, respectively, and the ball's radius to the contact point makes an angle $\theta$ with the horizontal (see Fig. 8.45). Assume that the ball's moment of inertia around its center is $I = (2/3)mr^2$. Find $\Omega$.

21. **Rolling lollipop** ∗∗∗

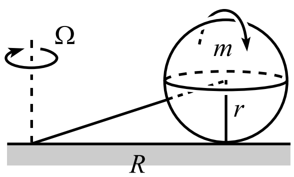


**Figure 8.46**

Consider a lollipop made of a solid sphere of mass $m$ and radius $r$, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground (see Fig. 8.46). The sphere rolls on the ground without slipping, with its center moving in a circle of radius $R$, with frequency $\Omega$.

(a) Find the angular velocity vector, $\boldsymbol{\omega}$.

(b) What is the normal force between the ground and the sphere?

22. **Rolling coin** ∗∗∗

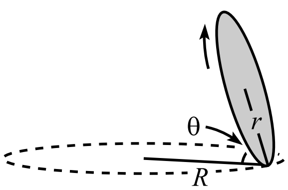


**Figure 8.47**

Initial conditions have been set up so that a coin of radius $r$ rolls around in a circle, as shown in Fig. 8.47. The contact point on the ground traces out a circle of radius $R$, and the coin makes a constant angle $\theta$ with the horizontal. The coin rolls without slipping. (Assume that the friction with the ground is as large as needed.) What is the frequency of the circular motion of the contact point on the ground? Show that such motion exists only if $R > (5/6)r\cos\theta$.

23. **Wobbling coin** ∗∗∗∗

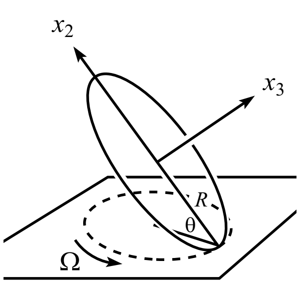


**Figure 8.48**

If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will decrease, and eventually the coin will come to rest. Assume that this process is slow, and consider the motion when the coin makes an angle $\theta$ with the table (see Fig. 8.48). You may assume that the CM is essentially motionless. Let $R$ be the radius of the coin, and let $\Omega$ be the frequency at which the point of contact on the table traces out its circle. Assume that the coin rolls without slipping.

(a) Show that the angular velocity vector of the coin is $\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ points upward along the coin, directly away from the contact point (see Fig. 8.27).

(b) Show that

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}\,. \tag{8.94}$$

(c) Show that Abe (or Tom, Franklin, George, John, Dwight, Sue, or Sacagawea) appears to rotate, when viewed from above, with frequency

$$2(1-\cos\theta)\sqrt{\frac{g}{R\sin\theta}}\,. \tag{8.95}$$

24. **Nutation cusps** $**$

   (a) Using the notation and initial conditions of the example in Section 8.7.6, prove that kinks occur in nutation if and only if $\Delta\Omega = \pm\Omega_s$. (A kink is where the plot of $\theta(t)$ vs. $\phi(t)$ has a discontinuity in its slope.)

   (b) Prove that these kinks are in fact cusps. (A cusp is a kink where the plot reverses direction in the $\phi$-$\theta$ plane.)

25. **Nutation circles** $**$

   (a) Using the notation and initial conditions of the example in Section 8.7.6, and assuming $\omega_3 \gg \Delta\Omega \gg \Omega_s$, find (approximately) the direction of the angular momentum right after the sideways kick takes place.

   (b) Use eqs. (8.88) to then show that the CM travels (approximately) in a circle around $\mathbf{L}$. And show that this "circular" motion is just what you would expect from the reasoning in Section 8.6.2 (in particular, eq. (8.53)), concerning the free top.

*Additional problems*

26. **Rolling straight?** $**$

   In some situations, for example the rolling-coin setup in Problem 22, the velocity of the CM of a rolling object changes direction as time goes by. Consider now a uniform sphere that rolls on the ground without slipping. Is it possible for the velocity of its CM to change direction? Justify your answer rigorously.

27. **Ball on paper** $***$

   A ball rolls without slipping on a table. It rolls onto a piece of paper. You slide the paper around in an arbitrary (horizontal) manner. (It's fine if there are abrupt, jerky motions, so that the ball slips with respect to the paper.) After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity.

28. **Ball on turntable** $****$

   A ball with uniform mass density rolls without slipping on a turntable. Show that the ball moves in a circle (as viewed from the inertial lab frame), with a frequency equal to 2/7 times the frequency of the turntable.

## 8.10 Solutions

1. **Fixed points on a sphere**

**First solution:** For the purposes of Theorem 8.1, we only need to show that two points remain fixed for an *infinitesimal* transformation. But since it's possible to prove this result for a general transformation, we'll consider the general case here.

Consider the point $A$ that ends up farthest away from where it started. (If there is more than one such point, pick any one of them.) Label the ending point $B$. Draw the great circle, $C_{AB}$, through $A$ and $B$. Draw the great circle, $C_A$, that is perpendicular to $C_{AB}$ at $A$; and draw the great circle, $C_B$, that is perpendicular to $C_{AB}$ at $B$.

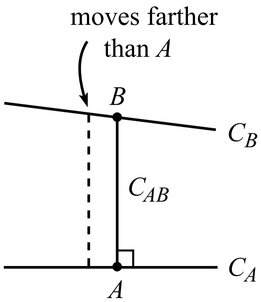


**Figure 8.49**

We claim that the transformation must take $C_A$ to $C_B$. This is true for the following reason. The image of $C_A$ is certainly a great circle through $B$, and this great circle must be perpendicular to $C_{AB}$, because otherwise there would exist another point that ended up farther away from its starting point than $A$ did (see Fig. 8.49). Since there is only one great circle through $B$ that is perpendicular to $C_{AB}$, the image of $C_A$ must in fact be $C_B$.

Now consider the two points, $P_1$ and $P_2$, where $C_A$ and $C_B$ intersect. (Any two great circles must intersect.) Let's look at $P_1$. The distances $P_1A$ and $P_1B$ are equal. Therefore, the point $P_1$ is not moved by the transformation. This is true because $P_1$ ends up on $C_B$ (because $C_B$ is the image of $C_A$, which is where $P_1$ started), and if it ends up at a point other than $P_1$, then its final distance from $B$ would be different from its initial distance from $A$. This would contradict the fact that distances are preserved on a rigid sphere. Likewise for $P_2$.

Note that for a non-infinitesimal transformation, every point on the sphere may move at some time during the transformation. What we just showed is that two of the points end up back where they started.

**Second solution:** In the spirit of the above solution, we can give simpler solution, but which is valid only in the case of infinitesimal transformations.

Pick any point, $A$, that moves during the transformation. Draw the great circle that passes through $A$ and is perpendicular to the direction of $A$'s motion. (Note that this direction is well-defined, because we are considering an infinitesimal transformation.) All points on this great circle must move (if they move at all) perpendicularly to the great circle, because otherwise their distances to $A$ would change. But they cannot all move in the same direction, because then the center of the sphere would move (but it is assumed to remain fixed). Therefore, at least one point on the great circle moves in the direction opposite to the direction in which $A$ moves. Therefore (by continuity), some point (and hence also its diametrically opposite point) on the great circle must remain fixed.

2. **Many different $\vec{\omega}$'s**

We want to find all the vectors, $\boldsymbol{\omega}$, such that $\boldsymbol{\omega} \times a\hat{\mathbf{x}} = v\hat{\mathbf{y}}$. Since $\boldsymbol{\omega}$ is orthogonal to this cross product, $\boldsymbol{\omega}$ must lie in the $x$-$z$ plane. We claim that if $\boldsymbol{\omega}$ makes an angle $\theta$ with the $x$-axis, and has magnitude $v/(a\sin\theta)$, then it will satisfy $\boldsymbol{\omega} \times a\hat{\mathbf{x}} = v\hat{\mathbf{y}}$. Indeed,

$$\boldsymbol{\omega} \times a\hat{\mathbf{x}} = |\boldsymbol{\omega}||a\hat{\mathbf{x}}|\sin\theta\hat{\mathbf{y}} = v\hat{\mathbf{y}}. \tag{8.96}$$

Alternatively, note that $\boldsymbol{\omega}$ may be written as

$$\boldsymbol{\omega} = \frac{v}{a\sin\theta}(\cos\theta, 0, \sin\theta) = \left(\frac{v}{a\tan\theta}\,,\,0,\,\frac{v}{a}\right), \tag{8.97}$$

and only the $z$-component is relevant in the cross product with $a\hat{\mathbf{x}}$.

It makes sense that the magnitude of $\boldsymbol{\omega}$ is $v/(a\sin\theta)$, because the particle is traveling in a circle of radius $a\sin\theta$ around $\boldsymbol{\omega}$, at speed $v$.

A few possible $\boldsymbol{\omega}$'s are drawn in Fig. 8.50. Technically, it is possible to have $\pi < \theta < 2\pi$, but then the $v/(a\sin\theta)$ coefficient in eq. (8.97) is negative, so $\boldsymbol{\omega}$ really points upward in the $x$-$z$ plane. ($\boldsymbol{\omega}$ must point upward if the particle's velocity is to be in the positive $y$-direction.) Note that $\omega_z$ is independent of $\theta$, so all the possible $\boldsymbol{\omega}$'s look like those in Fig. 8.51.

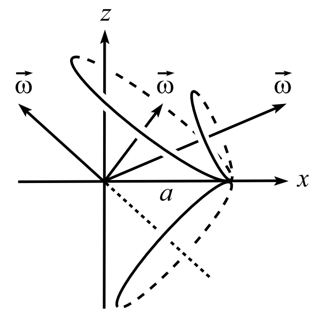


**Figure 8.50**

For $\theta = \pi/2$, we have $\omega = v/a$, which makes sense. If $\theta$ is very small, then $\omega$ is very large. This makes sense, because the particle is traveling around in a very small circle at the given speed $v$.

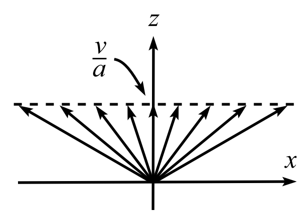


**Figure 8.51**

Remark: The point of this problem is that the particle may actually be in the process of having its position vector trace out a cone around one of many possible axes (or perhaps it may be undergoing some other complicated motion). If we are handed only the given initial information on position and velocity, then it is impossible to determine which of these motions is happening. And it is likewise impossible to uniquely determine $\boldsymbol{\omega}$. (This is true for a collection of points that lie on at most one line through the origin. If the points, along with the origin, span more than a 1-D line, then $\boldsymbol{\omega}$ is in fact uniquely determined.) ♣

3. **Rolling cone**

At the risk of overdoing it, we'll present three solutions. The second and third solutions are the type that tend to make your head hurt, so you may want to reread them after studying the discussion on the angular velocity vector in Section 8.7.2.

**First solution:** Without doing any calculations, we know that $\boldsymbol{\omega}$ points along the line of contact of the cone with the table, because these are the points on the cone that are instantaneously at rest. And we know that as time goes by, $\boldsymbol{\omega}$ rotates around in the horizontal plane with angular speed $v/(h\cos\alpha)$, because $P$ travels at speed $v$ in a circle of radius $h\cos\alpha$ around the $z$-axis.

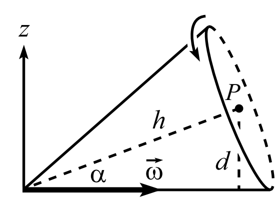


**Figure 8.52**

The magnitude of $\boldsymbol{\omega}$ can be found as follows. At a given instant, $P$ may be considered to be rotating in a circle of radius $d = h\sin\alpha$ around $\boldsymbol{\omega}$. (see Fig. 8.52). Since $P$ moves with speed $v$, the angular speed of this rotation is $v/d$. Therefore,

$$\omega = \frac{v}{h\sin\alpha}\,. \tag{8.98}$$

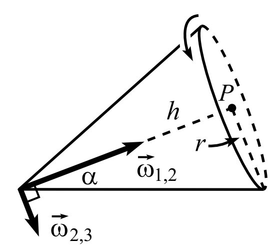


**Figure 8.53**

**Second solution:** We can use Theorem 8.3 with the following frames. $S_1$ is fixed in the cone; $S_3$ is the lab frame; and $S_2$ is the frame that (instantaneously) rotates around the tilted $\boldsymbol{\omega}_{2,3}$ axis shown in Fig. 8.53, at the speed such that the axis of the cone remains fixed in it. (The tip of $\boldsymbol{\omega}_{2,3}$ will trace out a circle as it precesses around the $z$-axis, so after the cone moves a little, we will need to use a new $S_2$ frame. But at any moment, $S_2$ instantaneously rotates around the axis perpendicular to the axis of the cone.) In the language of Theorem 8.3, $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown. We must find their magnitudes and then add the vectors to find the angular velocity of $S_1$ with respect to $S_3$.

First, we have

$$|\boldsymbol{\omega}_{2,3}| = \frac{v}{h}\,, \tag{8.99}$$

because point $P$ moves (instantaneously) with speed $v$ in a circle of radius $h$ around $\boldsymbol{\omega}_{2,3}$.

We now claim that

$$|\boldsymbol{\omega}_{1,2}| = \frac{v}{r}\,, \tag{8.100}$$

where $r = h\tan\alpha$ is the radius of the base of the cone. This is true because someone fixed in $S_2$ will see the endpoint of the radius (the one drawn) moving "backwards" at speed $v$, because it is stationary with respect to the table. Hence, the cone must be spinning with frequency $v/r$ in $S_2$.

The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.54. The result has magnitude $v/(h\sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \tan\alpha$).

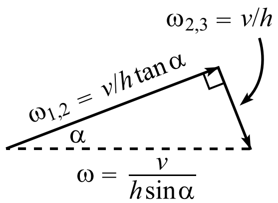


**Figure 8.54**

**Third solution:** We can use Theorem 8.3 with the following frames. $S_1$ is fixed in the cone; and $S_3$ is the lab frame (as in the second solution). But now let $S_2$ be the frame that rotates around the (negative) $z$-axis, at the speed such that the axis of the cone remains fixed in it. (Note that we can keep using this same $S_2$ frame as time goes by, unlike the $S_2$ frame in the second solution.) $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown in Fig. 8.55. As above, we must find their magnitudes and then add the vectors to find the angular velocity of $S_1$ with respect to $S_3$.

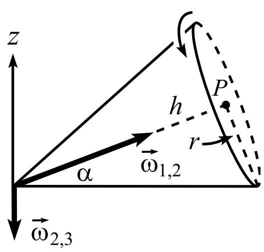


**Figure 8.55**

First, we have

$$|\boldsymbol{\omega}_{2,3}| = \frac{v}{h\cos\alpha}\,, \tag{8.101}$$

because point $P$ moves with speed $v$ in a circle of radius $h\cos\alpha$ around $\boldsymbol{\omega}_{2,3}$.

It's a little tricker to find $|\boldsymbol{\omega}_{1,2}|$. Consider the circle of contact points on the table where the base of the cone touches it. This circle has a radius $h/\cos\alpha$. From the point of view of someone spinning around with $S_2$, the table rotates backwards with frequency $|\boldsymbol{\omega}_{2,3}| = v/(h\cos\alpha)$, so this person sees the circle of contact points move with speed $v/\cos^2\alpha$ around the vertical. Since there is no slipping, the contact point on the cone must also move with this speed around the axis of the cone (which is fixed in $S_2$). And since the radius of the base is $r$, this means that the cone rotates with angular speed $v/(r\cos^2\alpha)$ with respect to $S_1$. Therefore,

$$|\boldsymbol{\omega}_{1,2}| = \frac{v}{r\cos^2\alpha} = \frac{v}{h\sin\alpha\cos\alpha}\,. \tag{8.102}$$

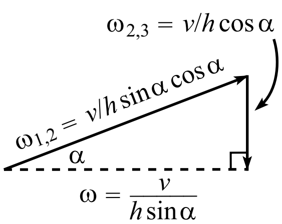


**Figure 8.56**

The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.56. The result has magnitude $v/(h\sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \sin\alpha$).

4. **Parallel-axis theorem**

Consider one of the diagonal entries in $\mathbf{I}$, say $\mathbf{I}_{11} = \int(y^2+z^2)$. In terms of the new variables, this equals

$$\begin{aligned}\mathbf{I}_{11} &= \int\Big((Y+y')^2 + (Z+z')^2\Big)\\ &= \int(Y^2+Z^2) + \int(y'^2+z'^2)\\ &= M(Y^2+Z^2) + \int(y'^2+z'^2),\end{aligned} \tag{8.103}$$

where we have used the fact that the cross terms vanish because, for example, $\int Yy' = Y\int y' = 0$, by definition of the CM.

Similarly, consider an off-diagonal entry in $\mathbf{I}$, say $\mathbf{I}_{12} = -\int xy$. We have

$$\mathbf{I}_{12} = -\int(X+x')(Y+y')$$

$$
\begin{aligned}
&= -\int XY - \int x'y' \\
&= -M(XY) - \int x'y', \qquad (8.104)
\end{aligned}
$$

where the cross terms have likewise vanished. We therefore see that all of the terms in **I** take the form of those in eq. (8.17), as desired.

5. **Existence of principal axes for a pancake**

For a pancake object, the inertia tensor **I** takes the form in eq. (8.8), with $z = 0$. Therefore, if we can find a set of axes for which $\int xy = 0$, then **I** will be diagonal, and we will have found our principal axes. We can prove, using a continuity argument, that such a set of axes exists.

Pick a set of axes, and write down the integral $\int xy \equiv I_0$. If $I_0 = 0$, then we are done. If $I_0 \neq 0$, then rotate these axes by an angle $\pi/2$, so that the new $\hat{\mathbf{x}}$ is the old $\hat{\mathbf{y}}$, and the new $\hat{\mathbf{y}}$ is the old $-\hat{\mathbf{x}}$ (see Fig. 8.57). Write down the new integral $\int xy \equiv I_{\pi/2}$. Since the new and old coordinates are related by $x_{\text{new}} = y_{\text{old}}$ and $y_{\text{new}} = -x_{\text{old}}$, we have $I_{\pi/2} = -I_0$. Therefore, since $\int xy$ switched sign during the rotation of the axes, there must exist some intermediate angle for which the integral $\int xy$ is zero.

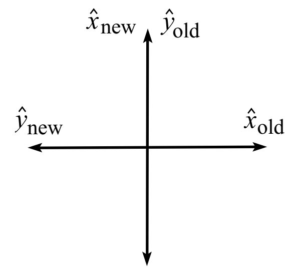


**Figure 8.57**

6. **Symmetries and principal axes for a pancake**

**First Solution:** In view of the form of the inertia tensor given in eq. (8.8), we want to show that if a pancake object has a symmetry under a rotation through $\theta \neq \pi$, then $\int xy = 0$ for any set of axes (through the origin).

Take an arbitrary set of axes and rotate them through an angle $\theta \neq \pi$. The new coordinates are $x' = (x\cos\theta + y\sin\theta)$ and $y' = (-x\sin\theta + y\cos\theta)$, so the new matrix entries, in terms of the old ones, are

$$
\begin{aligned}
I'_{xx} \equiv \int x'^2 &= I_{xx}\cos^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\
I'_{yy} \equiv \int y'^2 &= I_{xx}\sin^2\theta - 2I_{xy}\sin\theta\cos\theta + I_{yy}\cos^2\theta, \\
I'_{xy} \equiv \int x'y' &= -I_{xx}\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) + I_{yy}\sin\theta\cos\theta. \qquad (8.105)
\end{aligned}
$$

If the object looks exactly like it did before the rotation, then $I'_{xx} = I_{xx}$, $I'_{yy} = I_{yy}$, and $I'_{xy} = I_{xy}$. The first two of these are actually equivalent statements, so we'll just use the first and third. Using $\cos^2\theta - 1 = -\sin^2\theta$, these give

$$
\begin{aligned}
0 &= -I_{xx}\sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\
0 &= -I_{xx}\sin\theta\cos\theta - 2I_{xy}\sin^2\theta + I_{yy}\sin\theta\cos\theta. \qquad (8.106)
\end{aligned}
$$

Multiplying the first of these by $\cos\theta$ and the second by $\sin\theta$, and subtracting, gives

$$
2I_{xy}\sin\theta = 0. \qquad (8.107)
$$

Under the assumption $\theta \neq \pi$ (and $\theta \neq 0$, of course), we must therefore have $I_{xy} = 0$. Our initial axes were arbitrary; hence, any set of axes (through the origin) in the plane is a set of principal axes.

REMARK: If you don't trust this result, then you may want to show explicitly that the moments around two orthogonal axes are equal for, say, an equilateral triangle centered at the origin (which implies, by Theorem 8.5, all axes in the plane are principal axes). ♣

**Second Solution:** If an object is invariant under a rotation through an angle $\theta$, then $\theta$ must be of the form $\theta = 2\pi/N$, for some integer $N$ (convince yourself of this).[24]

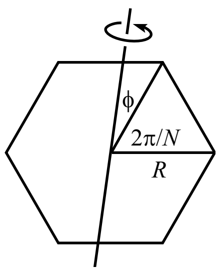


**Figure 8.58**

Consider a regular $N$-gon with "radius" $R$, with point-masses $m$ located at the vertices. Any object that is invariant under a rotation through $\theta = 2\pi/N$ can be considered to be built out of regular point-mass $N$-gons of various sizes. Therefore, if we can show that any axis in the plane of a regular point-mass $N$-gon is a principal axis, then we're done. We can do this as follows.

In Fig. 8.58, let $\phi$ be the angle between the axis and the nearest mass to its right. Label the $N$ masses clockwise from 0 to $N-1$, starting with this one. Then the angle between the axis and mass $k$ is $\phi + 2\pi k/N$. And the distance from mass $k$ to the axis is $r_k = |R\sin(\phi + 2\pi k/N)|$.

The moment of inertia around the axis is $I_\phi = \sum_{k=0}^{N-1} mr_k^2$. In view of Theorem 8.5, if we can show that $I_\phi = I_{\phi'}$ for some $\phi \neq \phi'$ (with $\phi \neq \phi' + \pi$), then we have shown that every axis is a principal axis. We will do this by demonstrating that $I_\phi$ is independent of $\phi$. We'll use a nice math trick here, which involves writing a trig function as the real part of a complex exponential. We have

$$\begin{aligned}
I_\phi &= mR^2 \sum_{k=0}^{N-1} \sin^2\left(\phi + \frac{2\pi k}{N}\right) \\
&= \frac{mR^2}{2} \sum_{k=0}^{N-1} \left(1 - \cos\left(2\phi + \frac{4\pi k}{N}\right)\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \cos\left(2\phi + \frac{4\pi k}{N}\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \mathrm{Re}\left(e^{i(2\phi + 4\pi k/N)}\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(1 + e^{4\pi i/N} + e^{8\pi i/N} + \cdots + e^{4(N-1)\pi i/N}\right)\right) \\
&= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(\frac{e^{4N\pi i/N} - 1}{e^{4\pi i/N} - 1}\right)\right), \qquad (8.108)
\end{aligned}$$

where we have summed the geometric series to obtain the last line. The numerator in the parentheses equals $e^{4\pi i} - 1 = 0$. And if $N \neq 2$, the denominator is not zero. Therefore, if $N \neq 2$ (which is equivalent to the $\theta \neq \pi$ restriction), then

$$I_\phi = \frac{NmR^2}{2}, \qquad (8.109)$$

which is independent of $\phi$. Hence, the moments around all axes in the plane are equal, so every axis in the plane is a principal axis, by Theorem 8.5

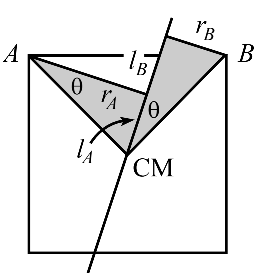


**Figure 8.59**

REMARKS: Given that the moments around all the axes in the plane are equal, they must be equal to $NmR^2/2$, because the perpendicular-axis theorem says that they all must be one-half of the moment around the axis perpendicular to the plane (which is $NmR^2$). ♣

7. **Rotating square**

Label two of the masses $A$ and $B$, as shown in Fig. 8.59. Let $\ell_A$ be the distance along

[24] If $N$ is divisible by 4, then a quick application of Theorem 8.5 shows that any axis in the plane is a principal axis. But if $N$ isn't divisible by 4, this isn't so obvious.

the axis from the CM to $A$'s string, and let $r_A$ be the length of $A$'s string. Likewise for $B$. The force, $F_A$, in $A$'s string must account for the centripetal acceleration of $A$. Hence, $F_A = mr_A\omega^2$. The torque around the CM due to $F_A$ is therefore

$$\tau_A = mr_A\ell_A\omega^2. \tag{8.110}$$

Likewise, the torque around the CM due to $B$'s string is $\tau_B = mr_B\ell_B\omega^2$, in the opposite direction.

But the two shaded triangles in Fig. 8.59 are congruent (they have the same hypotenuse and the same angle $\theta$). Therefore, $\ell_A = r_B$ and $\ell_B = r_A$. Hence, $\tau_A = \tau_B$, and the torques cancel. The torques from the other two masses likewise cancel. (Note that a uniform square is made up of many sets of these squares of point masses, so we've also shown that no torque is needed for a uniform square.)

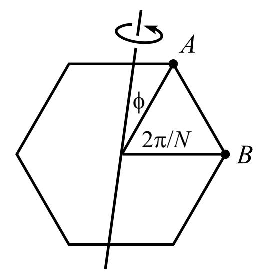

**Figure 8.60**

REMARK: For a general $N$-gon of point masses, Problem 6 shows that any axis in the plane is a principal axis. We should be able to use the above torque argument to prove this. This can be done as follows. (It's time for a nice math trick, involving the imaginary part of a complex exponential.) Using eq. (8.110), we see that the torque from mass $A$ in Fig. 8.60 is $\tau_A = m\omega^2R^2\sin\phi\cos\phi$. Likewise, the torque from mass $B$ is $\tau_B = m\omega^2R^2\sin(\phi+2\pi/N)\cos(\phi+2\pi/N)$, and so on. The total torque around the CM is therefore

$$\begin{aligned}
\tau &= mR^2\omega^2\sum_{k=0}^{N-1}\sin\left(\phi+\frac{2\pi k}{N}\right)\cos\left(\phi+\frac{2\pi k}{N}\right)\\
&= \frac{mR^2\omega^2}{2}\sum_{k=0}^{N-1}\sin\left(2\phi+\frac{4\pi k}{N}\right)\\
&= \frac{mR^2\omega^2}{2}\sum_{k=0}^{N-1}\mathrm{Im}\left(e^{i(2\phi+4\pi k/N)}\right)\\
&= \frac{mR^2\omega^2}{2}\mathrm{Im}\Big(e^{2i\phi}\left(1+e^{4\pi i/N}+e^{8\pi i/N}+\cdots++e^{4(N-1)\pi i/N}\right)\Big)\\
&= \frac{mR^2\omega^2}{2}\mathrm{Im}\left(e^{2i\phi}\left(\frac{e^{4N\pi i/N}-1}{e^{4\pi i/N}-1}\right)\right)\\
&= 0,
\end{aligned} \tag{8.111}$$

provided that $N \neq 2$ (but it's hard to have a 2-gon, anyway). This was essentially another proof of Problem 6. To prove that the torque was zero (which is one of the definitions of a principal axis), we showed here that $\sum r_i\ell_i = 0$. In terms of the chosen axes, this is equivalent to showing that $\sum xy = 0$, that is, showing that the off-diagonal terms in the inertia tensor vanish (which is simply another definition of the principal axes). ♣

8. **A nice cylinder**

Three axes that are certainly principal axes are the symmetry axis and two orthogonal diameters. The moments around the latter two are equal (call them $I$). Therefore, by Theorem 8.5, if the moment around the symmetry axis also equals $I$, then every axis is a principal axis.

Let the mass of the cylinder be $M$. Let its radius be $R$ and its height be $h$. Then the moment around the symmetry axis is $MR^2/2$.

Let $D$ be a diameter through the CM. The moment around $D$ can be calculated as follows. Slice the cylinder into horizontal disks of thickness $dy$. Let $\rho$ be the mass per unit height (so $\rho = M/h$). The mass of each disk is then $\rho\,dy$, so the moment around a diameter through the disk is $(\rho\,dy)R^2/4$. Therefore, by the parallel-axis theorem, the moment of a disk at height $y$ (where $-h/2 \le y \le h/2$) around $D$ is $(\rho\, dy)R^2/4 + (\rho\, dy)y^2$. Hence, the moment of the entire cylinder around $D$ is

$$I = \int_{-h/2}^{h/2} \left( \frac{\rho R^2}{4} + \rho y^2 \right) dy = \frac{\rho R^2 h}{4} + \frac{\rho h^3}{12} = \frac{MR^2}{4} + \frac{Mh^2}{12} \,. \tag{8.112}$$

We want this to equal $MR^2/2$. Therefore,

$$h = \sqrt{3}R. \tag{8.113}$$

You can show that if the origin was instead taken to be the center of one of the circular faces, then the answer would be $h = \sqrt{3}R/2$.

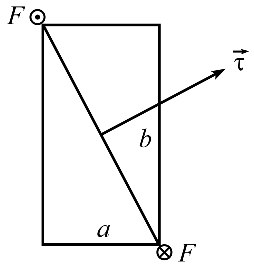

**Figure 8.61**

9. **Rotating rectangle**

If the force is out of the page at the upper left corner and into the page at the lower right corner, then the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ points upward to the right, as shown in Fig. 8.61, with $\boldsymbol{\tau} \propto (b, a)$. The angular momentum equals $\int \boldsymbol{\tau}\, dt$. Therefore, immediately after the strike, $\mathbf{L}$ is proportional to $(b, a)$.

The principal moments are $I_x = mb^2/12$ and $I_y = ma^2/12$. The angular momentum may be written as $\mathbf{L} = (I_x\omega_x, I_y\omega_y)$. Therefore, since we know $\mathbf{L} \propto (b, a)$, we have

$$(\omega_x, \omega_y) \propto \left( \frac{b}{I_x}, \frac{a}{I_y} \right) \propto \left( \frac{b}{b^2}, \frac{a}{a^2} \right) \propto (a, b), \tag{8.114}$$

which is the direction of the other diagonal. This answer checks in the special case $a = b$, and also in the limit where either $a$ or $b$ goes to zero.

10. **Rotating stick**

The angular momentum around the CM may be found as follows. Break $\boldsymbol{\omega}$ up into its components along the principal axes of the stick (which are parallel and perpendicular to the stick). The moment of inertia around the stick is zero. Therefore, to compute $\mathbf{L}$, we need to know only the component of $\boldsymbol{\omega}$ perpendicular to the stick. This component is $\omega \sin\theta$, and the associated moment of inertia is $m\ell^2/12$. Hence, the angular momentum at any time has magnitude

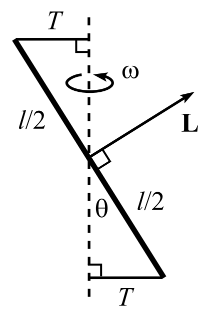

**Figure 8.62**

$$L = \frac{1}{12} m\ell^2 \omega \sin\theta, \tag{8.115}$$

and it points as shown in Fig. 8.62. The tip of the vector $\mathbf{L}$ traces out a circle in a horizontal plane, with frequency $\omega$. The radius of this circle is the horizontal component of $\mathbf{L}$, which is $L_\perp \equiv L\cos\theta$. The rate of change of $\mathbf{L}$ therefore has magnitude

$$\left| \frac{d\mathbf{L}}{dt} \right| = \omega L_\perp = \omega L \cos\theta = \omega \left( \frac{1}{12} m\ell^2 \omega \sin\theta \right) \cos\theta, \tag{8.116}$$

and it is directed into the page at the instant shown.

Let the tension in the strings be $T$. Then the torque due to the strings is $\boldsymbol{\tau} = 2T(\ell/2)\cos\theta$, directed into the page at the instant shown. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$T\ell\cos\theta = \omega \left( \frac{1}{12} m\ell^2 \omega \sin\theta \right) \cos\theta, \tag{8.117}$$

and so

$$T = \frac{1}{12} m\ell\omega^2 \sin\theta. \tag{8.118}$$

REMARKS: For $\theta \to 0$, this goes to zero, which makes sense. For $\theta \to \pi/2$, it goes to the finite value $m\ell\omega^2/12$, which isn't entirely obvious.

Note that if we instead had a massless stick with equal masses of $m/2$ on the ends (so that the relevant moment of inertia is now $m\ell^2/4$), then our answer would be $T = m\ell\omega^2 \sin\theta/4$. This makes sense if we write it as $T = (m/2)(\ell \sin\theta/2)\omega^2$, because each tension is simply responsible for keeping a mass of $m/2$ moving in a circle of radius $(\ell/2)\sin\theta$ at frequency $\omega$. ♣

11. **Another rotating stick**

As in Problem 10, the angular momentum around the CM may be found by breaking $\boldsymbol{\omega}$ up into its components along the principal axes of the stick (which are parallel and perpendicular to the stick). The moment of inertia around the stick is zero. Therefore, to compute $\mathbf{L}$, we need to know only the component of $\boldsymbol{\omega}$ perpendicular to the stick. This component is $\omega \sin\theta$, and the associated moment of inertia is $m\ell^2/12$. Hence, the angular momentum at any time has magnitude

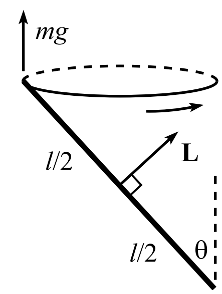


Figure 8.63

$$L = \frac{1}{12} m\ell^2 \omega \sin\theta, \tag{8.119}$$

and it points as shown in Fig. 8.63. The change in $\mathbf{L}$ comes from the horizontal component. This has length $L\cos\theta$ and travels in a circle at frequency $\omega$. Hence, $|d\mathbf{L}/dt| = \omega L \cos\theta$, and it is directed into the page at the instant shown.

The torque around the CM has magnitude $mg(\ell/2)\sin\theta$, and it points into the page at the instant shown. (This torque arises from the vertical force from the rail. There is no horizontal force from the rail, because the CM does not move.) Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\frac{mg\ell \sin\theta}{2} = \omega \left( \frac{m\ell^2 \omega \sin\theta}{12} \right) \cos\theta, \tag{8.120}$$

and so

$$\omega = \sqrt{\frac{6g}{\ell \cos\theta}}\,. \tag{8.121}$$

REMARKS: For $\theta \to \pi/2$, this goes to infinity, which makes sense. For $\theta \to 0$, it goes to the constant $\sqrt{6g/\ell}$, which isn't so obvious.

The motion in this problem is *not* possible if the bottom end of the stick, instead of the top end, slides along a rail. The magnitudes of all quantities are the same as in the original problem, but the direction of the torque (as you can check) is in the wrong direction.

If we instead had a massless stick with equal masses of $m/2$ on the ends (so that the relevant moment of inertia is now $m\ell^2/4$), then our answer would be $\omega = \sqrt{2g/(\ell\cos\theta)}$. This is simply the $\omega = \sqrt{g/[(\ell/2)\cos\theta]}$ answer for a point-mass spherical pendulum of length $\ell/2$ (see Problem 12), because the middle of the stick is motionless.

Note that the original massive stick *cannot* be treated like two sticks of length $\ell/2$. That is, the answer in eq. (8.121) is not obtained by using $\ell/2$ for the length in eq. (8.35). This is because there are internal forces in the stick that provide torques; if a free pivot were placed at the CM of the stick in this problem, the stick would not remain straight. ♣

12. **Spherical pendulum**

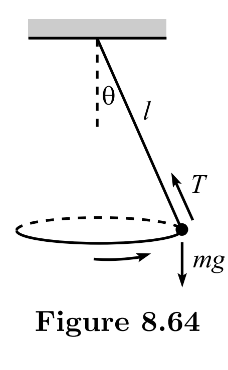


Figure 8.64

(a) The forces on the mass are gravity and the tension from the rod (see Fig. 8.64). Since there is no vertical acceleration, we have $T\cos\theta = mg$. The unbalanced horizontal force from the tension is therefore $T\sin\theta = mg\tan\theta$. This force accounts for the centripetal acceleration, $m(\ell\sin\theta)\Omega^2$. Hence,

$$\Omega = \sqrt{\frac{g}{\ell\cos\theta}}\,. \tag{8.122}$$

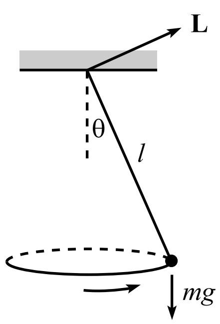

**Figure 8.65**

REMARK: For $\theta \approx 0$, this is the same as the $\sqrt{g/\ell}$ frequency for a simple pendulum. For $\theta \approx \pi/2$, it goes to infinity, which makes sense. Note that $\theta$ must be less than $\pi/2$ for circular motion to be possible. (This restriction does not hold for a gyroscope with extended mass.) ♣

(b) The only force that applies a torque relative to the pivot is the gravitational force. The torque is $\tau = mg\ell\sin\theta$, directed into the page (see Fig. 8.65).

At this instant in time, the mass has a speed $(\ell\sin\theta)\Omega$, directed into the page. Therefore, $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ has magnitude $m\ell^2\Omega\sin\theta$, and is directed upward to the right, as shown.

The tip of $\mathbf{L}$ traces out a circle of radius $L\cos\theta$, at frequency $\Omega$. Therefore, $d\mathbf{L}/dt$ has magnitude $\Omega L\cos\theta$, and is directed into the page.

Hence, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives $mg\ell\sin\theta = \Omega(m\ell^2\Omega\sin\theta)\cos\theta$. This yields eq. (8.122).

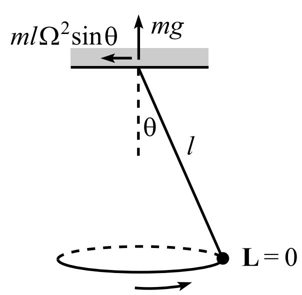

**Figure 8.66**

(c) The only force that applies a torque relative to the mass is that from the pivot. There are two components to this force (see Fig. 8.66).

There is the vertical piece, which is $mg$. Relative to the mass, this provides a torque of $mg(\ell\sin\theta)$, which is directed into the page.

There is also the horizontal piece, which accounts for the centripetal acceleration of the mass. This equals $m(\ell\sin\theta)\Omega^2$. Relative to the mass, this provides a torque of $m\ell\Omega^2\sin\theta(\ell\cos\theta)$, which is directed out of the page.

Relative to the mass, there is no angular momentum. Therefore, $d\mathbf{L}/dt = 0$. Hence, there must be no torque; so the above two torques cancel. This implies that $mg(\ell\sin\theta) = m\ell\Omega^2\sin\theta(\ell\cos\theta)$, which yields eq. (8.122).

REMARK: In problems that are more complicated than this one, it is often easier to work with a fixed pivot as the origin (if there is one) instead of the CM, because then you don't have to worry about messy pivot forces contributing to the torque. ♣

13. **Rolling in a cone**

(a) The forces on the particle are gravity ($mg$) and the normal force ($N$) from the cone. Since there is no net force in the vertical direction, we have

$$N\sin\theta = mg. \tag{8.123}$$

The inward horizontal force is therefore $N\cos\theta = mg/\tan\theta$. This force accounts for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\Omega^2$, and so

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{h}}\,. \tag{8.124}$$

(b) The forces on the ring are gravity ($mg$), the normal force ($N$) from the cone, and a friction force ($F$) pointing up along the cone. Since there is no net force in the vertical direction, we have

$$N\sin\theta + F\cos\theta = mg. \tag{8.125}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\Omega^2. \tag{8.126}$$

The previous two equations may be solved to yield $F$. The result is

$$F = mg\cos\theta - m\Omega^2(h\tan\theta)\sin\theta. \tag{8.127}$$

The torque on the ring (relative to the CM) is due solely to this $F$ (because gravity provides no torque, and $N$ points though the center of the ring, by the second assumption in the problem). Therefore, $\tau = rF$ equals

$$\tau = r\big(mg\cos\theta - m\Omega^2(h\tan\theta)\sin\theta\big). \tag{8.128}$$

and $\boldsymbol{\tau}$ points horizontally.

We must now find $d\mathbf{L}/dt$. Since are assuming $r \ll h\tan\theta$, the frequency of the spinning of the ring (call it $\omega$) is much greater than the frequency of precession, $\Omega$. We will therefore neglect the latter in finding $\mathbf{L}$. In this approximation, $L$ is simply $mr^2\omega$, and $\mathbf{L}$ points upward (or downward, depending on the direction of the precession) along the cone. The horizontal component of $\mathbf{L}$ has magnitude $L_\perp \equiv L\sin\theta$, and it traces out a circle at frequency $\Omega$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega L\sin\theta = \Omega(mr^2\omega)\sin\theta, \tag{8.129}$$

and it points horizontally, in the same direction as $\boldsymbol{\tau}$.

The non-slipping condition is $r\omega = (h\tan\theta)\Omega$,[25] which gives $\omega = (h\tan\theta)\Omega/r$. Using this in eq. (8.129) yields

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega^2 mrh\tan\theta\sin\theta, \tag{8.130}$$

Equating this $|d\mathbf{L}/dt|$ with the torque in eq. (8.128) gives

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{2h}}\,. \tag{8.131}$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).

Remark: If you consider an object with moment of inertia $\eta mr^2$ (our ring has $\eta = 1$), then you can show by the above reasoning that the "2" in eq. (8.131) is simply replaced by $(1+\eta)$. ♣

14. **Tennis racket theorem**

Presumably the experiment worked out as it was supposed to, without too much harm to the book. Now let's demonstrate the result mathematically.

*Rotation around* $\hat{\mathbf{x}}_1$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_1$ axis, then the initial $\omega_2$ and $\omega_3$ are much smaller than $\omega_1$. To emphasize this, relabel $\omega_2 \to \epsilon_2$ and

[25] This is technically not quite correct, for the same reason that the earth spins around 366 times instead of 365 times in a year. But it's valid enough in the limit of small $r$.

$\omega_3 \to \epsilon_3$. Then eqs. (8.43) become (with the torque equal to zero, because only gravity acts on the racket)

$$\begin{aligned} 0 &= \dot{\omega}_1 - A\epsilon_2\epsilon_3, \\ 0 &= \dot{\epsilon}_2 + B\omega_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_1\epsilon_2, \end{aligned} \tag{8.132}$$

where we have defined (for convenience)

$$A \equiv \frac{I_2 - I_3}{I_1}, \qquad B \equiv \frac{I_1 - I_3}{I_2}, \qquad C \equiv \frac{I_1 - I_2}{I_3}. \tag{8.133}$$

Note that $A$, $B$, and $C$ are all positive (this fact will be very important).

Our goal here is to show that if the $\epsilon$'s start out small, then they remain small. Assuming that they are small (which is true initially), the first equation says that $\dot{\omega}_1 \approx 0$ (to first order in the $\epsilon$'s). Therefore, we may assume that $\omega_1$ is essentially constant (when the $\epsilon$'s are small). Taking the derivative of the second equation then gives $0 = \ddot{\epsilon}_2 + B\omega_1\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_2 = -(BC\omega_1^2)\epsilon_2. \tag{8.134}$$

Because of the negative coefficient on the right-hand side, this equation describes simple harmonic motion. Therefore, $\epsilon_2$ oscillates sinusoidally around zero. Hence, if it starts small, it remains small. By the same reasoning, $\epsilon_3$ remains small.

We therefore see that $\boldsymbol{\omega} \approx (\omega_1, 0, 0)$ at all times, which implies that $\mathbf{L} \approx (I_1\omega_1, 0, 0)$ at all times. That is, $\mathbf{L}$ always points (nearly) along the $\hat{\mathbf{x}}_1$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (because there is no torque). Therefore, the direction of $\hat{\mathbf{x}}_1$ must also be (nearly) fixed in the lab frame. In other words, the racket doesn't wobble.

*Rotation around* $\hat{\mathbf{x}}_3$: The calculation goes through exactly as above, except with "1" and "3" interchanged. We find that if $\epsilon_1$ and $\epsilon_2$ start small, they remain small. And $\boldsymbol{\omega} \approx (0, 0, \omega_3)$ at all times.

*Rotation around* $\hat{\mathbf{x}}_2$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_2$ axis, then the initial $\omega_1$ and $\omega_3$ are much smaller than $\omega_2$. As above, let's emphasize this by relabeling $\omega_1 \to \epsilon_1$ and $\omega_3 \to \epsilon_3$. Then as above, eqs. (8.43) become

$$\begin{aligned} 0 &= \dot{\epsilon}_1 - A\omega_2\epsilon_3, \\ 0 &= \dot{\omega}_2 + B\epsilon_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_2\epsilon_1, \end{aligned} \tag{8.135}$$

Our goal here is to show that if the $\epsilon$'s start out small, then they do *not* remain small. Assuming that they are small (which is true initially), the second equation says that $\dot{\omega}_2 \approx 0$ (to first order in the $\epsilon$'s). So we may assume that $\omega_2$ is essentially constant (when the $\epsilon$'s are small). Taking the derivative of the first equation then gives $0 = \ddot{\epsilon}_1 - A\omega_2\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_1 = (AC\omega_2^2)\epsilon_1. \tag{8.136}$$

Because of the positive coefficient on the right-hand side, this equation describes an exponentially growing motion, instead of an oscillatory one. Therefore, $\epsilon_1$ grows quickly from its initial small value. Hence, even if it starts small, it becomes large. By the same reasoning, $\epsilon_3$ becomes large. (Of course, once the $\epsilon$'s become large, then our assumption of $\dot{\omega}_2 \approx 0$ isn't valid anymore. But once the $\epsilon$'s become large, we've shown what we wanted to.)

We see that $\boldsymbol{\omega}$ does *not* remain (nearly) equal to $(0, \omega_2, 0)$ at all times, which implies that $\mathbf{L}$ does *not* remain (nearly) equal to $(0, I_2\omega_2, 0)$ at all times. That is, $\mathbf{L}$ does not always point (nearly) along the $\hat{\mathbf{x}}_2$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (because there is no torque). Therefore, the direction of $\hat{\mathbf{x}}_2$ must change in the lab frame. In other words, the racket wobbles.

15. **Free-top angles**

In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned} \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.137}$$

Let $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) \equiv \omega_\perp\hat{\boldsymbol{\omega}}_\perp$ be the component of $\boldsymbol{\omega}$ orthogonal to $\boldsymbol{\omega}_3$. Then, by definition, we have

$$\tan\beta = \frac{\omega_\perp}{\omega_3}, \qquad \text{and} \qquad \tan\alpha = \frac{I\omega_\perp}{I_3\omega_3}. \tag{8.138}$$

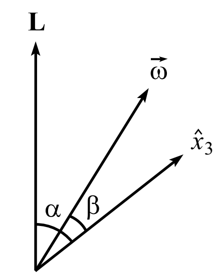

**Figure 8.67**

Therefore,

$$\frac{\tan\alpha}{\tan\beta} = \frac{I}{I_3}. \tag{8.139}$$

If $I > I_3$, then $\alpha > \beta$, and we have the situation shown in Fig. 8.67. A top with this property is called a "prolate top". An example is a football or a pencil.

If $I < I_3$, then $\alpha < \beta$, and we have the situation shown in Fig. 8.68. A top with this property is called an "oblate top". An example is a coin or a Frisbee.

16. **Gyroscope**

(a) In order for there to exist real solutions for $\Omega$ in eq. (8.79), the discriminant must be non-negative. If $\theta \geq \pi/2$, then $\cos\theta \leq 0$, so the discriminant is automatically positive, and any value of $\omega_3$ is allowed. But if $\theta < \pi/2$, then the lower limit on $\omega_3$ is

$$\omega_3 \geq \frac{\sqrt{4MIg\ell\cos\theta}}{I_3} \equiv \tilde{\omega}_3. \tag{8.140}$$

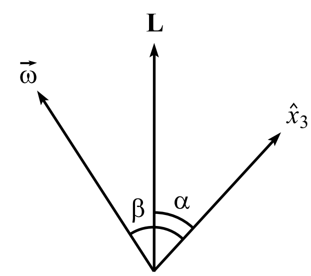

**Figure 8.68**

Note that at this critical value, eq. (8.79) gives

$$\Omega_+ = \Omega_- = \frac{I_3\tilde{\omega}_3}{2I\cos\theta} = \sqrt{\frac{Mg\ell}{I\cos\theta}} \equiv \Omega_0. \tag{8.141}$$

(b) Since $\omega_3$ has units, "large $\omega_3$" is a meaningless description. What we really mean is that the fraction in the square root in eq. (8.79) is very small compared to 1. That is, $\epsilon \equiv (4MIg\ell\cos\theta)/(I_3^2\omega_3^2) \ll 1$. In this case, we may use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2 + \cdots$ to write

$$\Omega_\pm \approx \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \left(1 - \frac{2MIg\ell\cos\theta}{I_3^2\omega_3^2}\right)\right). \tag{8.142}$$

Therefore, the two solutions for $\Omega$ are (to leading order in $\omega_3$)

$$\Omega_+ \approx \frac{I_3\omega_3}{I\cos\theta}, \qquad \text{and} \qquad \Omega_- \approx \frac{Mg\ell}{I_3\omega_3}. \tag{8.143}$$

These are known as the “fast” and “slow” frequencies of precession, respectively. $\Omega_-$ is the approximate answer we found in eq. (8.75), and it was obtained here under the assumption $\epsilon \ll 1$, which is equivalent to

$$\omega_3 \gg \frac{\sqrt{4MIg\ell\cos\theta}}{I_3} \qquad \text{(that is, } \omega_3 \gg \tilde{\omega}_3\text{)}. \tag{8.144}$$

This, therefore, is the condition for the result in eq. (8.75) to be a good approximation. Note that if $I$ is of the same order as $I_3$ (so that they are both of the order $M\ell^2$), and if $\cos\theta$ is of order 1, then this condition may be written as $\omega \gg \sqrt{g/\ell}$, which is the frequency of a pendulum of length $\ell$.

Remarks: The $\Omega_+$ solution is a fairly surprising result. Two strange features of $\Omega_+$ are that it grows with $\omega_3$, and that it is independent of $g$. To see what is going on with this precession, note that $\Omega_+$ is the value of $\Omega$ that makes the $L_\perp$ in eq. (8.77) essentially equal to zero. So **L** points nearly along the vertical axis. The rate of change of **L** is the product of a very small radius (of the circle the tip traces out) and a very large $\Omega$ (if we've picked $\omega_3$ to be large). The product of these equals the “medium sized” torque $Mg\ell\sin\theta$.

In the limit of large $\omega_3$, the fast precession should look basically like the motion of a free top (because **L** is essentially constant), discussed in Section 8.6.2. And indeed, $\Omega_+$ is independent of $g$. We'll leave it to you to show that $\Omega_+ \approx L/I$, which is the precession frequency of a free top (eq. (8.53)), as viewed from a fixed frame.

We can plot the $\Omega_\pm$ of eq. (8.79) as functions of $\omega_3$. With the definitions of $\tilde{\omega}_3$ and $\Omega_0$ in eqs. (8.140) and (8.141), we can rewrite eq. (8.79) as

$$\Omega_\pm = \frac{\omega_3\Omega_0}{\tilde{\omega}_3}\left(1 \pm \sqrt{1 - \frac{\tilde{\omega}_3^2}{\omega_3^2}}\right). \tag{8.145}$$

It is easier to work with dimensionless quantities, so let's rewrite this as

$$y_\pm = x \pm \sqrt{x^2 - 1}, \qquad \text{with } \; y_\pm \equiv \frac{\Omega_\pm}{\Omega_0}, \quad x \equiv \frac{\omega_3}{\tilde{\omega}_3}. \tag{8.146}$$

A rough plot of $y_\pm$ vs. $x$ is shown in Fig. 8.69. ♣

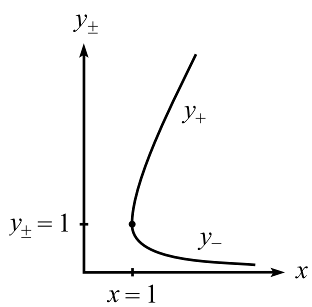


**Figure 8.69**

17. **Many gyroscopes**

The system is made up of $N$ rigid bodies, each consisting of a plate and the massless stick glued to it on its left (see Fig. 8.70). Label these sub-systems as $S_i$, with $S_1$ being the one closest to the pole.

Let each plate have mass $m$ and moment of inertia $I$, and let each stick have length $\ell$. Let the angular speeds be $\omega_i$. The relevant angular momentum of $S_i$ is then $L_i = I\omega_i$, and it points horizontally.[26] Let the desired precession frequency be $\Omega$. Then the magnitude of $d\mathbf{L}_i/dt$ is $L_i\Omega = (I\omega_i)\Omega$, and this points perpendicularly to $\mathbf{L}_i$.

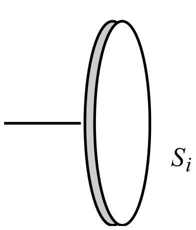


**Figure 8.70**

Consider the torque $\boldsymbol{\tau}_i$ on $S_i$, around its CM. Let's first look at $S_1$. The pole provides an upward force of $Nmg$ (this force is what keeps all the gyroscopes up), so it provides a torque of $Nmg\ell$ around the CM of $S_1$. The downward force from the stick to the right provides no torque around the CM (because it acts at the CM). Therefore, $\boldsymbol{\tau}_1 = d\mathbf{L}_1/dt$ gives $Nmg\ell = (I\omega_1)\Omega$, and so

$$\omega_1 = \frac{Nmg\ell}{I\Omega}. \tag{8.147}$$

[26]We are ignoring the angular momentum arising from the precession. This part of **L** points vertically (because the gyroscopes all point horizontally) and therefore does not change. Hence, it does not enter into $\vec{\tau} = d\mathbf{L}/dt$.

Now look at $S_2$. $S_1$ provides an upward force of $(N-1)mg$ (this force is what keeps $S_2$ through $S_N$ up), so it provides a torque of $(N-1)mg\ell$ around the CM of $S_2$. The downward force from the stick to the right provides no torque around the CM of $S_2$. Therefore, $\boldsymbol{\tau}_2 = d\mathbf{L}_2/dt$ gives $(N-1)mg\ell = (I\omega_2)\Omega$, and so

$$\omega_2 = \frac{(N-1)mg\ell}{I\Omega}. \tag{8.148}$$

Similar reasoning applies to the other $S_i$, and we arrive at

$$\omega_i = \frac{(N+1-i)mg\ell}{I\Omega}. \tag{8.149}$$

The $\omega_i$ are therefore in the ratio

$$\omega_1 : \omega_2 : \cdots : \omega_{N-1} : \omega_N = N : (N-1) : \cdots : 2 : 1. \tag{8.150}$$

Note that we needed to apply $\boldsymbol{\tau} = d\mathbf{L}/dt$ many times, using each CM as an origin. Using only the pivot point on the pole as the origin would have given only one piece of information, whereas we needed $N$ pieces.

REMARKS: As a double-check, we can verify that these $\omega$'s make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum relative to the pivot on the pole. (Using the CM of the entire system as the origin would give the same equation.) The CM of the entire system is $(N+1)\ell/2$ from the wall, so the torque due to gravity is

$$\tau = Nmg\frac{(N+1)\ell}{2}. \tag{8.151}$$

The total angular momentum is, using eq. (8.149),

$$\begin{aligned} L &= I(\omega_1 + \omega_2 + \cdots + \omega_N) \\ &= \frac{mg\ell}{\Omega}\Big(N + (N-1) + (N-2) + \cdots + 2 + 1\Big) \\ &= \frac{mg\ell}{\Omega}\frac{N(N+1)}{2}. \end{aligned} \tag{8.152}$$

So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$.

We can also pose this problem for the setup where all the $\omega_i$ are equal (call them $\omega$), and the goal is the find the lengths of the sticks that will allow the desired motion. We can use the same reasoning as above, and eq. (8.149) takes the modified form

$$\omega = \frac{(N+1-i)mg\ell_i}{I\Omega}, \tag{8.153}$$

where $\ell_i$ is the length of the $i$th stick. Therefore, the $\ell_i$ are in the ratio

$$\ell_1 : \ell_2 : \cdots : \ell_{N-1} : \ell_N = \frac{1}{N} : \frac{1}{N-1} : \cdots : \frac{1}{2} : 1. \tag{8.154}$$

Note that since the sum $\sum 1/n$ diverges, it is possible to make the setup extend arbitrarily far from the pole.

Again, we can verify that these $\ell$'s make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum relative to the pivot on the pole. As an exercise, you can show that the CM happens to be a distance $\ell_N$ from the pole. So the torque due to gravity is, using eq. (8.153) to obtain $\ell_N$,

$$\tau = Nmg\ell_N = Nmg(\omega I\Omega/mg) = NI\omega\Omega. \tag{8.155}$$

The total angular momentum is simply $L = NI\omega$. So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$. ♣

18. **Heavy top on slippery table**

In Section 8.7.3, we looked at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the pivot point. Such quantities are of no use here, because we can't use $\boldsymbol{\tau} = d\mathbf{L}/dt$ relative to the pivot point (because it is accelerating). We will therefore look at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the CM, which is always a legal origin around which we can apply $\boldsymbol{\tau} = d\mathbf{L}/dt$.

The only force the floor applies is the normal force $Mg$. Therefore, the torque relative to the CM has the same magnitude, $Mg\ell\sin\theta$, and the same direction as in Section 8.7.3. If we choose the CM as the origin of our coordinate system, then all the Euler angles are the same as before. The only change in the whole analysis is the change in the $I_1 = I_2 \equiv I$ moment of inertia. We are now measuring these moments with respect to the CM, instead of the pivot point. By the parallel axis theorem, they are now equal to

$$I' = I - M\ell^2. \tag{8.156}$$

Therefore, changing $I$ to $I - M\ell^2$ is the only modification needed.

19. **Fixed highest point**

For the desired motion, the important thing to note is that every point in the top moves in a fixed circle around the $\hat{\mathbf{z}}$-axis. Therefore, $\boldsymbol{\omega}$ points vertically. Hence, if $\Omega$ is the frequency of precession, we have $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}}$.

(Another way to see that $\boldsymbol{\omega}$ points vertically is to view things in the frame that rotates with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the top has no motion whatsoever. It is not even spinning, because the point $P$ is always the highest point. In the language of Fig. 8.27, we therefore have $\omega' = 0$. Hence, $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}}$.)

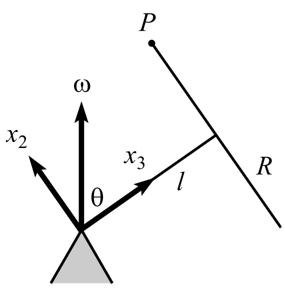

**Figure 8.71**

The principal moments are (with the pivot as the origin; see Fig. 8.71)

$$I_3 = \frac{MR^2}{2}, \qquad \text{and} \qquad I \equiv I_1 = I_2 = M\ell^2 + \frac{MR^2}{4}, \tag{8.157}$$

where we have used the parallel-axis theorem to obtain the latter. The components of $\boldsymbol{\omega}$ along the principal axes are $\omega_3 = \Omega\cos\theta$, and $\omega_2 = \Omega\sin\theta$. Therefore (keeping things in terms of the general moments, $I_3$ and $I$),

$$\mathbf{L} = I_3\Omega\cos\theta\hat{\mathbf{x}}_3 + I\Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.158}$$

The horizontal component of $\mathbf{L}$ is then $L_\perp = (I_3\Omega\cos\theta)\sin\theta - (I\Omega\sin\theta)\cos\theta$, and so $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_\perp\Omega = \Omega^2\sin\theta\cos\theta(I_3 - I), \tag{8.159}$$

and is directed into the page (or out of the page, if this quantity is negative). This must equal the torque, which has magnitude $|\boldsymbol{\tau}| = Mg\ell\sin\theta$, and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I)\cos\theta}}. \tag{8.160}$$

We see that for a general symmetric top, such precessional motion (where the same "side" always points up) is possible only if

$$I_3 > I. \tag{8.161}$$

Note that this condition is independent of $\theta$. For the problem at hand, $I_3$ and $I$ are given in eq. (8.157), and we find

$$\Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos\theta}}\,, \tag{8.162}$$

and the necessary condition for such motion is $R > 2\ell$.

REMARKS:

(a) It is intuitively clear that $\Omega$ should become very large as $\theta \to \pi/2$, although it is by no means intuitively clear that such motion should exist at all for angles near $\pi/2$.

(b) $\Omega$ approaches a non-zero constant as $\theta \to 0$, which isn't entirely obvious.

(c) If both $R$ and $\ell$ are scaled up by the same factor, then $\Omega$ decreases. (This also follows from dimensional analysis.)

(d) The condition $I_3 > I$ can be understood in the following way. If $I_3 = I$, then $\mathbf{L} \propto \vec{\omega}$, and so $\mathbf{L}$ points vertically along $\vec{\omega}$. If $I_3 > I$, then $\mathbf{L}$ points somewhere to the right of the $\hat{\mathbf{z}}$-axis (at the instant shown in Fig. 8.71). This means that the tip of $\mathbf{L}$ is moving into the page, along with the top. This is what we need, because $\vec{\tau}$ points into the page. If, however, $I_3 < I$, then $\mathbf{L}$ points somewhere to the left of the $\hat{\mathbf{z}}$-axis, so $d\mathbf{L}/dt$ points out of the page, and hence cannot be equal to $\vec{\tau}$. ♣

20. **Basketball on rim**

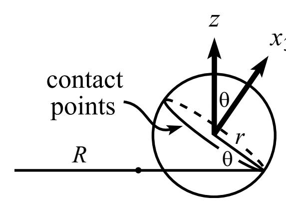

**Figure 8.72**

Consider the setup in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the center of the ball is at rest. Therefore, if the contact points are to form a great circle, the ball must be spinning around the (negative) $\hat{\mathbf{x}}_3$ axis shown in Fig. 8.72. Let the frequency of this spinning be $\omega'$ (in the language of Fig. 8.27). Then the nonslipping condition says that $\omega' r = \Omega R$, and so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the ball in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - (R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.163}$$

Let us choose the center of the ball as the origin around which $\boldsymbol{\tau}$ and $\mathbf{L}$ are calculated. Then every axis in the ball is a principal axis, with moment of inertia $I = (2/3)mr^2$. The angular momentum is therefore

$$\mathbf{L} = I\boldsymbol{\omega} = I\Omega\hat{\mathbf{z}} - I(R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.164}$$

Only the $\hat{\mathbf{x}}_3$ piece has a horizontal component which will contribute to $d\mathbf{L}/dt$. This component has length $L_\perp = I(R/r)\Omega\sin\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \frac{2}{3}\Omega^2 mrR\sin\theta, \tag{8.165}$$

and points out of the page.

The torque (relative to the center of the ball) comes from the force at the contact point. There are two components of this force. The vertical component is $mg$, and the horizontal component is $m(R - r\cos\theta)\Omega^2$ (pointing to the left), because the center of the ball moves in a circle of radius $(R - r\cos\theta)$. We then find the torque to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.166}$$

with outward from the page taken to be positive. Using the previous two equations, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega^2 = \frac{g\cos\theta}{\sin\theta\left(\frac{5}{3}R - r\cos\theta\right)}\,. \tag{8.167}$$

REMARKS:

(a) $\Omega \to \infty$ as $\theta \to 0$, which makes sense.

(b) Also, $\Omega \to \infty$ when $R = (3/5)r\cos\theta$. This case, however, is not physical, because we must have $R > r\cos\theta$ in order for the other side of the rim to be outside the basketball.

(c) You can also work out the problem for the case where the contact points trace out a circle other than a great circle (say, one that makes an angle $\beta$ with respect to the great circle). The expression for the torque in eq. (8.166) is unchanged, but the value of $\omega'$ and the angle of the $\hat{\mathbf{x}}_3$-axis both change. Eq. (8.165) is therefore modified. The resulting $\Omega$, however, is a bit messy. ♣

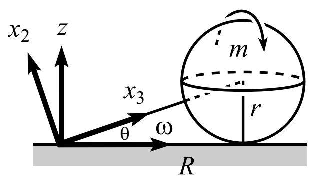


**Figure 8.73**

21. **Rolling lollipop**

(a) We claim that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown in Fig. 8.73), with magnitude $(R/r)\Omega$. This can be seen in (at least) two ways.

The first method is to note that we essentially have the same scenario as in the "Rolling cone" setup of Problem 3. The sphere's contact point with the ground is at rest (the non-slipping condition), so $\boldsymbol{\omega}$ must pass through this point (horizontally). The center of the sphere moves with speed $\Omega R$. But since the center may also be considered to be instantaneously moving with frequency $\omega$ in a circle of radius $r$ around the horizontal axis, we have $\omega r = \Omega R$. Therefore, $\omega = (R/r)\Omega$.

The second method is to write $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (in the language of Fig. 8.27), where $\omega'$ is the frequency of the spinning as viewed by someone rotating around the (negative) $\hat{\mathbf{z}}$ axis with frequency $\Omega$. The contact points form a circle of radius $R$ on the ground. But they also form a circle of radius $r\cos\theta$ on the sphere (where $\theta$ is the angle between the stick and the ground). The non-slipping condition then implies $\Omega R = \omega'(r\cos\theta)$. Therefore, $\omega' = \Omega R/(r\cos\theta)$, and

$$\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = -\Omega\hat{\mathbf{z}} + \left(\frac{\Omega R}{r\cos\theta}\right)(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}) = (R/r)\Omega\hat{\mathbf{x}}, \tag{8.168}$$

where we have used $\tan\theta = r/R$.

(b) Choose the pivot as the origin. The principal axes are then $\hat{\mathbf{x}}_3$ along the stick, along with any two directions orthogonal to the stick. Choose $\hat{\mathbf{x}}_2$ to be in the plane of the paper. Then the components of $\boldsymbol{\omega}$ along the principal axes are

$$\omega_3 = (R/r)\Omega\cos\theta, \qquad \text{and} \qquad \omega_2 = -(R/r)\Omega\sin\theta. \tag{8.169}$$

The principal moments are

$$I_3 = (2/5)mr^2, \qquad \text{and} \qquad I_2 = (2/5)mr^2 + m(r^2 + R^2), \tag{8.170}$$

where we have used the parallel-axis theorem. The angular momentum is $\mathbf{L} = I_3\omega_3\hat{\mathbf{x}}_3 + I_2\omega_2\hat{\mathbf{x}}_2$, so its horizontal component has length $L_\perp = I_3\omega_3\cos\theta - I_2\omega_2\sin\theta$. Therefore, the magnitude of $d\mathbf{L}/dt$ is

$$\begin{aligned}\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\ &= \Omega(I_3\omega_3\cos\theta - I_2\omega_2\sin\theta) \\ &= \Omega\left(\left(\frac{2}{5}mr^2\right)\left(\frac{R}{r}\Omega\cos\theta\right)\cos\theta \right.\\ &\qquad\left. -\left(\frac{2}{5}mr^2 + m(r^2+R^2)\right)\left(-\frac{R}{r}\Omega\sin\theta\right)\sin\theta\right) \\ &= \Omega^2 m\frac{R}{r}\left(\frac{2}{5}r^2 + (r^2+R^2)\sin^2\theta\right) \\ &= \frac{7}{5}mrR\Omega^2, \end{aligned} \tag{8.171}$$

where we have used $\sin\theta = r/\sqrt{r^2+R^2}$. The direction of $d\mathbf{L}/dt$ is out of the page.

Remark: There is actually a quicker way to calculate $d\mathbf{L}/dt$. At a given instant, the sphere is rotating around the horizontal $x$-axis with frequency $\omega = (R/r)\Omega$. The moment of inertia around this axis is $I_x = (7/5)mr^2$, from the parallel-axis theorem. Therefore, the horizontal component of $\mathbf{L}$ has magnitude

$$L_x = I_x\omega = \frac{7}{5}mrR\Omega. \tag{8.172}$$

Multiplying this by frequency (namely $\Omega$) at which $\mathbf{L}$ swings around the $z$-axis gives the result for $|d\mathbf{L}/dt|$ in eq. (8.171). Note that there is also a vertical component of $\mathbf{L}$ relative to the pivot,[27] but this component doesn't change, so it doesn't come into $d\mathbf{L}/dt$. ♣

The torque (relative to the pivot) is due to the gravitational force acting at the CM, along with the normal force, $N$, acting at the contact point. (Any horizontal friction at the contact point will yield zero torque relative to the pivot.) Therefore, $\boldsymbol{\tau}$ points out of the page with magnitude $|\boldsymbol{\tau}| = (N - mg)R$. Equating this with the $|d\mathbf{L}/dt|$ in eq. (8.171) gives

$$N = mg + \frac{7}{5}mr\Omega^2. \tag{8.173}$$

This has the interesting property of being independent of $R$ (and hence $\theta$).

Remark: The pivot must provide a downward force of $N - mg = (7/5)mr\Omega^2$, to make the net vertical force on the lollipop equal to zero. This result is slightly larger than the $mr\Omega^2$ result for the "sliding" situation in Exercise 6.

The sum of the horizontal forces at the pivot and the contact point must equal the required centripetal force of $mR\Omega^2$. But it is impossible to say how this force is divided up, without being given more information. ♣

22. **Rolling coin**

Choose the CM as the origin. The principal axes are then $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ (as shown in Fig. 8.74), along with $\hat{\mathbf{x}}_1$ pointing into the paper. Let $\Omega$ be the desired frequency.

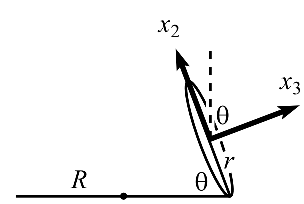


**Figure 8.74**

[27]This vertical component equals $-mR^2\Omega$, which makes intuitive sense. It can be obtained via the inertia tensor relative to the pivot, or by calculating $L_y = I_3\omega_3\sin\theta + I_2\omega_2\cos\theta$.

Look at things in the frame rotating around the $\hat{\mathbf{z}}$-axis with frequency $\Omega$. In this frame, the CM remains fixed, and the coin rotates with frequency $\omega'$ (in the language of Fig. 8.27) around the (negative) $\hat{\mathbf{x}}_3$-axis. The non-slipping condition says that $\omega' r = \Omega R$, and so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - \frac{R}{r}\Omega\hat{\mathbf{x}}_3. \tag{8.174}$$

But $\hat{\mathbf{z}} = \sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3$, so we may write $\boldsymbol{\omega}$ in terms of the principal axes as

$$\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2 - \Omega\left(\frac{R}{r} - \cos\theta\right)\hat{\mathbf{x}}_3. \tag{8.175}$$

The principal moments are

$$I_3 = (1/2)mr^2, \qquad \text{and} \qquad I_2 = (1/4)mr^2. \tag{8.176}$$

The angular momentum is $\mathbf{L} = I_2\omega_2\hat{\mathbf{x}}_2 + I_3\omega_3\hat{\mathbf{x}}_3$, so its horizontal component has length $L_\perp = I_2\omega_2\cos\theta - I_3\omega_3\sin\theta$, with leftward taken to be positive. Therefore, the magnitude of $d\mathbf{L}/dt$ is

$$\begin{aligned}\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\ &= \Omega(I_2\omega_2\cos\theta - I_3\omega_3\sin\theta) \\ &= \Omega\left(\left(\frac{1}{4}mr^2\right)\Big(\Omega\sin\theta\Big)\cos\theta - \left(\frac{1}{2}mr^2\right)\Big(-\Omega(R/r-\cos\theta)\Big)\sin\theta\right) \\ &= \frac{1}{4}mr\Omega^2\sin\theta(2R - r\cos\theta),\end{aligned} \tag{8.177}$$

with a positive quantity corresponding to $d\mathbf{L}/dt$ pointing out of the page (at the instant shown).

The torque (relative to the CM) comes from the force at the contact point. There are two components of this force. The vertical component is $mg$, and the horizontal component is $m(R - r\cos\theta)\Omega^2$ (pointing to the left), because the CM moves in a circle of radius $(R - r\cos\theta)$. We then find the torque to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.178}$$

with outward from the page taken to be positive. Equating this $|\boldsymbol{\tau}|$ with the $|d\mathbf{L}/dt|$ from eq. (8.177) gives

$$\Omega^2 = \frac{g}{\frac{3}{2}R\tan\theta - \frac{5}{4}r\sin\theta}. \tag{8.179}$$

The right-hand side must be positive if a solution for $\Omega$ is to exist. Therefore, the condition for the desired motion to be possible is

$$R > \frac{5}{6}r\cos\theta. \tag{8.180}$$

Remarks:

(a) For $\theta \to \pi/2$, eq. (8.179) gives $\Omega \to 0$, as it should. And for $\theta \to 0$, we obtain $\Omega \to \infty$, which also makes sense.

(b) Note that for $(5/6)r\cos\theta < R < r\cos\theta$, the CM of the coin lies to the *left* of the center of the contact-point circle (at the instant shown). The centripetal force, $m(R - r\cos\theta)\Omega^2$, is therefore negative (which means that it is directed radially outward, to the right), but the motion is still possible. As $R$ gets close to $(5/6)r\cos\theta$, the frequency $\Omega$ goes to infinity, which means that the radially outward force also goes to infinity. The coefficient of friction between the coin and the ground must therefore be correspondingly large.

(c) We may consider a more general coin, whose density depends on only the distance from the center, and which has $I_3 = \eta mr^2$. (For example, a uniform coin has $\eta = 1/2$, and a coin with all its mass on the edge has $\eta = 1$.) By the perpendicular axis theorem, $I_1 = I_2 = (1/2)\eta mr^2$, and you can show that the above methods yield

$$\Omega^2 = \frac{g}{(1+\eta)R\tan\theta - (1+\eta/2)r\sin\theta}. \tag{8.181}$$

The condition for such motion to exist is then

$$R > \left(\frac{1+\eta/2}{1+\eta}\right) r\cos\theta. \;\clubsuit \tag{8.182}$$

23. **Wobbling coin**

(a) Look at the setup in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the location of the contact point remains fixed, and the coin rotates with frequency $\omega'$ (in the language of Fig. 8.27) around the negative $\hat{\mathbf{x}}_3$ axis. The radius of the circle of contact points on the table is $R\cos\theta$. Therefore, the non-slipping condition says that $\omega' R = \Omega(R\cos\theta)$, and so $\omega' = \Omega\cos\theta$. Hence, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega(\sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3) - (\Omega\cos\theta)\hat{\mathbf{x}}_3 = \Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.183}$$

In retrospect, it makes sense that $\boldsymbol{\omega}$ must point in the $\hat{\mathbf{x}}_2$ direction. Both the CM and the instantaneous contact point on the coin are at rest, so $\boldsymbol{\omega}$ must lie along the line containing these two points (that is, along the $\hat{\mathbf{x}}_2$-axis).

(b) Choose the CM as the origin. The principal moment around the $\hat{\mathbf{x}}_2$-axis is $I = mR^2/4$. The angular momentum is $\mathbf{L} = I\omega_2\hat{\mathbf{x}}_2$, so its horizontal component has length $L_\perp = L\cos\theta = (I\omega_2)\cos\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega\left(\frac{mR^2}{4}\right)(\Omega\sin\theta)\cos\theta, \tag{8.184}$$

and it points out of the page.

The torque (relative to the CM) is due to the normal force at the contact point (there is no sideways friction force at the contact point, because the CM is motionless), so it has magnitude

$$|\boldsymbol{\tau}| = mgR\cos\theta, \tag{8.185}$$

and it also points out of the page. Using the previous two equations, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}. \tag{8.186}$$

REMARKS:

i. $\Omega \to \infty$ as $\theta \to 0$. This is quite evident if you do the experiment; the contact point travels very quickly around the circle.

ii. $\Omega \to 2\sqrt{g/R}$ as $\theta \to \pi/2$. This isn't so intuitive (to me, at least). In this case, **L** points nearly vertically, and it traces out a tiny cone, due to a tiny torque.

iii. In this limit $\theta \to \pi/2$, $\Omega$ is also the frequency at which the plane of the coin spins around the vertical axis. If you spin a coin very fast about a vertical diameter, it will initially undergo a pure spinning motion with only one contact point. It will then gradually lose energy due to friction, until the spinning frequency slows down to $2\sqrt{g/R}$, at which time it will begin to wobble. (We're assuming, of course, that the coin is very thin, so that it can't balance on its edge.) In the case where the coin is a quarter (with $R \approx .012\,\text{m}$), this critical frequency of $2\sqrt{g/R}$ turns out to be $\Omega \approx 57\,\text{rad/s}$, which corresponds to about 9 Hertz.

iv. The result in eq. (8.186) is a special case of the result in eq. (8.179) of Problem 22. The CM of the coin in Problem 22 will be motionless if $R = r\cos\theta$. Plugging this into eq. (8.179) gives $\Omega^2 = 4g/(r\sin\theta)$, which agrees with eq. (8.186), because $r$ was the coin's radius in Problem 22. ♣

(c) Consider one revolution of the point of contact around the $\hat{\mathbf{z}}$-axis. Since the radius of the circle on the table is $R\cos\theta$, the contact point moves a distance $2\pi R\cos\theta$ around the coin during this time. Hence, the new point of contact on the coin is a distance $2\pi R - 2\pi R\cos\theta$ away from the original point of contact. The coin therefore appears to have rotated by a fraction $(1-\cos\theta)$ of a full turn during this time. The frequency with which you see it turn is therefore

$$(1-\cos\theta)\Omega = \frac{2(1-\cos\theta)}{\sqrt{\sin\theta}}\sqrt{\frac{g}{R}}\,. \tag{8.187}$$

REMARKS:

i. If $\theta \approx \pi/2$, then the frequency of Abe's rotation is essentially equal to $\Omega$. This makes sense, because the top of Abe's head will be, say, always near the top of the coin, and this point will trace out a small circle around the $\hat{\mathbf{z}}$-axis, with nearly the same frequency as the contact point.

ii. As $\theta \to 0$, Abe appears to rotate with frequency $\theta^{3/2}\sqrt{g/R}$ (using $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$). Therefore, although the contact point moves infinitely quickly in this limit, we nevertheless see Abe rotating infinitely slowly.

iii. All of the results for frequencies in this problem have to look like some multiple of $\sqrt{g/R}$, by dimensional analysis. But whether the multiplication factor is zero, infinite, or something in between, is not at all obvious.

iv. An incorrect answer for the frequency of Abe's turning (when viewed from above) is that it equals the vertical component of $\vec{\omega}$, which is $\omega_z = \omega\sin\theta = (\Omega\sin\theta)\sin\theta = 2(\sin\theta)^{3/2}\sqrt{g/R}$. This does not equal the result in eq. (8.187). (It agrees at $\theta = \pi/2$, but is off by a factor of 2 for $\theta \to 0$.) This answer is incorrect because there is simply no reason why the vertical component of $\vec{\omega}$ should equal the frequency of revolution of, say, Abe's nose, around the vertical axis. For example, at moments when $\vec{\omega}$ passes through the nose, then the nose isn't moving at all, so it certainly cannot be described as moving around the vertical axis with frequency $\omega_z = \omega\sin\theta$.

The result in eq. (8.187) is a sort of average measure of the frequency of rotation. Even though any given point on the coin is not undergoing uniform circular motion, your eye will see the whole coin (approximately) rotating uniformly. ♣

24. **Nutation cusps**

(a) Because both $\dot\phi$ and $\dot\theta$ are continuous functions of time, we must have $\dot\phi = \dot\theta = 0$ at a kink. (Otherwise, either $d\theta/d\phi = \dot\theta/\dot\phi$ or $d\phi/d\theta = \dot\phi/\dot\theta$ would be well-defined at the kink.) Let the kink occur at $t = t_0$. Then the second of eqs. (8.89) gives $\sin(\omega_n t_0) = 0$. Therefore, $\cos(\omega_n t_0) = \pm 1$, and the first of eqs. (8.89) gives

$$\Delta\Omega = \mp\Omega_s, \qquad (8.188)$$

as we wanted to show.

REMARK: Note that if $\cos(\omega_n t_0) = 1$, then $\Delta\Omega = -\Omega_s$, so eq. (8.88) says that the kink occurs at the smallest value of $\theta$, that is, at the highest point of the top's motion. And if $\cos(\omega_n t_0) = -1$, then $\Delta\Omega = \Omega_s$, so eq. (8.88) again says that the kink occurs at the highest point of the top's motion. ♣

(b) To show that these kinks are cusps, we will show that the slope of the $\theta$ vs. $\phi$ plot is infinite on either side of the kink. That is, we will show that $d\theta/d\phi = \dot\theta/\dot\phi = \pm\infty$. For simplicity, we will look at the case where $\cos(\omega_n t_0) = 1$ and $\Delta\Omega = -\Omega_s$. (The $\cos(\omega_n t_0) = -1$ case proceeds the same.) With $\Delta\Omega = -\Omega_s$, eqs. (8.89) give

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \sin\omega_n t}{1 - \cos\omega_n t}\,. \qquad (8.189)$$

Letting $t = t_0 + \epsilon$, we have (using $\cos(\omega_n t_0) = 1$ and $\sin(\omega_n t_0) = 0$, and expanding to lowest order in $\epsilon$)

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \omega_n \epsilon}{\omega_n^2 \epsilon^2/2} = \frac{2\sin\theta_0}{\omega_n \epsilon}\,. \qquad (8.190)$$

For infinitesimal $\epsilon$, this switches from $-\infty$ to $+\infty$ as $\epsilon$ passes through zero.

25. **Nutation circles**

(a) Note that a change in angular speed of $\Delta\Omega$ around the fixed $\hat{\mathbf{z}}$-axis corresponds to a change in angular speed of $\sin\theta_0\Delta\Omega$ around the $\hat{\mathbf{x}}_2$-axis. The kick therefore produces an angular momentum component (relative to the pivot) of $I\sin\theta_0\Delta\Omega$ in the $\hat{\mathbf{x}}_2$ direction.

The original $\mathbf{L}$ pointed along the $\hat{\mathbf{x}}_3$ direction, with magnitude $I_3\omega_3$. (These two statements hold to a good approximation if $\omega_3 \gg \Omega_s$.) By definition, $\mathbf{L}$ made an angle $\theta_0$ with the vertical $\hat{\mathbf{z}}$-axis. Therefore, from Fig. 8.75, the angle that the new $\mathbf{L}$ makes with the $\hat{\mathbf{z}}$-axis is (using $\Delta\Omega \ll \omega_3$)

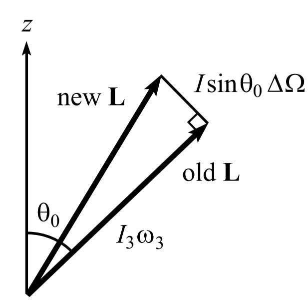


**Figure 8.75**

$$\theta_0' = \theta_0 - \frac{I\sin\theta_0\Delta\Omega}{I_3\omega_3} \equiv \theta_0 - \frac{\sin\theta_0\Delta\Omega}{\omega_n}\,, \qquad (8.191)$$

where we have used the definition of $\omega_n$ from eq. (8.83). We see that the effect of the kick is to make $\mathbf{L}$ quickly change its $\theta$ value. (It changes by only a small amount, because we are assuming $\omega_n \sim \omega_3 \gg \Delta\Omega$). The $\phi$ value doesn't immediately change.

(b) The torque (relative to the pivot) has magnitude $Mg\ell\sin\theta$ and is directed horizontally. Because $\theta$ doesn't change appreciably, the magnitude of the torque is essentially constant, so $\mathbf{L}$ traces out a circle at a constant rate. This rate is simply $\Omega_s$. (None of the relevant quantities in $\boldsymbol{\tau} = d\mathbf{L}/dt$ changed much from the original circular-precession case, so the precession frequency remains basically the same.) Therefore, the new $\mathbf{L}$ has its $(\phi, \theta)$ coordinates given by

$$(\phi(t), \theta(t))_{\mathbf{L}} = \left( \Omega_s t,\ \theta_0 - \frac{\sin\theta_0 \Delta\Omega}{\omega_n} \right). \tag{8.192}$$

Looking at eqs. (8.88), we see that the coordinates of the CM relative to $\mathbf{L}$ are

$$(\phi(t), \theta(t))_{\text{CM}-\mathbf{L}} = \left( \left( \frac{\Delta\Omega}{\omega_n} \right) \sin\omega_n t, \left( \frac{\Delta\Omega}{\omega_n} \sin\theta_0 \right) \cos\omega_n t \right). \tag{8.193}$$

The $\sin\theta_0$ factor in $\theta(t)$ is exactly what is needed for the CM to trace out a circle around $\mathbf{L}$, because a change in $\phi$ corresponds to a CM spatial change of $\ell\Delta\phi\sin\theta_0$, whereas a change in $\theta$ corresponds to a CM spatial change of $\ell\Delta\theta$.

This circular motion is exactly what we expect from the results in Section 8.6.2, for the following reason. For $\omega_n$ very large and $\Omega_s$ very small, $\mathbf{L}$ is essentially motionless, and the CM traces out a circle around it at frequency $\omega_n$. Since $\mathbf{L}$ is essentially constant, the top should therefore behave very much like a free top, as viewed from a fixed frame.

Eq. (8.53) in Section 8.6.2 says that the frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ for a free top is $L/I$. But the frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ in the present problem is $\omega_n$, so this had better be equal to $L/I$. And indeed, $L$ is essentially equal to $I_3\omega_3$, so $L/I = I_3\omega_3/I \equiv \omega_n$.

Therefore, for short enough time scales (short enough so that $\mathbf{L}$ doesn't move much), a nutating top with $\omega_3 \gg \Delta\Omega \gg \Omega_s$ looks very much like a free top.

Remark: We need the $\Delta\Omega \gg \Omega_s$ condition so that the nutation motion looks like circles. This can be seen by the following reasoning. The time for one period of the nutation motion is $2\pi/\omega_n$. From eq. (8.88), $\phi(t)$ increases by $\Delta\phi = 2\pi\Omega_s/\omega_n$ in this time. And also from eq. (8.88), the width, $w$, of the "circle" along the $\phi$ axis is roughly $w = 2\Delta\Omega/\omega_n$. The motion looks like basically like a circle if $w \gg \Delta\phi$, that is, if $\Delta\Omega \gg \Omega_s$. ♣

26. **Rolling straight?**

Intuitively, it is fairly clear that the sphere cannot change direction. But it is a little tricky to prove. Qualitatively, we can reason as follows. Assume there is a nonzero friction force at the contact point. (The normal force is irrelevant here, because it doesn't provide a torque about the CM. This is what is special about a sphere.) Then the ball will accelerate in the direction of this force. However, you can show with the right-hand rule that this force will produce a torque that will cause the angular momentum to change in a way that corresponds to the ball accelerating in the direction *opposite* to the friction force. There is thus a contradiction, unless the friction force equals zero.

Let's now be rigorous. Let the angular velocity of the ball be $\boldsymbol{\omega}$. The non-slipping condition says that the ball's velocity equals

$$\mathbf{v} = \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.194}$$

where $a$ is the radius of the sphere. The ball's angular momentum is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{8.195}$$

The friction force from the ground (if it exists) at the contact point will change both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{8.196}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{8.197}$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will now show that the preceding four equations imply $\dot{\boldsymbol{\omega}} = \mathbf{0}$, that is, $\mathbf{v}$ is constant. Use the $\mathbf{v}$ from eq. (8.194) in eq. (8.196), and then plug the resulting $\mathbf{F}$ into eq. (8.197). Also, use the $\mathbf{L}$ from eq. (8.195) in eq. (8.197). The result is

$$(-a\hat{\mathbf{z}}) \times \big(m\dot{\boldsymbol{\omega}} \times (a\hat{\mathbf{z}})\big) = I\dot{\boldsymbol{\omega}}. \tag{8.198}$$

Because the vector $\dot{\boldsymbol{\omega}}$ lies in the horizontal plane, you can work out that $\hat{\mathbf{z}}\times(\dot{\boldsymbol{\omega}}\times\hat{\mathbf{z}}) = \dot{\boldsymbol{\omega}}$. Therefore, we have

$$-ma^2\dot{\boldsymbol{\omega}} = \frac{2}{5}ma^2\dot{\boldsymbol{\omega}}, \tag{8.199}$$

and hence $\dot{\boldsymbol{\omega}} = \mathbf{0}$, as we wanted to show.

27. **Ball on paper**

Our strategy will be to produce (and equate) two different expressions for the total change in the angular momentum of the ball (relative to its center). The first comes from the effects of the friction force on the ball. The second comes from looking at the initial and final motion.

To produce our first expression for $\Delta\mathbf{L}$, note that the normal force provides no torque, so we may ignore it. The friction force, $\mathbf{F}$, from the paper changes both $\mathbf{p}$ and $\mathbf{L}$, according to,

$$\begin{aligned} \Delta\mathbf{p} &= \int \mathbf{F}\,dt, \\ \Delta\mathbf{L} &= \int \boldsymbol{\tau}\,dt = \int (-R\hat{\mathbf{z}}) \times \mathbf{F}\,dt = (-R\hat{\mathbf{z}}) \times \int \mathbf{F}\,dt. \end{aligned} \tag{8.200}$$

Both of these integrals run over the entire slipping time, which may include time on the table after the ball leaves the paper. In the second line above, we have used the fact that the friction force always acts at the same location, namely $(-R\hat{\mathbf{z}})$, relative to the center of the ball. The two above equations yield

$$\Delta\mathbf{L} = (-R\hat{\mathbf{z}}) \times \Delta\mathbf{p}. \tag{8.201}$$

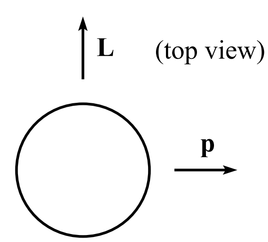

Figure 8.76

To produce our second equation for $\Delta\mathbf{L}$, let's examine how $\mathbf{L}$ is related to $\mathbf{p}$ when the ball is rolling without slipping, which is the case at both the start and the finish. When the ball is not slipping, we have the situation show in Fig. 8.76 (assuming that the ball is rolling to the right). The magnitudes of $p$ and $L$ are given by

$$\begin{aligned} p &= mv, \\ L &= I\omega = \frac{2}{5}mR^2\omega = \frac{2}{5}Rm(R\omega) = \frac{2}{5}Rmv = \frac{2}{5}Rp, \end{aligned} \tag{8.202}$$

where we have used the non-slipping condition, $v = R\omega$. (The actual $I = (2/5)mR^2$ value for a solid sphere will not be important for the final result.) We now note that the directions of $\mathbf{L}$ and $\mathbf{p}$ can be combined with the above $L = 2Rp/5$ scalar relation to give

$$\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}} \times \mathbf{p}, \tag{8.203}$$

where $\hat{\mathbf{z}}$ points out of the page. Since this relation is true at both the start and the finish, it must also be true for the differences in $\mathbf{L}$ and $\mathbf{p}$. That is,

$$\Delta\mathbf{L} = \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p}. \tag{8.204}$$

Eqs. (8.201) and (8.204) give

$$\begin{aligned} (-R\hat{\mathbf{z}}) \times \Delta\mathbf{p} &= \frac{2}{5}R\hat{\mathbf{z}} \times \Delta\mathbf{p} \\ \Longrightarrow \quad 0 &= \hat{\mathbf{z}} \times \Delta\mathbf{p}. \end{aligned} \tag{8.205}$$

There are three ways this cross product can be zero:

- $\Delta\mathbf{p}$ is parallel to $\hat{\mathbf{z}}$. But it isn't, because $\Delta\mathbf{p}$ lies in the horizontal plane.
- $\hat{\mathbf{z}} = 0$. Not true.
- $\Delta\mathbf{p} = 0$. This one must be true. Therefore, $\Delta\mathbf{v} = 0$, as we wanted to show.

REMARKS:

(a) As stated in the problem, it's fine if you move the paper in a jerky motion, so that the ball slips around on it. We assumed nothing about the nature of the friction force in the above reasoning. And we used the non-slipping condition only at the initial and final times. The intermediate motion is arbitrary.

(b) As a special case, if you start a ball at rest on a piece of paper, then no matter how you choose to (horizontally) slide the paper out from underneath the ball, the ball will be at rest in the end.

(c) You are encouraged to experimentally verify that all these crazy claims are true. Make sure that the paper doesn't wrinkle (this would allow a force to be applied at a point other than the contact point). And balls that don't squish are much better, of course (for the same reason). ♣

28. **Ball on turntable**

Let the angular velocity of the turntable be $\Omega\hat{\mathbf{z}}$, and let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity with respect to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$, where $a$ is the radius of the ball. The ball's velocity with respect to the lab frame is therefore

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.206}$$

The important point to realize in this problem is that the friction force from the turntable is responsible for changing both the ball's linear momentum and its angular momentum. In particular, $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,. \tag{8.207}$$

And the angular momentum of the ball is $\mathbf{L} = I\boldsymbol{\omega}$, so $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = I\frac{d\boldsymbol{\omega}}{dt}\,, \tag{8.208}$$

because the force is applied at position $-a\hat{\mathbf{z}}$ relative to the center.

We will now use the previous three equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form,

$$\frac{d\mathbf{v}}{dt} = \Omega'\hat{\mathbf{z}} \times \mathbf{v}, \tag{8.209}$$

since this describes circular motion, with frequency $\Omega'$ (to be determined). Plugging the expression for $\mathbf{F}$ from eq. (8.207) into eq. (8.208) gives

$$\begin{aligned} (-a\hat{\mathbf{z}}) \times \left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\,. \end{aligned} \tag{8.210}$$

Taking the derivative of eq. (8.206) gives

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\right) \times (a\hat{\mathbf{z}}). \end{aligned} \tag{8.211}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{ma^2}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (ma^2/I)}\right)\hat{\mathbf{z}} \times \mathbf{v}. \end{aligned} \tag{8.212}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \mathbf{v}. \tag{8.213}$$

Therefore, in view of eq. (8.209), we see that the ball undergoes circular motion, with a frequency equal to $2/7$ times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

Remarks:

(a) Integrating eq. (8.213) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \tag{8.214}$$

This may be written (as you can show) in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \tag{8.215}$$

This equation describes circular motion, with the center located at the point,

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0), \tag{8.216}$$

and with radius,

$$R = |\mathbf{r}_0 - \mathbf{r}_c| = \frac{7}{2\Omega}|\hat{\mathbf{z}} \times \mathbf{v}_0| = \frac{7v_0}{2\Omega}\,. \tag{8.217}$$

(b) There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then $R = 0$ and the ball remains in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$. The radius of the circle is therefore $R = (7/2)r_0$, and its center is located at (from eq. (8.216))

$$\mathbf{r}_c = \mathbf{r}_0 + \frac{7}{2\Omega}(-\Omega\mathbf{r}_0) = -\frac{5\mathbf{r}_0}{2}\,. \tag{8.218}$$

  The point on the circle diametrically opposite to the initial point is therefore at a radius $r_c + R = (5/2)r_0 + (7/2)r_0 = 6r_0$.
- If we want the center of the circle be the center of the turntable, then eq. (8.216) says that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

(c) The fact that the frequency $(2/7)\Omega$ is a rational multiple of $\Omega$ means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. From the point of view of someone on the turntable, the ball will "spiral" around five times before returning to the original position.

(d) If we look at a ball with moment of inertia $I = \eta ma^2$ (so a uniform sphere has $\eta = 2/5$), then you can show that the "2/7" in eq. (8.213) gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to $\infty$ (as long as $v_0 \neq 0$). ♣

# Chapter 9

# Accelerated Frames of Reference



Newton's laws hold only in inertial frames of reference. However, there are many non-inertial (that is, accelerated) frames of reference that we might reasonably want to study, such as elevators, merry-go-rounds, and so on. Is there any possible way to modify Newton's laws so that they hold in non-inertial frames, or do we have to give up entirely on $\mathbf{F} = m\mathbf{a}$?

It turns out that we can indeed hold onto our good friend $\mathbf{F} = m\mathbf{a}$, provided that we introduce some new "fictitious" forces. These are forces that a person in the accelerated frame thinks exist. If she applies $\mathbf{F} = m\mathbf{a}$ while including these new forces, then she will get the correct answer for the acceleration, $\mathbf{a}$, as measured with respect to her frame.

To be quantitative about all this, we'll have to spend some time determining how the coordinates (and their derivatives) in an accelerated frame relate to those in an inertial frame. But before diving into that, let's look at a simple example which demonstrates the basic idea of fictitious forces.

---

**Example (The train):** Imagine that you are standing on a train that is accelerating to the right with acceleration $a$. If you wish to remain at the same spot on the train, there must be a friction force between the floor and your feet, with magnitude $F_\text{f} = ma$, pointing to the right. Someone standing in the inertial frame of the ground will simply interpret the situation as, "The friction force, $F_\text{f} = ma$, causes your acceleration, $a$."

How do you interpret the situation, in the frame of the train? (Imagine that there are no windows, so that all you see is the inside of the train.) As we will show below in eq. (9.11), you will feel a fictitious *translation* force, $F_\text{trans} = -ma$, pointing to the left. You will therefore interpret the situation as, "In my frame (the frame of the train), the friction force $F_\text{f} = ma$ pointing to my right exactly cancels the mysterious $F_\text{trans} = -ma$ force pointing to my left, resulting in zero acceleration (in my frame)."

Of course, if the floor of the train is frictionless so that there is no force at your feet, then you will say that the net force on you is $F_\text{trans} = -ma$, pointing to the left. You will therefore accelerate with acceleration $a$ to the left, with respect to your frame (the train). In other words, you will remain motionless with respect to the inertial frame of the ground, which is all quite obvious to someone standing on the ground.

In the case where the friction force at your feet is nonzero, but not large enough to balance out the whole $F_{\text{trans}} = -ma$ force, you will end up being jerked toward the back of the train. This undesired motion will continue until you make some adjustments with your feet or hands in order to balance out all of the $F_{\text{trans}}$ force. Such adjustments are generally necessary on a subway train, at least here in Boston, where hands are often crucial.

---

Let's now derive the fictitious forces in their full generality. The main task here is to relate the coordinates in an accelerated frame to those in an inertial frame, so this endeavor will require a bit of math.

## 9.1 Relating the coordinates

Consider an inertial coordinate system with axes $\hat{\mathbf{x}}_{\text{I}}$, $\hat{\mathbf{y}}_{\text{I}}$, and $\hat{\mathbf{z}}_{\text{I}}$, and let there be another (possibly accelerating) coordinate system with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. These axes will be allowed to change in an arbitrary manner with respect to the inertial frame. That is, the origin may undergo acceleration, and the axes may rotate. (This is the most general possible motion, as we saw in Section 8.1.) These axes may be considered to be functions of the inertial axes. Let $O_{\text{I}}$ and $O$ be the origins of the two coordinate systems.

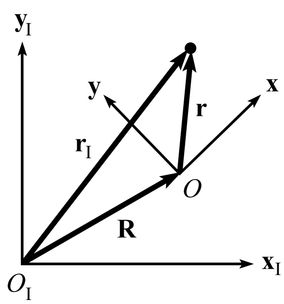

**Figure 9.1**

Let the vector from $O_{\text{I}}$ to $O$ be $\mathbf{R}$. Let the vector from $O_{\text{I}}$ to a given particle be $\mathbf{r}_{\text{I}}$. And let the vector from $O$ to the particle be $\mathbf{r}$. (See Fig. 9.1 for the 2-D case.) Then

$$\mathbf{r}_{\text{I}} = \mathbf{R} + \mathbf{r}. \tag{9.1}$$

These vectors have an existence that is independent of any specific coordinate system, but let us write them in terms of some definite coordinates. We may write

$$\begin{aligned} \mathbf{R} &= X\hat{\mathbf{x}}_{\text{I}} + Y\hat{\mathbf{y}}_{\text{I}} + Z\hat{\mathbf{z}}_{\text{I}}, \\ \mathbf{r}_{\text{I}} &= x_{\text{I}}\hat{\mathbf{x}}_{\text{I}} + y_{\text{I}}\hat{\mathbf{y}}_{\text{I}} + z_{\text{I}}\hat{\mathbf{z}}_{\text{I}}, \\ \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \end{aligned} \tag{9.2}$$

For reasons that will become clear, we have chosen to write $\mathbf{R}$ and $\mathbf{r}_{\text{I}}$ in terms of the inertial-frame coordinates, and $\mathbf{r}$ in terms of the accelerated-frame coordinates. If desired, eq. (9.1) may be written as

$$x_{\text{I}}\hat{\mathbf{x}}_{\text{I}} + y_{\text{I}}\hat{\mathbf{y}}_{\text{I}} + z_{\text{I}}\hat{\mathbf{z}}_{\text{I}} = (X\hat{\mathbf{x}}_{\text{I}} + Y\hat{\mathbf{y}}_{\text{I}} + Z\hat{\mathbf{z}}_{\text{I}}) + (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\,. \tag{9.3}$$

Our goal is to take the second time derivative of eq. (9.1), and to interpret the result in an $\mathbf{F} = m\mathbf{a}$ form. The second derivative of $\mathbf{r}_{\text{I}}$ is simply the acceleration of the particle with respect to the inertial system, and so Newton's second law tells us that $\mathbf{F} = m\ddot{\mathbf{r}}_{\text{I}}$. The second derivative of $\mathbf{R}$ is the acceleration of the origin of the moving system. The second derivative of $\mathbf{r}$ is the tricky part. Changes in $\mathbf{r}$ can come about in two ways. First, the coordinates $(x, y, z)$ of $\mathbf{r}$ (which are measured with respect to the moving axes) may change. And second, the axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ themselves may change. So even if $(x, y, z)$ remain fixed, the $\mathbf{r}$ vector may still change.[1] Let's be quantitative about this.

### Calculation of $d^2\mathbf{r}/dt^2$

We should clarify our goal here. We would like to obtain $d^2\mathbf{r}/dt^2$ in terms of the coordinates in the *moving* frame, because we want to be able to work entirely in terms of the coordinates of the accelerated frame, so that a person in this frame can write down an $\mathbf{F} = m\mathbf{a}$ equation in terms of only her coordinates, without having to consider the underlying inertial frame at all.[2]

The following exercise in taking derivatives works for a general vector $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ in the moving frame (it's not necessary that it be a position vector). So we'll work with a general $\mathbf{A}$ and then set $\mathbf{A} = \mathbf{r}$ when we're done.

To take $d/dt$ of $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$, we can use the product rule to obtain

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}\hat{\mathbf{x}} + \frac{dA_y}{dt}\hat{\mathbf{y}} + \frac{dA_z}{dt}\hat{\mathbf{z}}\right) + \left(A_x\frac{d\hat{\mathbf{x}}}{dt} + A_y\frac{d\hat{\mathbf{y}}}{dt} + A_z\frac{d\hat{\mathbf{z}}}{dt}\right). \tag{9.4}$$

Yes, the product rule works with vectors too. We're doing nothing more than expanding $(A_x + dA_x)(\hat{\mathbf{x}} + d\hat{\mathbf{x}}) - A_x\hat{\mathbf{x}}$, etc., to first order.

The first of the two terms in eq. (9.4) is simply the rate of change of $\mathbf{A}$, as measured with respect to the moving frame. We will denote this quantity by $\delta\mathbf{A}/\delta t$.

The second term arises because the coordinate axes are moving. In what manner are they moving? We have already extracted the motion of the origin of the moving system (by introducing the vector $\mathbf{R}$), so the only thing left is a rotation about some axis $\boldsymbol{\omega}$ through the origin (see Theorem 8.1). This axis may be changing in time, but at any instant a unique axis of rotation describes the system. The fact that the axis may change will be relevant in finding the second derivative of $\mathbf{r}$, but not in finding the first derivative.

We saw in Theorem 8.2 that a vector $\mathbf{B}$ of fixed length (the coordinate axes here do indeed have fixed length), rotating with angular velocity $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, changes at a rate $d\mathbf{B}/dt = \boldsymbol{\omega} \times \mathbf{B}$. In particular, $d\hat{\mathbf{x}}/dt = \boldsymbol{\omega} \times \hat{\mathbf{x}}$, etc. So in eq. (9.4), the $A_x(d\hat{\mathbf{x}}/dt)$ term, for example, equals $A_x(\boldsymbol{\omega} \times \hat{\mathbf{x}}) = \boldsymbol{\omega} \times (A_x\hat{\mathbf{x}})$. Adding on the $y$ and $z$ terms gives $\boldsymbol{\omega} \times (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) = \boldsymbol{\omega} \times \mathbf{A}$. Therefore, eq. (9.4) yields

$$\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega} \times \mathbf{A}. \tag{9.5}$$

This agrees with the result obtained in Section 8.5, eq. (8.39). We've basically given the same proof here, but with a little more mathematical rigor.

[1] Remember, the $\mathbf{r}$ vector is *not* simply the ordered triplet $(x, y, z)$. It is the whole expression, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. So even if $(x, y, z)$ are fixed, meaning that $\mathbf{r}$ doesn't change with respect to the moving system, $\mathbf{r}$ can still change with respect to the inertial system if the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes are themselves moving.

[2] In terms of the *inertial* frame, $d^2\mathbf{r}/dt^2$ is simply $d^2(\mathbf{r}_{\mathrm{I}} - \mathbf{R})/dt^2$, but this is not very enlightening by itself.

We still have to take one more time derivative. The time derivative of eq. (9.5) yields

$$\frac{d^2\mathbf{A}}{dt^2} = \frac{d}{dt}\left(\frac{\delta\mathbf{A}}{\delta t}\right) + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A} + \boldsymbol{\omega}\times\frac{d\mathbf{A}}{dt}\,. \tag{9.6}$$

Applying eq. (9.5) to the first term (with $\delta\mathbf{A}/\delta t$ in place of $\mathbf{A}$), and plugging eq. (9.5) into the third term, gives

$$\begin{aligned}\frac{d^2\mathbf{A}}{dt^2} &= \left(\frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t}\right) + \left(\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A}\right) + \left(\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})\right)\\ &= \frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A}) + 2\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A}.\end{aligned} \tag{9.7}$$

At this point, we will now set $\mathbf{A} = \mathbf{r}$, so we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\delta^2\mathbf{r}}{\delta t^2} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}, \tag{9.8}$$

where $\mathbf{v} \equiv \delta\mathbf{r}/\delta t$ is the velocity of the particle, as measured with respect to the moving frame.

## 9.2 The fictitious forces

From eq. (9.1) we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}_{\mathrm{I}}}{dt^2} - \frac{d^2\mathbf{R}}{dt^2}\,. \tag{9.9}$$

Let us equate this expression for $d^2\mathbf{r}/dt^2$ with the one in eq. (9.8), and then multiply through by the mass $m$ of the particle. Recognizing that the $m(d^2\mathbf{r}_{\mathrm{I}}/dt^2)$ term is simply the force $\mathbf{F}$ acting on the particle ($\mathbf{F}$ may be gravity, a normal force, friction, tension, etc.), we may write the result as

$$\begin{aligned}m\frac{\delta^2\mathbf{r}}{\delta t^2} &= \mathbf{F} - m\frac{d^2\mathbf{R}}{dt^2} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}\\ &\equiv \mathbf{F} + \mathbf{F}_{\text{translation}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{azimuthal}},\end{aligned} \tag{9.10}$$

where the *fictitious forces* are defined as

$$\begin{aligned}\mathbf{F}_{\text{trans}} &\equiv -m\frac{d^2\mathbf{R}}{dt^2}\,,\\ \mathbf{F}_{\text{cent}} &\equiv -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}),\\ \mathbf{F}_{\text{cor}} &\equiv -2m\boldsymbol{\omega}\times\mathbf{v},\\ \mathbf{F}_{\text{az}} &\equiv -m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}.\end{aligned} \tag{9.11}$$

We have taken the liberty of calling these quantities "forces", because the left-hand side of eq. (9.10) is simply $m$ times the acceleration, as measured by someone in the accelerated frame. This person should therefore be able to interpret the right-hand side as some effective force. In other words, if a person in the accelerated frame wishes to calculate $m\mathbf{a}_{\text{acc}} \equiv m(\delta^2\mathbf{r}/\delta t^2)$, she simply needs to take the true force $\mathbf{F}$, and then add on all the other terms on the right-hand side, which she will then quite reasonably interpret as forces (in her frame). She will interpret eq. (9.10) as an $\mathbf{F} = m\mathbf{a}$ statement in the form,

$$m\mathbf{a}_{\text{acc}} = \sum \mathbf{F}_{\text{acc}}. \tag{9.12}$$

Note that the extra terms in eq. (9.10) are not actual forces. The constituents of $\mathbf{F}$ are the only real forces in the problem. All we are saying is that if our friend in the moving frame assumes the extra terms are real forces, and if she then adds them to $\mathbf{F}$, then she will get the correct answer for $m(\delta^2\mathbf{r}/\delta t^2)$, the mass times acceleration in her frame.

For example, consider a box (far away from other objects, in outer space) that accelerates at a rate of $g = 10\,\text{m/s}^2$ in some direction. A person in the box will feel a fictitious force of $\mathbf{F}_{\text{trans}} = mg$ down into the floor. For all she knows, she is in a box on the surface of the earth. If she performs various experiments under this assumption, the results will always be what she expects. The surprising fact that no local experiment can distinguish between the fictitious force in the accelerated box and the real gravitational force on the earth is what led Einstein to his Equivalence Principle and his theory of General Relativity (discussed in Chapter 13). These fictitious forces are more meaningful than you might think.

As Einstein explored elevators,
And studied the spinning ice-skaters,
He eyed as suspicious,
The forces, fictitious,
Of gravity's great imitators.

Let's now look at each of the fictitious forces in detail. The translational and centrifugal forces are fairly easy to understand. The Coriolis force is a little more difficult. And the azimuthal force can be easy or difficult, depending on how exactly $\boldsymbol{\omega}$ is changing (we'll mainly deal with the easy case).

### 9.2.1 Translation force: $-md^2\mathbf{R}/dt^2$

This is the most intuitive of the fictitious forces. We've already discussed this force in the train example in the introduction to this chapter. If $\mathbf{R}$ is the position of the train, then $\mathbf{F}_{\text{trans}} \equiv -m\,d^2\mathbf{R}/dt^2$ is the fictitious force you feel in the accelerated frame.

### 9.2.2 Centrifugal force: $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$

This force goes hand-in-hand with the $mv^2/r = mr\omega^2$ centripetal acceleration as viewed by someone in an inertial frame.

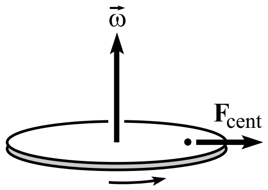


**Figure 9.2**

**Example 1 (Standing on a carousel):** Consider a person standing motionless on a carousel. Let the carousel rotate in the $x$-$y$ plane with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ (see Fig. 9.2). What is the centrifugal force felt by a person standing at a distance $r$ from the center?

**Solution:** $\boldsymbol{\omega} \times \mathbf{r}$ has magnitude $\omega r$ and points in the tangential direction, in the direction of motion ($\boldsymbol{\omega} \times \mathbf{r}$ is simply the velocity as viewed by someone on the ground, after all). Therefore, $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ has magnitude $mr\omega^2$ and points radially inward. Hence, the centrifugal force, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, points radially outward with magnitude $mr\omega^2$.

Remark: If the person is not moving with respect to the carousel, and if $\boldsymbol{\omega}$ is constant, then the centrifugal force is the only non-zero fictitious force in eq. (9.10). Since the person is not accelerating in her rotating frame, the net force (as measured in her frame) must be zero. The forces in her frame are (1) gravity pulling downward, (2) a normal force pushing upward (which cancels the gravity), (3) a friction force pushing inward at her feet, and (4) the centrifugal force pulling outward. We conclude that the last two of these must cancel. That is, the friction force points inward with magnitude $mr\omega^2$.

Of course, someone standing on the ground will observe only the first three of these forces, so the net force will not be zero. And indeed, there is a centripetal acceleration, $v^2/r = r\omega^2$, due to the friction force. To sum up: in the inertial frame, the friction force exists to provide an acceleration. In the rotating frame, the friction force exists to balance out the mysterious new centrifugal force, in order to yield zero acceleration. ♣

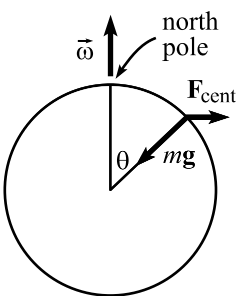


**Figure 9.3**

**Example 2 (Effective gravity force, $m\mathbf{g}_{\text{eff}}$):** Consider a person standing motionless on the earth, at a polar angle $\theta$. (See Fig. 9.3. The way we've defined it, $\theta$ equals $\pi/2$ minus the latitude angle.) She will feel a force due to gravity, $m\mathbf{g}$, directed toward the center of the earth.[3] But in her rotating frame, she will also feel a centrifugal force, directed away from the rotation axis. The sum of these two forces (that is, what she thinks is gravity) will not point radially, unless she is at the equator or at a pole. Let us denote the sum of these forces as $m\mathbf{g}_{\text{eff}}$.

To calculate $m\mathbf{g}_{\text{eff}}$, we must calculate $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. The $\boldsymbol{\omega} \times \mathbf{r}$ part has magnitude $R\omega\sin\theta$, where $R$ is the radius of the earth, and it is directed tangentially along the latitude circle of radius $R\sin\theta$. So $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points outward from the $z$-axis, with magnitude $mR\omega^2\sin\theta$, which is just what we expect for something traveling at frequency $\omega$ in a circle of radius $R\sin\theta$. Therefore, the effective gravitational force,

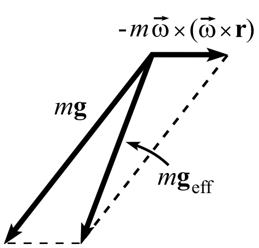


**Figure 9.4**

$$m\mathbf{g}_{\text{eff}} \equiv m\Big(\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\Big), \tag{9.13}$$

points slightly in the southerly direction (for someone in the northern hemisphere), as shown in Fig. 9.4. The magnitude of the correction term, $mR\omega^2\sin\theta$, is small compared to $g$. Using $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$ (that is, one revolution per day, which is $2\pi$ radians per 86,400 seconds) and $R \approx 6.4 \cdot 10^6\,\text{m}$, we find $R\omega^2 \approx .03\,\text{m/s}^2$. Therefore, the correction to $g$ is about 0.3% at the equator. But it is zero at the poles. Exercise 1 and Problem 1 deal further with $\mathbf{g}_{\text{eff}}$.

[3]Note that we are using $\mathbf{g}$ to denote the acceleration due solely to the gravitational force. This isn't the "$g$-value" that the person measures, as we will shortly see.

REMARKS: In the construction of buildings, and in similar matters, it is of course $\mathbf{g}_{\text{eff}}$, and not $\mathbf{g}$, that determines the "upward" direction in which the building should point. The exact direction to the center of the earth is irrelevant. A plumb bob hanging from the top of a skyscraper touches exactly at the base. Both the bob and the building point in a direction slightly different from the radial, but no one cares.

If you look in table and find that the acceleration due to gravity in New York is $9.803\,\text{m/s}^2$, remember that this is the $g_{\text{eff}}$ value and not the $g$ value (which describes only the gravitational force, in our terminology). The way we've defined it, the $g$ value is the acceleration with which things would fall if the earth kept its same shape but somehow stopped spinning. The exact value of $g$ is therefore generally irrelevant. ♣

### 9.2.3 Coriolis force: $-2m\vec{\omega} \times \mathbf{v}$

While the centrifugal force is very intuitive concept (everyone has gone around a corner in a car), the same thing cannot be said about the Coriolis force. This force requires a non-zero velocity $\mathbf{v}$ relative to the accelerated frame, and people normally don't move appreciably with respect to their car while rounding a corner. To get a feel for this force, let's look at two special cases.

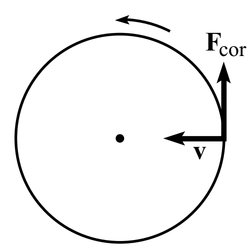


**Figure 9.5**

**Case 1 (Moving radially on a carousel):** A carousel rotates with constant angular speed $\omega$. Consider someone walking radially inward on the carousel, with speed $v$ (relative to the carousel) at radius $r$ (see Fig. 9.5). $\boldsymbol{\omega}$ points out of the page, so the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points tangentially in the direction of the motion of the carousel (that is, to the person's right, in our scenario), with magnitude

$$F_{\text{cor}} = 2m\omega v. \tag{9.14}$$

Let's assume that the person counters this force with a tangential friction force of $2m\omega v$ (pointing to his left) at his feet, so that he continues to walk on the same radial line.[4]

Why does this Coriolis force (and hence the tangential friction force) exist? It exists so that the resultant friction force changes the angular momentum of the person (measured with respect to the lab frame) in the proper way. To see this, take $d/dt$ of $L = mr^2\omega$, where $\omega$ is the person's angular speed with respect to the lab frame, which is also the carousel's angular speed. Using $dr/dt = -v$, we have

$$\frac{dL}{dt} = -2mr\omega v + mr^2(d\omega/dt). \tag{9.15}$$

But $d\omega/dt = 0$, because the person is staying on one radial line, and we're assuming that the carousel is arranged to keep a constant $\omega$. Eq. (9.15) then gives $dL/dt = -2mr\omega v$. So the $L$ of the person changes at a rate of $-(2m\omega v)r$. This is simply the radius times the tangential friction force applied by the carousel. In other words, it is the torque applied to the person.

[4]There is also the centrifugal force, which is countered by a radial friction force at the person's feet. This effect won't be important here.

REMARK: What if the person does not apply a tangential friction force at his feet? Then the Coriolis force of $2m\omega v$ produces a tangential acceleration of $2\omega v$ in his frame, and hence the lab frame also. This acceleration exists essentially to keep the angular momentum (measured with respect to the lab frame) of the person constant. (It *is* constant in this scenario, because there are no tangential forces in the lab frame.) To see that this tangential acceleration is consistent with conservation of angular momentum, set $dL/dt = 0$ in eq. (9.15) to obtain $2\omega v = r(d\omega/dt)$. The right-hand side of this is by definition the tangential acceleration. Therefore, saying that $L$ is conserved is the same as saying that $2\omega v$ is the tangential acceleration (for this situation where the inward radial speed is $v$). ♣

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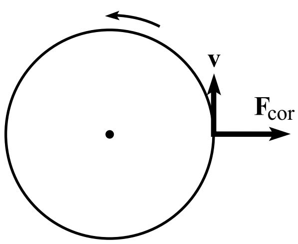


**Figure 9.6**

**Case 2 (Moving tangentially on a carousel):** Now consider someone walking tangentially on a carousel, in the direction of the carousel's motion, with speed $v$ (relative to the carousel) at constant radius $r$ (see Fig. 9.6). The Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points radially outward with magnitude $2m\omega v$. Assume that the person applies the friction force necessary to continue moving at radius $r$.

There is a simple way to see why this outward force of $2m\omega v$ exists. Let $V \equiv \omega r$ be the speed of a point on the carousel at radius $r$, as viewed by an outside observer. If the person moves tangentially (in the same direction as the spinning) with speed $v$ relative to the carousel, then his speed as viewed by the outside observer is $V + v$. The outside observer therefore sees the person walking in a circle of radius $r$ at speed $V + v$. The acceleration of the person in the ground frame is therefore $(V + v)^2/r$. This acceleration must be caused by an inward-pointing friction force at the person's feet, so

$$F_{\text{friction}} = \frac{m(V+v)^2}{r} = \frac{mV^2}{r} + \frac{2mVv}{r} + \frac{mv^2}{r}. \tag{9.16}$$

This friction force is of course the same in any frame. How, then, does our person on the carousel interpret the three pieces of the inward-pointing friction force in eq. (9.16)? The first term simply balances the outward centrifugal force due to the rotation of the frame, which he always feels. The third term is simply the inward force his feet must apply if he is to walk in a circle of radius $r$ at speed $v$, which is exactly what he is doing in the rotating frame. The middle term is the additional inward friction force he must apply to balance the outward Coriolis force of $2m\omega v$ (using $V \equiv \omega r$).

Said in an equivalent way, the person on the carousel will write down an $F = ma$ equation of the form (taking radially inward to be positive),

$$\begin{aligned} m\frac{v^2}{r} &= \frac{m(V+v)^2}{r} - \frac{mV^2}{r} - \frac{2mVv}{r}, \qquad \text{or} \\ m\mathbf{a} &= \mathbf{F}_{\text{friction}} + \mathbf{F}_{\text{cent}} + \mathbf{F}_{\text{cor}}. \end{aligned} \tag{9.17}$$

We see that the net force he feels is indeed equal to his $ma$ (where $a$ is measured with respect to his rotating frame).

---

For cases in between the two special ones above, things aren't so clear, but that's the way it goes. Note that no matter what direction you move on a carousel, the Coriolis force always points in the same perpendicular direction relative to your motion. Whether it's to the right or to the left depends on the direction of the rotation. But given $\boldsymbol{\omega}$, you're stuck with the same relative direction of force.

On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
'Twas like he was stalked
By some fiend always pushing him right.

Let's do some more examples...

---

**Example 1 (Dropped ball):** A ball is dropped from a height $h$, at a polar angle $\theta$ (measured down from the north pole). How far to the east is the ball deflected, by the time it hits the ground?

**Solution:** Note that the ball is indeed deflected to the east, independent of which hemisphere it is in. The angle between $\boldsymbol{\omega}$ and $\mathbf{v}$ is $\pi - \theta$, so the Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, is directed eastward with magnitude $2m\omega v \sin\theta$, where $v = gt$ is the speed at time $t$ ($t$ runs from 0 to the usual $\sqrt{2h/g}$).[5] The eastward acceleration at time $t$ is therefore $2\omega gt \sin\theta$. Integrating this to obtain the eastward speed (with an initial eastward speed of 0) gives $v_{\text{east}} = \omega gt^2 \sin\theta$. Integrating once more to obtain the eastward deflection (with an initial eastward deflection of 0) gives $d_{\text{east}} = \omega gt^3 \sin\theta/3$. Plugging in $t = \sqrt{2h/g}$ gives

$$d_{\text{east}} = h\left(\frac{2\sqrt{2}}{3}\right)\left(\omega\sqrt{\frac{h}{g}}\right)\sin\theta. \tag{9.18}$$

This is valid up to second-order effects in the small dimensionless quantity $\omega\sqrt{h/g}$. For an everyday value of $h$, this quantity is indeed small, since $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$.

---

**Example 2 (Foucault's pendulum):** This is the classic example of a consequence of the Coriolis force. It unequivocally shows that the earth rotates. The basic idea is that due to the rotation of the earth, the plane of a swinging pendulum rotates slowly, with a calculable frequency.

In the special case where the pendulum is at one of the poles, this rotation is easy to understand. Consider the north pole. An external observer, hovering above the north pole and watching the earth rotate, sees the pendulum's plane stay fixed (with respect to the distant stars) while the earth rotates counterclockwise beneath it.[6] Therefore, to an observer on the earth, the pendulum's plane rotates clockwise (viewed from above). The frequency of this rotation is of course just the frequency of the earth's rotation, so the earth-based observer sees the pendulum's plane make one revolution each day.

What if the pendulum is not at one of the poles? What is the frequency of the precession? Let the pendulum be located at a polar angle $\theta$. We will work in the approximation where the velocity of the pendulum bob is horizontal. This is essentially

[5] Technically, $v = gt$ isn't quite correct. Due to the Coriolis force, the ball will pick up a small sideways velocity component (this is the point of the problem). This component will then produce a second-order Coriolis force that effects the vertical speed (see Exercise 2). We may, however, ignore this small effect in this problem.

[6] Assume that the pivot of the pendulum is a frictionless bearing, so that it can't provide any torque to twist the pendulum's plane.

true if the pendulum's string is very long; the correction due to the rising and falling of the bob is negligible. The Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, points in some complicated direction, but fortunately we are concerned only with the component that lies in the horizontal plane. The vertical component serves only to modify the apparent force of gravity and is therefore negligible. (Although the frequency of the pendulum depends on $g$, the resulting modification is very small.)

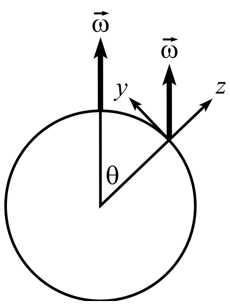


**Figure 9.7**

With this in mind, let's break $\boldsymbol{\omega}$ into vertical and horizontal components in a coordinate system located at the pendulum. From Fig. 9.7, we see that

$$\boldsymbol{\omega} = \omega\cos\theta\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}. \tag{9.19}$$

We'll ignore the $y$-component, because it produces a Coriolis force in the $\hat{\mathbf{z}}$ direction (since $\mathbf{v}$ lies in the horizontal $x$-$y$ plane). So for our purposes, $\boldsymbol{\omega}$ is essentially equal to $\omega\cos\theta\hat{\mathbf{z}}$. From this point on, the problem of finding the frequency of precession can be done in numerous ways. We'll present two solutions.

**First solution (The slick way):** The horizontal component of the Coriolis force has magnitude

$$F^{\text{horiz}}_{\text{cor}} = |-2m(\omega\cos\theta\hat{\mathbf{z}}) \times \mathbf{v}| = 2m(\omega\cos\theta)v, \tag{9.20}$$

and it is perpendicular to $\mathbf{v}(t)$. Therefore, as far as the pendulum is concerned, it is located at the north pole of a planet called Terra Costhetica which has rotational frequency $\omega\cos\theta$. But as we saw above, the precessional frequency of a Foucault pendulum located at the north pole of such a planet is simply

$$\omega_F = \omega\cos\theta, \tag{9.21}$$

in the clockwise direction. So that's our answer.[7]

**Second solution (In the pendulum's frame):** Let's work in the frame of the vertical plane that the Foucault pendulum sweeps through. Our goal is to find the rate of precession of this frame. With respect to a frame fixed on the earth (with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$), we know that this plane rotates with frequency $\boldsymbol{\omega}_F = -\omega\hat{\mathbf{z}}$ if we're at the north pole ($\theta = 0$), and frequency $\boldsymbol{\omega}_F = 0$ if we're at the equator ($\theta = \pi/2$). So if there's any justice in the world, the general answer has got to be $\boldsymbol{\omega}_F = -\omega\cos\theta\hat{\mathbf{z}}$, and that's what we'll now show.

Working in the frame of the plane of the pendulum is useful, because we can take advantage of the fact that *the pendulum feels no sideways forces in this frame*, because otherwise it would move outside of the plane (which it doesn't, by definition).

The frame fixed on the earth rotates with frequency $\boldsymbol{\omega} = \omega\cos\theta\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}$, with respect to the inertial frame. Let the pendulum's plane rotate with frequency $\boldsymbol{\omega}_F = \omega_F\hat{\mathbf{z}}$ with respect to the earth frame. Then the angular velocity of the pendulum's frame with respect to the inertial frame is

$$\boldsymbol{\omega} + \boldsymbol{\omega}_F = (\omega\cos\theta + \omega_F)\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}. \tag{9.22}$$

To find the horizontal component of the Coriolis force in this rotating frame, we only care about the $\hat{\mathbf{z}}$ part of this frequency. The horizontal Coriolis force therefore has magnitude $2m(\omega\cos\theta + \omega_F)v$. But in the frame of the pendulum, there is no horizontal force, so this must be zero. Therefore,

$$\omega_F = -\omega\cos\theta. \tag{9.23}$$

[7]As mentioned above, the setup isn't *exactly* like the one on the new planet. There will be a vertical component of the Coriolis force for the pendulum on the earth, but this effect is negligible.

This agrees with eq. (9.21), where we just wrote down the magnitude of $\omega_F$.

---

### 9.2.4 Azimuthal force: $-m(d\boldsymbol{\omega}/dt) \times \mathbf{r}$

In this section, we will restrict ourselves to the simple and intuitive case where $\boldsymbol{\omega}$ changes only in magnitude (that is, not in direction).[8] In this case, the azimuthal force may be written as

$$\mathbf{F}_{\text{az}} = -m\dot{\omega}\hat{\boldsymbol{\omega}} \times \mathbf{r}. \tag{9.24}$$

This force is easily understood by considering a person standing at rest with respect to a rotating carousel. If the carousel speeds up, then the person must feel a tangential friction force at his feet if he is to remain fixed on the carousel. This friction force equals $ma_{\text{tan}}$, where $a_{\text{tan}} = r\dot{\omega}$ is the tangential acceleration as measured in the ground frame. But from the person's point of view in the rotating frame, he is not moving, so there must be some other mysterious force that balances the friction. This is the azimuthal force. Quantitatively, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, we have $|\hat{\boldsymbol{\omega}} \times \mathbf{r}| = r$, so the azimuthal force in eq. (9.24) has magnitude $mr\dot{\omega}$. This is the same as the magnitude of the friction force, as it should be.

What we have here is exactly the same effect that we had with the translation force on the accelerating train. If the floor speeds up beneath you, then you must apply a friction force if you don't want to be thrown backwards with respect to the floor. If you shut your eyes and ignore the centrifugal force, then you can't tell if you're on a linearly accelerating train, or on an angularly accelerating carousel. The translation and azimuthal forces both arise from the acceleration of the floor. (Well, for that matter, the centrifugal force does, too.)

We can also view things in terms of rotational quantities, as opposed to the linear $a_{\text{tan}}$ acceleration above. If the carousel speeds up, then a torque must be applied to the person if he is to remain fixed on the carousel, because his angular momentum in the fixed frame increases. Therefore, he must feel a friction force at his feet.

Let's show that this friction force, which produces the change in angular momentum of the person in the fixed frame, exactly cancels the azimuthal force in the rotating frame, thereby yielding zero net force in the rotating frame.[9] Since $L = mr^2\omega$, we have $dL/dt = mr^2\dot{\omega}$ (assuming $r$ is fixed). And since $dL/dt = \tau = rF$, we see that the required friction force is $F = mr\dot{\omega}$. And as we saw above, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, the azimuthal force in eq. (9.24) also equals $mr\dot{\omega}$, in the direction opposite to the carousel's motion.

---

**Example (Spinning ice skater):** We have all seen ice skaters increase their angular speed by bringing their arms in close to their body. This is easily understood in terms of angular momentum; a smaller moment of inertia requires a larger $\omega$, to keep

[8]The more complicated case where $\boldsymbol{\omega}$ changes direction is left for Problem 9.

[9]This is basically the same calculation as the one above, with an extra $r$ thrown in.

$L$ constant. But let's analyze the situation here in terms of fictitious forces. We will idealize things by giving the skater massive hands at the end of massless arms attached to a massless body.[10] Let the hands have total mass $m$, and let them be drawn in radially.

Look at things in the skater's frame (which has an increasing $\omega$), defined by the vertical plane containing the hands. The crucial thing to realize is that the skater always remains in the skater's frame (a fine tautology, indeed). Therefore, the skater must feel zero net tangential force in her frame, because otherwise she would accelerate with respect to it. Her hands are being drawn in by a muscular force that works against the centrifugal force, but there is no net tangential force on the hands in the skater's frame.

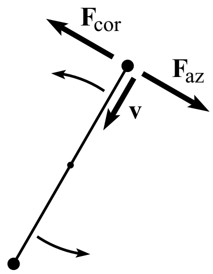

**Figure 9.8**

What are the tangential forces in the skater's frame? (See Fig. 9.8.) Let the hands be drawn in at speed $v$. Then there is a Coriolis force (in the same direction as the spinning) with magnitude $2m\omega v$. There is also an azimuthal force with magnitude $mr\dot{\omega}$ (in the direction opposite the spinning, as you can check). Since the net tangential force is zero in the skater's frame, we must have

$$2m\omega v = mr\dot{\omega}. \tag{9.25}$$

Does this relation make sense? Well, let's look at things in the ground frame. The total angular momentum of the hands in the ground frame is constant. Therefore, $d(mr^2\omega)/dt = 0$. Taking this derivative and using $dr/dt \equiv -v$ gives eq. (9.25).

---

A word of advice about using fictitious forces: Decide which frame you are going to work in (the lab frame or the accelerated frame), and then stick with it. The common mistake is to work a little in one frame and a little on the other, without realizing it. For example, you might introduce a centrifugal force on someone sitting at rest on a carousel, but then also give her a centripetal acceleration. This is incorrect. In the lab frame, there is a centripetal acceleration (caused by the friction force) and no centrifugal force. In the rotating frame, there is a centrifugal force (which cancels the friction force) and no centripetal acceleration (because the person is sitting at rest on the carousel, consistent with the fact that the net force is zero). Basically, if you ever mention the words "centrifugal" or "Coriolis", etc., then you had better be working in an accelerated frame.

[10] This reminds me of a joke about a spherical cow...

## 9.3 Exercises

*Section 9.2: The fictitious forces*

1. **Magnitude of $\mathbf{g}_{\text{eff}}$** $*$

   What is the magnitude of $\mathbf{g}_{\text{eff}}$? Give you answer to the leading-order correction in $\omega$.

2. **Corrections to gravity** $**$

   A mass is dropped from rest from a point directly above the equator. Let the initial distance from the center of the earth be $R$, and let the distance fallen be $d$. If we consider only the centrifugal force, then the correction to $g$ is $\omega^2(R-d)$. There is, however, also a second-order Coriolis effect. What is the sum of these corrections?[11]

3. **Southern deflection** $**$

   A ball is dropped from a height $h$ (small compared to the radius of the earth), at a polar angle $\theta$. How far to the *south* (in the northern hemisphere) is it deflected away from the $\mathbf{g}_{\text{eff}}$ direction, by the time it hits the ground? (This is a second order Coriolis effect.)

4. **Oscillations across equator** $*$

   A bead lies on a frictionless wire which lies in the north-south direction across the equator. The wire takes the form of an arc of a circle; all points are the same distance from the center of the earth. The bead is released from rest at a short distance from the equator. Because $\mathbf{g}_{\text{eff}}$ does not point directly toward the earth's center, the bead will head toward the equator and undergo oscillatory motion. What is the frequency of these oscillations?

5. **Roche limit** $*$

   Exercise 4.29 dealt with the Roche limit for a particle falling in radially toward a planet. Show that the Roche limit for a object in a circular orbit is

$$d = R\left(\frac{3\rho_{\text{p}}}{\rho_{\text{r}}}\right)^{1/3}. \qquad (9.26)$$

6. **Spinning bucket** $**$

   An upright bucket of water is spun at frequency $\omega$ around the vertical axis. If the water is at rest with respect to the bucket, find the shape of the water's surface.

7. **Coin on turntable** $***$

   A coin stands upright at an arbitrary point on a rotating turntable, and rotates (without slipping) with the required speed to make its center remain motionless

[11] $g$ will also vary with height, but let's not worry about that here.

in the lab frame. In the frame of the turntable, the coin will roll around in a circle with the same frequency as that of the turntable. In the frame of the turntable, show that

(a) $\mathbf{F} = d\mathbf{p}/dt$, and

(b) $\boldsymbol{\tau} = d\mathbf{L}/dt$.

8. **Precession viewed from rotating frame** ***

Consider a top made of a wheel with all its mass on the rim. A massless rod (perpendicular to the plane of the wheel) connects the CM to the pivot. Initial conditions have been set up so that the top undergoes precession, with the rod always horizontal.

In the language of Figure 8.27, we may write the angular velocity of the top as $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (where $\hat{\mathbf{x}}_3$ is horizontal here). Consider things in the frame rotating around the $\hat{\mathbf{z}}$-axis with angular speed $\Omega$. In this frame, the top spins with angular speed $\omega'$ around its *fixed* symmetry axis. Therefore, in this frame $\boldsymbol{\tau} = 0$, because there is no change in $\mathbf{L}$.

Verify explicitly that $\boldsymbol{\tau} = 0$ (calculated with respect to the pivot) in this rotating frame (you will need to find the relation between $\omega'$ and $\Omega$). In other words, show that the torque due to gravity is exactly canceled by the torque do to the Coriolis force. (You can easily show that the centrifugal force provides no net torque.)

## 9.4 Problems

*Section 9.2: The fictitious forces*

1. **$\mathbf{g}_{\text{eff}}$ vs. $\mathbf{g}$** $*$

   For what $\theta$ is the angle between $m\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ maximum?

2. **Longjumping in $\mathbf{g}_{\text{eff}}$** $*$

   If a longjumper can jump 8 meters at the north pole, how far can he jump at the equator? (Ignore effects of wind resistance, temperature, and runways made of ice. And assume that the jump is made in the north-south direction at the equator, so that there is no Coriolis force.)

3. **Lots of circles** $**$

   (a) Two circles in a plane, $C_1$ and $C_2$, each rotate with frequency $\omega$ (relative to an inertia frame). See Fig. 9.9. The center of $C_1$ is fixed in an inertial frame, and the center of $C_2$ is fixed on $C_1$. A mass is fixed on $C_2$. The position of the mass relative to the center of $C_1$ is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

   (b) $N$ circles in a plane, $C_i$, each rotate with frequency $\omega$ (relative to an inertia frame). See Fig. 9.10. The center of $C_1$ is fixed in an inertial frame, and the center of $C_i$ is fixed on $C_{i-1}$ (for $i = 2, \dots, N$). A mass is fixed on $C_N$. The position of the mass relative to the center of $C_1$ is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

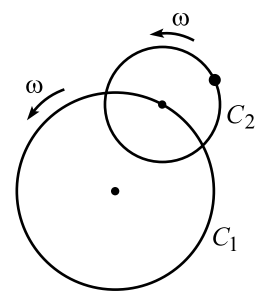


**Figure 9.9**

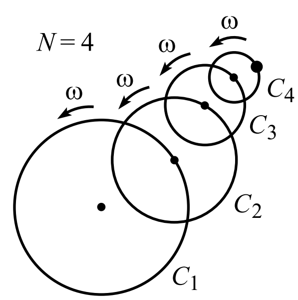


**Figure 9.10**

4. **Which way down?** $*$

   You are floating high up in a balloon, at rest with respect to the earth. Give three quasi-reasonable definitions for which point on the ground is right "below" you.

5. **Mass on a turntable** $*$

   A mass rests motionless with respect to the lab frame, while a frictionless turntable rotates beneath it. The frequency of the turntable is $\omega$, and the mass is located at radius $r$. In the frame of the turntable, what are the forces acting on the mass?

6. **Released mass** $*$

   A mass is bolted down to a frictionless turntable. The frequency of rotation is $\omega$, and the mass is located at a radius $a$. The mass is released. Viewed from an inertial frame, it travels in a straight line. In the rotating frame, what path does the mass take? Specify $r(t)$ and $\theta(t)$, where $\theta$ is the angle with respect to the initial radius.

7. **Coriolis circles** $*$

A puck slides with speed $v$ on frictionless ice. The surface is "level", in the sense that it is orthogonal to $\mathbf{g}_{\text{eff}}$ at all points. Show that the puck moves in a circle, as seen in the earth's rotating frame. What is the radius of the circle? What is the frequency of the motion? Assume that the radius of the circle is small compared to the radius of the earth.

8. **Shape of the earth** $***$

The earth bulges slightly at the equator, due to the centrifugal force in the earth's rotating frame. Show that the height of a point on the earth (relative to a spherical earth), is given by

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta - 2), \tag{9.27}$$

where $\theta$ is the polar angle (the angle down from the north pole), and $R$ is the radius of the earth.

9. **Changing $\boldsymbol{\omega}$'s direction** $***$

Consider the special case where a reference frame's $\boldsymbol{\omega}$ changes only in direction (and not in magnitude). In particular, consider a cone rolling on a table, which is a natural example of such a situation.

The instantaneous $\boldsymbol{\omega}$ for a rolling cone is its line of contact with the table, because these are the points that are instantaneously at rest. This line precesses around the origin. Let the frequency of the precession be $\boldsymbol{\Omega}$. Let the origin of the cone frame be the tip of the cone. This point remains fixed in the inertial frame.

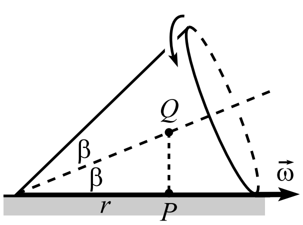

**Figure 9.11**

In order to isolate the azimuthal force, consider the special case of a point $P$ that lies on the instantaneous $\boldsymbol{\omega}$ and which is motionless with respect to the cone (see Fig. 9.11). From eq. (9.11), we then see that the centrifugal, Coriolis, and translation forces are zero. The only remaining fictitious force is the azimuthal force, and it arises from the fact that $P$ is accelerating up away from the table.

(a) Find the acceleration of $P$.

(b) Calculate the azimuthal force on a mass $m$ located at $P$, and show that the result is consistent with part (a).

## 9.5 Solutions

1. **$\mathbf{g}_{\text{eff}}$ vs. $\mathbf{g}$**

   The forces $m\mathbf{g}$ and $\mathbf{F}_{\text{cent}}$ are shown in Fig. 9.12. The magnitude of $\mathbf{F}_{\text{cent}}$ is $mR\omega^2 \sin\theta$, so the component of $\mathbf{F}_{\text{cent}}$ perpendicular to $m\mathbf{g}$ is $mR\omega^2 \sin\theta\cos\theta = mR\omega^2(\sin 2\theta)/2$. For small $\mathbf{F}_{\text{cent}}$, maximizing the angle between $\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ is equivalent to maximizing this perpendicular component. Therefore, we obtain the maximum angle when

$$\theta = \frac{\pi}{4}\,. \tag{9.28}$$

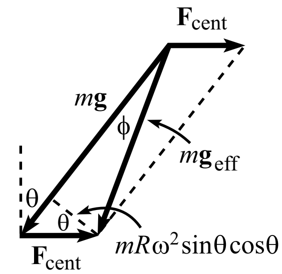

**Figure 9.12**

   The maximum angle turns out to be $\phi \approx \sin\phi \approx \Big(mR\omega^2(\sin\frac{\pi}{2})/2\Big)/mg = R\omega^2/2g \approx 0.0017$, which is about $0.1^\circ$. For this $\theta = \pi/4$ case, the line along $\mathbf{g}_{\text{eff}}$ misses the center of the earth by about 10 km, as you can show.

   REMARK: The above method works only when the magnitude of $\mathbf{F}_{\text{cent}}$ is much smaller than $mg$; we dropped higher order terms in the above calculation. One way of solving the problem exactly is to break $\mathbf{F}_{\text{cent}}$ into components parallel and perpendicular to $\mathbf{g}$. If $\phi$ is the angle between $\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$, then from Fig. 9.12 we have

$$\tan\phi = \frac{mR\omega^2 \sin\theta\cos\theta}{mg - mR\omega^2\sin^2\theta}\,. \tag{9.29}$$

   We can then maximize $\phi$ by taking a derivative. But be careful if $R\omega^2 > g$, in which case maximizing $\phi$ doesn't mean maximizing $\tan\phi$. We'll let you work this out, and instead we'll give the following slick geometric solution.

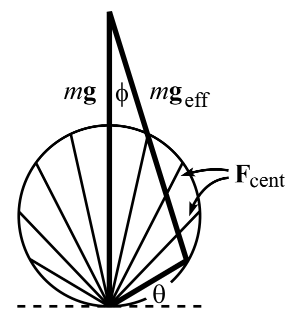

**Figure 9.13**

   Draw the $\mathbf{F}_{\text{cent}}$ vectors, for various $\theta$, relative to $m\mathbf{g}$. The result looks like Fig. 9.13. Since the lengths of the $\mathbf{F}_{\text{cent}}$ vectors are proportional to $\sin\theta$, you can show that the tips of the $\mathbf{F}_{\text{cent}}$ vectors form a circle. The maximum $\phi$ is therefore achieved when $\mathbf{g}_{\text{eff}}$ is tangent to this circle, as shown in Fig. 9.14. In the limit where $g \gg R\omega^2$ (that is, in the limit of a small circle), we want the point of tangency to be the rightmost point on the circle, so the maximum $\phi$ is achieved when $\theta = \pi/4$, in which case $\sin\phi \approx (R\omega^2/2)/g$. But in the general case, Fig. 9.14 shows that the maximum $\phi$ is given by

$$\sin\phi_{\text{max}} = \frac{\frac{1}{2}mR\omega^2}{mg - \frac{1}{2}mR\omega^2}\,. \tag{9.30}$$

   In the limit of small $\omega$, this is approximately $R\omega^2/2g$, as above.

   The above reasoning holds only if $R\omega^2 < g$. In the case where $R\omega^2 > g$ (that is, the circle extends above the top end of the $m\mathbf{g}$ segment), the maximum $\phi$ is simply $\pi$, and it is achieved at $\theta = \pi/2$. ♣

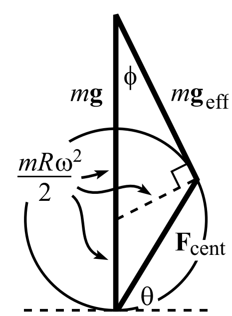

**Figure 9.14**

2. **Longjumping in $\mathbf{g}_{\text{eff}}$**

   Let the jumper take off with speed $v$, at an angle $\theta$. Then $g_{\text{eff}}(t/2) = v\sin\theta$ tells us that the time in the air is $t = 2v\sin\theta/g_{\text{eff}}$. The distance traveled is therefore

$$d = v_x t = vt\cos\theta = \frac{2v^2\sin\theta\cos\theta}{g_{\text{eff}}} = \frac{v^2\sin 2\theta}{g_{\text{eff}}}\,. \tag{9.31}$$

   This is maximum when $\theta = \pi/4$, as we well know. So we see that $d \propto 1/\sqrt{g_{\text{eff}}}$. Taking $g_{\text{eff}} \approx 10$ m/s$^2$ at the north pole, and $g_{\text{eff}} \approx (10 - 0.03)$ m/s$^2$ at the equator, we find that the jump at the equator is approximately 1.0015 times as long as the one on the north pole. So the longjumper gains about one centimeter.

   REMARK: For a longjumper, the optimal angle of takeoff is undoubtedly not $\pi/4$. The act of changing the direction abruptly from horizontal to such a large angle would entail a significant loss in speed. The optimal angle is some hard-to-determine angle less than $\pi/4$. But this won't change our general $d \propto 1/\sqrt{g_{\text{eff}}}$ result (which follows from dimensional analysis), so our answer still holds. ♣

3. **Lots of circles**

(a) The fictitious force, $\mathbf{F}_{\text{f}}$, on the mass has an $\mathbf{F}_{\text{cent}}$ part and an $\mathbf{F}_{\text{trans}}$ part, because the center of $C_2$ is moving. So the fictitious force is

$$\mathbf{F}_{\text{f}} = m\omega^2 \mathbf{r}_2 + \mathbf{F}_{\text{trans}}, \tag{9.32}$$

where $\mathbf{r}_2$ is the position of the mass in the frame of $C_2$. But $\mathbf{F}_{\text{trans}}$, which arises from the acceleration of the center of $C_2$, is simply the centrifugal force felt by any point on $C_1$. Therefore,

$$\mathbf{F}_{\text{trans}} = m\omega^2 \mathbf{r}_1, \tag{9.33}$$

where $\mathbf{r}_1$ is the position of the center of $C_2$, in the frame of $C_1$. Substituting this into eq. (9.32) gives

$$\begin{aligned} \mathbf{F}_{\text{f}} &= m\omega^2(\mathbf{r}_2 + \mathbf{r}_1) \\ &= m\omega^2 \mathbf{R}(t). \end{aligned} \tag{9.34}$$

(b) The fictitious force, $\mathbf{F}_{\text{f}}$, on the mass has an $\mathbf{F}_{\text{cent}}$ part and an $\mathbf{F}_{\text{trans}}$ part, because the center of the $N$th circle is moving. So the fictitious force is

$$\mathbf{F}_{\text{f}} = m\omega^2 \mathbf{r}_N + \mathbf{F}_{\text{trans},N}, \tag{9.35}$$

where $\mathbf{r}_N$ is the position of the mass in the frame of $C_N$. But $\mathbf{F}_{\text{trans},N}$ is simply the centrifugal force felt by a point on the $(N-1)$st circle, plus the translation force coming from the movement of the center of the $(N-1)$st circle. Therefore,

$$\mathbf{F}_{\text{trans},N} = m\omega^2 \mathbf{r}_{N-1} + \mathbf{F}_{\text{trans},N-1}. \tag{9.36}$$

Substituting this into eq. (9.35) and successively rewriting the $\mathbf{F}_{\text{trans},i}$ terms in a similar manner, gives

$$\begin{aligned} \mathbf{F}_{\text{f}} &= m\omega^2(\mathbf{r}_N + \mathbf{r}_{N-1} + \cdots + \mathbf{r}_1) \\ &= m\omega^2 \mathbf{R}(t). \end{aligned} \tag{9.37}$$

The main point in this problem is that $\mathbf{F}_{\text{cent}}$ is linear in $\mathbf{r}$.

Remark: There is actually a much easier way to see that $\mathbf{F}_{\text{f}} = m\omega^2 \mathbf{R}(t)$. Since all the circles rotate with the same $\omega$, they may as well be glued together. Such a rigid setup would indeed yield the same $\omega$ for all the circles. This is similar to the moon rotating once on its axis for every revolution is makes around the earth, thereby causing the same side to always face the earth. It is then clear that the mass simply moves in a circle at frequency $\omega$, yielding a fictitious centrifugal force of $m\omega^2 \mathbf{R}(t)$. And we see that the radius $R$ is in fact constant. ♣

4. **Which way down?**

Here are three possible definitions of the point "below" you on the ground: (1) The point that lies along the line between you and the center of the earth, (2) The point where a hanging plumb bob rests, and (3) The point where a dropped object hits the ground.

The second definition is the most reasonable, because it defines the upward direction in which buildings are constructed. It differs from the first definition due to the centrifugal force which makes $\mathbf{g}_{\text{eff}}$ point in a slightly different direction from $\mathbf{g}$. The third definition differs from the second due to the Coriolis force. The velocity of the falling object produces a Coriolis force which causes an eastward deflection.

Note that all three definitions are equivalent at the poles. Additionally, definitions 1 and 2 are equivalent at the equator.

5. **Mass on turntable**

In the lab frame, the net force on the mass is zero, because it is sitting at rest. (The normal force cancels the gravitational force.) But in the rotating frame, the mass travels in a circle of radius $r$, with frequency $\omega$. So in the rotating frame there must be a force of $m\omega^2 r$ inward to account for the centripetal acceleration. And indeed, the mass feels a centrifugal force of $m\omega^2 r$ outward, and a Coriolis force of $2m\omega v = 2m\omega^2 r$ inward, which sum to the desired force (see Fig. 9.15).

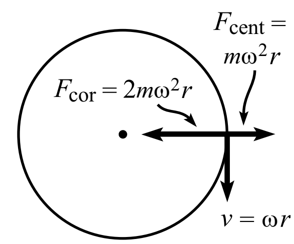

**Figure 9.15**

Remark: The net inward force in this problem is a little different from that for someone swinging around in a circle in an inertial frame. If a skater maintains a circular path by holding onto a rope whose other end is fixed, she has to use her muscles to maintain the position of her torso with respect to her arm, and her head with respect to her torso, etc. But if a person takes the place of the mass in this problem, she needs to exert no effort to keep her body moving in the circle (which is clear, when looked at from the inertial frame), because each atom in her body is moving at (essentially) the same speed and radius, and therefore feels the same Coriolis and centrifugal forces. So she doesn't really *feel* this force, in the same sense that someone doesn't feel gravity when in free-fall with no air resistance. ♣

6. **Released mass**

Let the $x'$- and $y'$-axes of the rotating frame coincide with the $x$- and $y$-axes of the inertial frame at the moment the mass is released (at $t = 0$). Let the mass initially be located on the $y'$-axis. Then after a time $t$, the situation looks like that in Fig. 9.16. The speed of the mass is $v = a\omega$, so it has traveled a distance $a\omega t$. The angle that its position vector makes with the inertial $x$-axis is therefore $\tan^{-1}\omega t$, with counterclockwise taken to be positive. Hence, the angle that its position vector makes with the rotating $y'$-axis is $\theta(t) = -(wt - \tan^{-1}\omega t)$. And the radius is simply $r(t) = a\sqrt{1+\omega^2 t^2}$. So for large $t$, $r(t) \approx a\omega t$ and $\theta(t) \approx -wt + \pi/2$, which make sense.

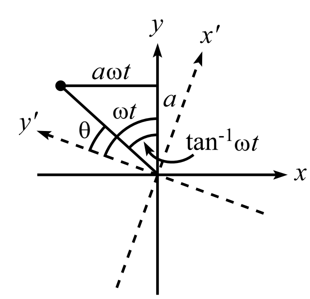

**Figure 9.16**

7. **Coriolis circles**

By construction, the normal force from the ice exactly cancels all effects of the gravitational and centrifugal forces in the rotating frame of the earth. We therefore need only concern ourselves with the Coriolis force. This force equals $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$.

Let the angle down from the north pole be $\theta$; we assume the circle is small enough so that $\theta$ is essentially constant throughout the motion. Then the component of the Coriolis force that points horizontally along the surface has magnitude $f = 2m\omega v \cos\theta$ and is perpendicular to the direction of motion. (The vertical component of the Coriolis force will simply modify the required normal force.) Because this force is perpendicular to the direction of motion, $v$ does not change. Therefore, $f$ is constant.

But a constant force perpendicular to the motion of a particle produces a circular path. The radius of the circle is given by

$$2m\omega v\cos\theta = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{v}{2\omega\cos\theta}\,. \tag{9.38}$$

The frequency of the circular motion is

$$\omega' = \frac{v}{r} = 2\omega\cos\theta. \tag{9.39}$$

REMARKS: To get a rough idea of the size of the circle, you can show (using $\omega \approx 7.3\cdot 10^{-5}\,\text{s}^{-1}$) that $r \approx 10\,\text{km}$ when $v = 1\,\text{m/s}$ and $\theta = 45^\circ$. Even the tiniest bit of friction will clearly make this effect essentially impossible to see.

For the $\theta \approx \pi/2$ (that is, near the equator), the component of the Coriolis force along the surface is negligible, so $r$ becomes large, and $\omega'$ goes to 0.

For the $\theta \approx 0$ (that is, near the north pole), the Coriolis force essentially points along the surface. The above equations give $r \approx v/(2\omega)$, and $\omega' \approx 2\omega$. For the special case where the center of the circle is the north pole, this $\omega' \approx 2\omega$ result might seem incorrect, because you might want to say that the circular motion should be achieved by having the puck remain motionless in the inertial fame, while the earth rotates beneath it (thus making $\omega' = \omega$). The error in this reasoning is that the "level" earth is not spherical, due to the non-radial direction of $\mathbf{g}_{\text{eff}}$. If the puck starts out motionless in the inertial frame, it will be drawn toward the north pole, due to the component of the gravitational force along the "level" surface. In order to not fall toward the pole, the puck needs to travel with frequency $\omega$ (relative to the inertial frame) in the direction opposite[12] to the earth's rotation.[13] The puck therefore moves at frequency $2\omega$ relative to the frame of the earth. ♣

8. **Shape of the earth**

In the reference frame of the earth, the forces on an atom at the surface are: earth's gravity, the centrifugal force, and the normal force from the ground below it. These three forces must sum to zero. Therefore, the sum of the gravity plus centrifugal forces must be normal to the surface. Said differently, the gravity-plus-centrifugal force must have no component along the surface. Said in yet another way, the potential energy function derived from the gravity-plus-centrifugal force must be constant along the surface. (Otherwise, a piece of the earth would want to move along the surface, which would mean we didn't have the correct surface to begin with.)

If $x$ is the distance from the earth's axis, then the centrifugal force is $F_c = m\omega^2 x$, directed outward. The potential energy function for this force is $V_c = -m\omega^2 x^2/2$, up to an arbitrary additive constant. The potential energy for the earth's gravitation force is simply $mgh$. (We've arbitrarily chosen the original spherical surface have zero potential; any other choice would add on an irrelevant constant. Also, we've assumed that the slight distortion of the earth won't make the $mgh$ result invalid. This is true to lowest order in $h/R$, which you can demonstrate if you wish.)

The equal-potential condition is therefore

$$mgh - \frac{m\omega^2 x^2}{2} = C, \tag{9.40}$$

[12] Of course, the puck could also move with frequency $\omega$ in the *same* direction as the earth's rotation. But in this case, the puck simply sits at one place on the earth.

[13] The reason for this is the following. In the rotating frame of the puck, the puck feels the same centrifugal force that it would feel if it were at rest on the earth, spinning with it. The puck therefore happily stays at the same $\theta$ value on the "level" surface, just as a puck at rest on the earth does.

where $C$ is a constant to be determined. Using $x = r\sin\theta$, we obtain

$$h = \frac{\omega^2 r^2 \sin^2\theta}{2g} + B, \tag{9.41}$$

where $B \equiv C/(mg)$ is another constant. We may replace the $r$ here with the radius of the earth, $R$, with negligible error.

Depending what the constant $B$ is, this equation describes a whole family of surfaces. We may determine the correct value of $B$ by demanding that the volume of the earth be the same as it would be in its spherical shape if the centrifugal force were turned off. This is equivalent to demanding that the integral of $h$ over the surface of the earth is zero. The integral of $(a\sin^2\theta + b)$ over the surface of the earth is (the integral is easy if we write $\sin^2\theta$ as $1-\cos^2\theta$)

$$\begin{aligned}\int_0^\pi \big(a(1-\cos^2\theta)+b\big)\,2\pi R^2\sin\theta\,d\theta &= \int_0^\pi \big(-a\cos^2\theta + (a+b)\big)\,2\pi R^2\sin\theta\,d\theta \\ &= 2\pi R^2\left(\frac{a\cos^3\theta}{3} - (a+b)\cos\theta\right)\Big|_0^\pi \\ &= 2\pi R^2\left(-\frac{2a}{3} + 2(a+b)\right). \end{aligned} \tag{9.42}$$

Hence, we need $b = -(2/3)a$ for this integral to be zero. Plugging this result into eq. (9.41) gives

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta - 2), \tag{9.43}$$

as desired.

9. **Changing $\boldsymbol{\omega}$'s direction**

   (a) Let $Q$ be the point which lies on the axis of the cone and which is directly above $P$ (see Fig. 9.17). If $P$ is a distance $r$ from the origin, and if the half-angle of the cone is $\beta$, then $Q$ is a height $y = r\tan\beta$ above $P$.

   Consider the situation an infinitesimal time $t$ later. Let $P'$ be the point that is now directly below $Q$ (see Fig. 9.17). The angular speed of the cone is $\omega$, so $Q$ moves horizontally at a speed $v_Q = \omega y = \omega r\tan\beta$. Therefore, in the infinitesimal time $t$, we see that $Q$ moves a distance $\omega yt$ to the side.

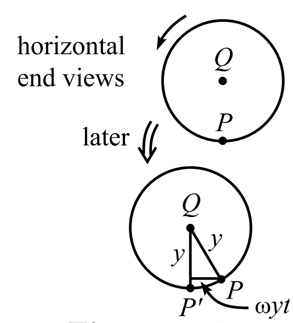


**Figure 9.17**

   This distance $\omega yt$ is also (essentially) the horizontal distance between $P$ and $P'$. Therefore, a little geometry tells us that $P$ is now a distance

$$h(t) = y - \sqrt{y^2 - (\omega yt)^2} \approx \frac{(\omega t)^2 y}{2} = \frac{1}{2}(\omega^2 y)t^2 \tag{9.44}$$

   above the table. Since $P$ started on the table with zero speed, this means that $P$ is undergoing an acceleration of $\omega^2 y$ in the vertical direction. A mass $m$ located at $P$ must therefore feel a force $F_P = m\omega^2 y$ in the upward direction, if it is to remain motionless with respect to the cone.

   (b) The precession frequency $\Omega$ (which is how fast $\boldsymbol{\omega}$ swings around the origin) is equal to the speed of $Q$ divided by $r$. This is true because $Q$ is always directly above $\boldsymbol{\omega}$, so it moves in a circle of radius $r$ around the $z$-axis. Therefore, $\boldsymbol{\Omega}$ has magnitude $v_Q/r = \omega y/r$, and it points in the downward vertical direction (for the situation shown in Fig. 9.11). Hence, $d\boldsymbol{\omega}/dt = \boldsymbol{\Omega}\times\boldsymbol{\omega}$ has magnitude $\omega^2 y/r$, and it points in the horizontal direction (out of the page). Therefore,

$\mathbf{F}_{\text{az}} = -m(d\boldsymbol{\omega}/dt) \times \mathbf{r}$ has magnitude $m\omega^2 y$, and it points in the downward vertical direction.

A person of mass $m$ at point $P$ therefore interprets the situation as, "I am not accelerating with respect to the cone. Therefore, the net force on me is zero. And indeed, the upward normal force $F_P$ from the cone, with magnitude $m\omega^2 y$, is exactly balanced by the mysterious downward force $F_{\text{az}}$, also with magnitude $m\omega^2 y$."

# Chapter 10

# Relativity (Kinematics)



We now come to Einstein's theory of relativity. This is where we find out that everything we've done so far in this book has been wrong. Well, perhaps "incomplete" would be a better word. The important point to realize is that Newtonian physics is a limiting case of the more correct Relativistic theory. Newtonian physics works perfectly fine when the speeds you are dealing with are much less than the speed of light (which is about $3 \cdot 10^8$m/s). It would be silly, to put it mildly, to use relativity to solve a problem involving the length of a baseball trajectory. But in problems involving large speeds, or in problems where a high degree of accuracy is required, you must use the Relativistic theory. This will be the topic of the next four chapters.[1]

The theory of Relativity is certainly one of the most exciting and talked-about topics in physics. It is well-known for its "paradoxes", which are quite conducive to discussion. There is, however, nothing at all paradoxical about it. The theory is logically and experimentally sound, and the whole subject is actually quite straightforward, provided that you proceed calmly and keep a firm hold of your wits.

The theory rests upon certain postulates. The one that most people find counterintuitive is that the speed of light has the same value in any inertial (that is, non-accelerating) reference frame. This speed is much greater than the speed of everyday objects, so most of the consequences of this new theory are not noticeable. If we lived in a world similar to ours, with the only difference being that the speed of light were 100 mph, then the consequences of relativity would be ubiquitous. We wouldn't think twice about time dilations, length contractions, and so on.

I have included a large number of puzzles and "paradoxes" in the text and in the problems. When attacking these, be sure to follow them through to completion, and do not say, "I could finish this one if I wanted to, but all I'd have to do would be such-and-such, so I won't bother," because the essence of the paradox may very

[1] At any rate, you shouldn't feel too bad about having spent so much time learning about a theory that is simply the limiting case of another theory, because you're now going to do it again. Relativity is also the limiting case of yet another theory (quantum field theory). And likewise, you shouldn't feel too bad about spending so much time on relativity, because quantum field theory is also the limiting case of yet another theory (string theory). And likewise... well, you get the idea. It just might actually be turtles all the way down.

well be contained in the "such-and-such", and you will have missed out on all the fun. Most of the paradoxes arise because different frames of reference *seem* to give different answers. Therefore, in explaining a paradox, you not only have to give the correct reasoning; you also have to say what's wrong with incorrect reasoning.

There are two main topics in relativity. One is Special Relativity (which does not deal with gravity), and the other is General Relativity (which does). We will deal mostly with the former, but Chapter 13 contains some of the latter.

Special Relativity may be divided into two topics, *kinematics* and *dynamics*. Kinematics deals with lengths, times, speeds, etc. It is basically concerned with only the space and time coordinates of an abstract particle, and not with masses, forces, energy, momentum, etc. Dynamics, on the other hand, does deal with these quantities.

This chapter will cover kinematics. Chapter 11 will cover dynamics. Most of the fun paradoxes fall into the kinematics part, so the present chapter will be the longer of the two. In Chapter 12, we will introduce the concept of 4-vectors, which ties much of the material in Chapters 10 and 11 together.

## 10.1 The postulates

Various approaches can be taken in deriving the consequences of the Special Relativity theory. Different approaches use different postulates. Some start with the invariance of the speed of light in any inertial frame. Others start with the invariant interval (discussed in Section 10.4). Others start with the invariance of the inner product of 4-momentum vectors (discussed in Chapter 12). Postulates in one approach are theorems in another. There is no "good" or "bad" route to take; they are all equally valid. However, some approaches are simpler and more intuitive (if there is such a thing as intuition in relativity) than others. I will choose to start with the speed-of-light postulate:

- *The speed of light has the same value in any inertial frame.*

I do not claim that this statement is obvious, or even believable. But I do claim that it is easy to understand what the statement says (even if you think it's too silly to be true). It says the following. Consider a train moving along the ground at constant velocity (that is, it is not accelerating; this is the definition of an inertial frame). Someone on the train shines a light from one point on the train to another. Let the speed of the light with respect to the train be $c$ ($\approx 3 \cdot 10^8$m/s). Then the above postulate says that a person on the ground also sees the light move at speed $c$.

This is a rather bizarre statement. It does not hold for everyday objects. If a baseball is thrown on a train, then the speed of the baseball is of course different in different frames. The observer on the ground must add the velocity of the train and the velocity of the ball (with respect to the train) to obtain the velocity of the ball with respect to the ground.[2]

[2]Actually, this isn't quite true, as the velocity-addition result in Section 10.3.3 shows. But it's true enough for the point we are making here.

The truth of the speed-of-light postulate cannot be demonstrated from first principles. No statement with any physical content in physics (that is, one that isn't purely mathematical, such as, "two apples plus two apples gives four apples") can be proven. In the end, we must rely on experiment. And indeed, all the consequences of the speed-of-light postulate have been verified countless times during this century. In particular, the consequences are being verified continually each day in high-energy particle accelerators, where elementary particles reach speeds very close to $c$.

The most well-known of the early experiments on the speed of light was the one performed by Michelson and Morley, who in 1887 tried to measure the effect of the earth's motion on the speed of light. If light moves at speed $c$ with respect to only one special frame (the frame of the "ether", analogous to the way sound travels through air), then the speed of light in a given direction on the earth should be faster or slower than $c$, depending on which way the earth is moving through the ether. In particular, if the speed of light in a given direction is measured at one time, and then measured again six months later, then the results should be different, due to the earth's motion around the sun. Michelson and Morley were not able to measure any such differences in the speed of light. Nor has anyone else been able to do so. The conclusion is that the ether simply does not exist. Light does not require a medium to support its motion.

The findings of Michelson–Morley
Allow us to say very surely,
"If this ether is real,
Then it has no appeal,
And shows itself off rather poorly."

The collection of all the data from various experiments over many years allows us to conclude with reasonable certainty that our starting assumption of an invariant speed of light is correct (or is at least the limiting case of a more accurate theory).

There is one more postulate in the Special Relativity theory, namely the "Relativity" postulate. It is much more believable than the speed-of-light postulate, so you might just take it for granted and forget to consider it. But like any postulate, of course, it is crucial. It can be stated in various ways, but we'll simply word it as:

- *All inertial frames are "equivalent".*

This postulate basically says that one inertial frame is no better than any another. There is no preferred reference frame. That is, it makes no sense to say that something is moving; it only makes sense to say that one thing is moving with respect to another. This is where the "Relativity" in Special Relativity comes from. There is no absolute frame; the motion of any frame is only defined relative to other frames.

This postulate also says that if the laws of physics hold in one inertial frame (and presumably they do hold in the frame in which I now sit),[3] then they hold

[3]Technically, the earth is spinning while revolving around the sun, and there are also little vibrations in the floor beneath my chair, etc., so I'm not *really* in an inertial frame. But it's close enough for me.

in all others. It also says that if we have two frames $S$ and $S'$, then $S$ should see things in $S'$ in exactly the same way that $S'$ sees things in $S$ (because we could simply switch the labels of $S$ and $S'$). It also says that empty space is homogeneous (that is, all points look the same), because we could pick any point to be, say, the origin of a coordinate system. It also says that empty space is isotropic (that is, all directions look the same), because we could pick any axis to be, say, the $x$-axis of a coordinate system.

Unlike the first postulate, this one is entirely reasonable. We've gotten used to having no special places in the universe. We gave up having the earth as the center, so let's not give any other point a chance, either.

Copernicus gave his reply
To those who had pledged to deny.
"All your addictions
To ancient convictions
Won't bring back your place in the sky."

Everything we've said here about our second postulate refers to empty space. If we have a chunk of mass, then there is certainly a difference between the position of the mass and a point a meter away. To incorporate mass into the theory, we would have to delve into General Relativity. But we won't have anything to say about that in this chapter. We will deal only with empty space, containing perhaps a few observant souls sailing along in rockets or floating aimlessly on little spheres. Though it may sound boring at first, it will turn out to be more exciting than you'd think.

## 10.2 The fundamental effects

The most striking effects of our two postulates are: (1) the loss of simultaneity, (2) length contraction, and (3) time dilation. In this section, we will discuss these three effects using some time-honored concrete examples. In the following section, we will derive the Lorentz transformations using these three results.

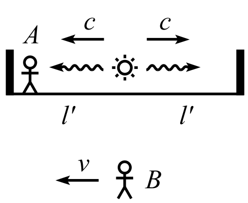

**Figure 10.1**

### 10.2.1 Loss of Simultaneity

Consider the following setup. In $A$'s reference frame, a light source is placed midway between two receivers, a distance $\ell'$ from each (see Fig. 10.1). The light source emits a flash. From $A$'s point of view, the light hits the two receivers at the same time, $\ell'/c$ seconds after the flash. Now consider another observer, $B$, who travels by to the left at speed $v$. From her point of view, does the light hit the receivers at the same time? We will show that it does not.

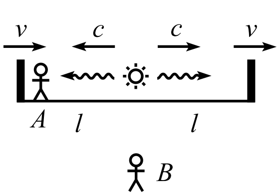

**Figure 10.2**

In $B$'s reference frame, the situation looks like that in Fig. 10.2. The receivers (along with everything else in $A$'s frame) are moving to the right at speed $v$, and the light is traveling in both directions at speed $c$ with respect to $B$ (*not* with respect to the light source, as measured in $B$'s frame; this is where the speed-of-light postulate comes into play). Therefore, the relative speed (as viewed by $B$) of the light and the left receiver is $c + v$, and the relative speed of the light and the right receiver is $c - v$.[4]

Let $\ell$ be the distance from the source to the receivers, as measured by $B$.[5] Then in $B$'s frame, the light hits the left receiver at $t_l$ and the right receiver at $t_r$, where

$$t_l = \frac{\ell}{c+v}\,, \qquad t_r = \frac{\ell}{c-v}\,. \tag{10.1}$$

These are not equal if $v \neq 0$.[6]

The moral of this exercise is that it makes no sense to say that one event happens at the same time as another, unless you state which frame you're talking about. Simultaneity depends on the frame in which the observations are made.

Of the many effects, miscellaneous,
The loss of events, simultaneous,
Allows $A$ to claim
There's no pause in $B$'s frame,

REMARKS:

1. The invariance of the speed of light was used in saying that the two relative speeds above were $c + v$ and $c - v$. If we were talking about baseballs instead of light beams, then the relative speeds wouldn't look like this. If $v_b$ is the speed at which the baseballs are thrown in $A$'s frame, then $B$ sees the balls move at speeds $v_b - v$ to the left and $v_b + v$ to the right.[7] These are not equal to $v_b$, as they would be in the case of the light beams. The relative speeds between the balls and the left and right receivers are therefore $(v_b - v) + v = v_b$ and $(v_b + v) - v = v_b$. These are equal, so $B$ sees the balls hit the receivers at the same time, as we know very well from everyday experience.

2. It is indeed legal in eq. (10.1) to obtain the times by simply dividing $\ell$ by the relative speeds, $c + v$ and $c - v$. But if you want a more formal method, then consider the following. In $B$'s frame, the position of the right photon is given by $ct$, and the position

[4]Yes, it is legal to simply add and subtract these speeds to obtain the relative speeds *as viewed by B*. If $v$ equals, say, $2 \cdot 10^8$ m/s, then in one second the left receiver will move $2 \cdot 10^8$ m to the right, while the left ray of light will move $3 \cdot 10^8$ m to the left. This means that they will now be $5 \cdot 10^8$ m closer than they were a second ago. In other words, the relative speed (as measured by $B$) is $5 \cdot 10^8$ m/s, which is simply $c + v$. Note that this implies that it is perfectly legal for the relative speed of two things, as measured by a third, to take any value up to $2c$. Both the $v$ and $c$ here are measured with respect to the *same* person, namely $B$, so our intuition works fine. We don't need to use the "velocity-addition formula", which we'll derive in Section 10.3.3, and which is relevant in a different setup. I include this footnote here just in case you've seen the velocity-addition formula and think it is relevant in this setup. But if it didn't occur to you, then never mind.

[5]We will see in Section 10.2.3 that $\ell$ is not equal to $\ell'$, due to length contraction. But this won't be important here. The only thing we assume now is that the light source is equidistant from the receivers. This follows from the fact that space is homogeneous, which implies that the length-contraction factor must be the same everywhere. More on this in Section 10.2.3.

[6]The one exception is when $\ell = 0$, in which case the two events happen at the same place and same time in all frames.

[7]The velocity-addition formula in Section 10.3.3 shows that these formulas aren't actually correct. But they're close enough for our purposes here.

of the right receiver (which had a head start of $\ell$) is given by $\ell + vt$. Equating these two positions gives $t_r = \ell/(c-v)$. Likewise for the left photon.

3. There is always a difference between the time an event happens and the time someone *sees* the event happen, because light takes time to travel from the event to the observer. What we calculated above were the times at which the events really happen. We could, of course, calculate the times at which $B$ *sees* the events occur, but such times are rarely important, and in general we will not be concerned with them. They can simply be calculated by adding on a (distance)/$c$ time difference for the path of the photons to $B$'s eye. Of course, if $B$ did the above experiment to find $t_r$ and $t_l$, she would do it by writing down the times at which she saw the events occur, and then subtracting off the relevant (distance)/$c$ time differences to find when the events really happened.

   To sum up, the $t_r \neq t_l$ result in eq. (10.1) is due to the fact that the events truly occur at different times. *It has nothing to do with the time it takes light to travel to your eye.* In this chapter, we will sometimes use sloppy language of the sort, "What time does Beth see event $Q$ happen?" But we don't really mean, "When do Beth's eyes register that $Q$ happened?" Instead, we mean, "What time does Beth *know* that event $Q$ happened in her frame?" If we ever want to use "see" in the former sense, we will explicitly say so (as in Section 10.6 on the Doppler effect). ♣

Where this last line is not so extraneous.

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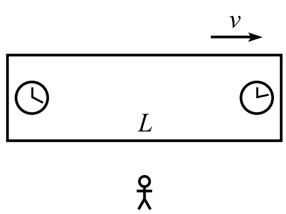


Figure 10.3

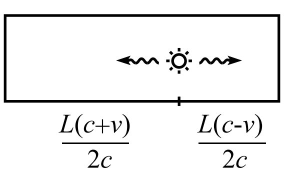


Figure 10.4

**Example (The head start):** Two clocks are positioned at the ends of a train of length $L$ (as measured in its own frame). They are synchronized in the train frame. The train travels past you at speed $v$. It turns out that if you observe the clocks at simultaneous times in your frame, you will see the rear clock showing a higher reading than the front clock (see Fig. 10.3). By how much?

**Solution:** As above, let's put a light source on the train, but let's now position it so that the light hits the clocks at the ends of the train at the same time *in your frame*. As above, the relative speeds of the photons and the clocks are $c+v$ and $c-v$ (as viewed in your frame). We therefore need to divide the train into lengths in this ratio, in your frame. But since length contraction (discussed in Section 10.2.3) is independent of position, this must also be the ratio in the train frame. So in the train frame, you can quickly show that two numbers that are in this ratio, and that add up to $L$, are $L(c+v)/2c$ and $L(c-v)/2c$.

The situation in the train frame therefore looks like that in Fig. 10.4. The light must travel an extra distance of $L(c+v)/2c - L(c-v)/2c = Lv/c$ to reach the rear clock. The extra time is therefore $Lv/c^2$. Hence, the rear clock reads $Lv/c^2$ more when it is hit by the backward photon, compared to what the front clock reads when it is hit by the forward photon.

Now, let the instant you look at the clocks be the instant the photons hit them (that's why we chose the hittings to be simultaneous in your frame). Then you observe the rear clock reading more than the front clock by an amount,

$$\text{(difference in readings)} = \frac{Lv}{c^2}\,. \tag{10.2}$$

Note that the $L$ that appears here is the length of the train in its own frame, and not the shortened length that you observe in your frame (see Section 10.2.3).

Remarks: The fact that the rear clock is *ahead* of the front clock in your frame means that the light hits the rear clock *after* it hits the front clock in the train frame.

Note that our result does *not* say that you see the rear clock ticking at a faster rate than the front clock. They run at the same rate. (Both have the same time-dilation factor relative to you; see Section 10.2.2.) The back clock is simply a fixed time ahead of the front clock, as seen by you.

It's easy to forget which of the clocks is the one that's ahead. But a helpful mnemonic for remembering "rear clock ahead" is that both the first and fourth letters in each word form the same acronym, "rca," which is an anagram for "car," which is sort of like a train. Sure. ♣

### 10.2.2 Time dilation

We present here the classic example of a light beam traveling vertically on a train. Let there be a light source on the floor of the train and a mirror on the ceiling, which is a height $h$ above the floor. Let observer $A$ be on the train, and observer $B$ be on the ground. The speed of the train with respect to the ground is $v$.[8] A flash of light is emitted. The light travels up to the mirror and then back down to the source.

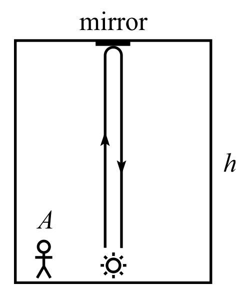

Figure 10.5

In $A$'s frame, the train is at rest. The path of the light is shown in Fig. 10.5. It takes the light a time $h/c$ to reach the ceiling and then a time $h/c$ to return to the source. The roundtrip time is therefore

$$t_A = \frac{2h}{c}\,. \tag{10.3}$$

In $B$'s frame, the train moves at speed $v$. The path of the light is shown in Fig. 10.6. The crucial fact to remember is that the speed of light in $B$'s frame is still $c$. This means that the light travels along its diagonally upward path at speed $c$. (The vertical component of the speed is *not* $c$, as would be the case if light behaved like a baseball.) Since the horizontal component of the light's velocity is $v$,[9] the vertical component must be $\sqrt{c^2 - v^2}$, as shown in Fig. 10.7.[10] The time it takes to reach the mirror is therefore $h/\sqrt{c^2 - v^2}$,[11] so the roundtrip time is

$$t_B = \frac{2h}{\sqrt{c^2 - v^2}}\,. \tag{10.4}$$

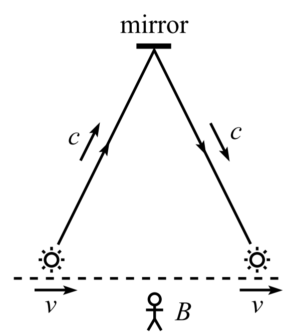

Figure 10.6

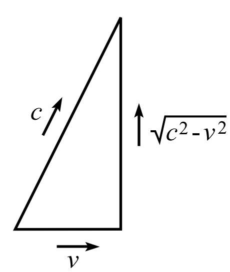

Figure 10.7

[8] Technically, the words, "with respect to...," should *always* be included when talking about speeds, because there is no absolute reference frame, and hence no absolute speed. But in the future, when it is clear what we mean (as in the case of a train moving with respect to the ground), we will occasionally be sloppy and drop the "with respect to...."

[9] Yes, it is still $v$. The light is always located on the vertical line between the source and the mirror. Since both of these move horizontally at speed $v$, the light does also.

[10] The Pythagorean theorem is indeed valid here. It is valid for distances, and since speeds are simply distances divided by time, it is also valid for speeds.

[11] We've assumed that the height of the train in $B$'s frame is still $h$. Although we will see in Section 10.2.3 that there is length contraction along the direction of motion, there is none in the direction perpendicular to the motion (see Problem 1).

Dividing eq. (10.4) by eq. (10.3) gives

$$t_B = \gamma t_A, \tag{10.5}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{10.6}$$

This $\gamma$ factor is ubiquitous in special relativity. Note that it is always greater than or equal to 1. This means that the roundtrip time is longer in $B$'s frame than in $A$'s frame.

What are the implications of this? For concreteness, let $v/c = 3/5$, which gives $\gamma = 5/4$. Then we may say the following. If $A$ is standing next to the light source, and if $B$ is standing on the ground, and if $A$ claps his hands at $t_A = 4$ second intervals, then $B$ will observe claps at $t_B = 5$ second intervals (after having subtracted off the time for the light to travel to her eye, of course). This is true because both $A$ and $B$ must agree on the number of roundtrips the light beam takes between claps. Assuming, for convenience, that a roundtrip takes one second in $A$'s frame, the four roundtrips between claps will take five seconds in $B$'s frame, using eq. (10.5). And if we have a train that does not contain one of our special clocks, that's no matter. We *could* have built one if we wanted to, so the same results concerning the claps must hold.

Therefore, $B$ will observe $A$ moving strangely slowly. $B$ will observe $A$'s heartbeat beating slowly; his blinks will be a bit lethargic; and his sips of coffee will be slow enough to suggest that he needs another cup.

> The effects of dilation of time
> Are magical, strange, and sublime.
> In your frame, this verse,
> Which you'll see is not terse,
> Can be read in the same amount of time it takes someone
> else in another frame to read a similar sort of rhyme.

Note that we can make these conclusions only if $A$ is at rest with respect to the train. If $A$ is moving with respect to the train, then eq. (10.5) does not hold, because we *cannot* say that both $A$ and $B$ must agree on the number of roundtrips the light beam takes between claps, because of the problem of simultaneity. It cannot be said which flash of light happens at the same time as a clap. Such a statement depends on the frame.

REMARKS:

1. The time dilation result derived in eq. (10.5) is a bit strange, no doubt, but there doesn't seem to be anything downright incorrect about it until we look at the situation from $A$'s point of view. $A$ sees $B$ flying by at a speed $v$ in the other direction. The ground is no more fundamental than a train, so the same reasoning applies. The time dilation factor, $\gamma$, doesn't depend on the sign of $v$, so $A$ sees the same time dilation factor that $B$ sees. That is, $A$ sees $B$'s clock running slow. But how can this be? Are

we claiming that $A$'s clock is slower than $B$'s, and also that $B$'s clock is slower than $A$'s? Well. . . yes and no.

Remember that the above time-dilation reasoning applies only to a situation where something is motionless in the appropriate frame. In the second situation (where $A$ sees $B$ flying by), the statement $t_A = \gamma t_B$ holds only when the events happen at the same place in $B$'s frame. But for two such events, they are not in the same place in $A$'s frame, so the $t_B = \gamma t_A$ result of eq. (10.5) does *not* hold. The conditions of being motionless in each frame never both hold for a given setup (unless $v = 0$, in which case $\gamma = 1$ and $t_A = t_B$). So, the answer to the question at the end of the previous paragraph is "yes" if you ask the questions in the appropriate frames, and "no" if you think the answer should be frame independent.

2. Concerning the fact that $A$ sees $B$'s clock run slow, *and* $B$ sees $A$'s clock run slow, consider the following statement. "This is a contradiction. It is essentially the same as saying, 'I have two apples on a table. The left one is bigger than the right one, and the right one is bigger than the left one.' " How would you respond to this?

   Well, it is not a contradiction. Observers $A$ and $B$ are using *different coordinates* to measure time. The times measured in each of their frames are quite different things. The seemingly contradictory time-dilation result is really no stranger than having two people run away from each other into the distance, and having them both say that the other person looks smaller.

   In short, we are not comparing apples and apples. We are comparing apples and oranges. A more correct analogy would be the following. An apple and an orange sit on a table. The apple says to the orange, "You are a much uglier apple than I am," and the orange says to the apple, "You are a much uglier orange than I am."

3. One might view the statement, "$A$ sees $B$'s clock running slowly, and also $B$ sees $A$'s clock running slowly," as somewhat unsettling. But in fact, it would be a complete disaster for the theory if $A$ and $B$ viewed each other in different ways. A critical postulate in the theory of relativity is that $A$ sees $B$ in exactly the same way as $B$ sees $A$. ♣

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**Example (Twin paradox):** Twin $A$ stays on the earth, while twin $B$ flies quickly to a distant star and back (see Fig. 10.8). Show that $B$ is younger when they meet up again.

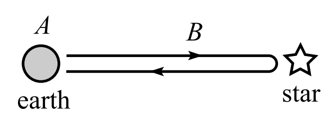


**Figure 10.8**

**Solution:** From $A$'s point of view, $B$'s clock is running slow by a factor $\gamma$, on both the outward and return parts of the trip. Therefore, $B$ is younger when they meet up again.

That's all there is to it. But although the above reasoning is correct, it leaves one large point unaddressed. The "paradox" part of this problem's title comes from the following reasoning. You might say that in $B$'s frame, $A$'s clock is running slow by a factor $\gamma$, and so $A$ is younger when they meet up again.

It is definitely true that when the two twins are standing next to each other (that is, when they are in the same frame), we can't have both $B$ younger than $A$, and $A$ younger than $B$. So what is wrong with the reasoning in the previous paragraph? The error lies in the fact that there is no "one frame" that $B$ is in. The inertial frame for the outward trip is different from the inertial frame for the return trip. The derivation of our time-dilation result applies only to one inertial frame.

Said in a different way, $B$ accelerates when she turns around, and our time-dilation result holds only from the point of view of an *inertial* observer.[12] The symmetry in the problem is broken by the acceleration. If both $A$ and $B$ are blindfolded, they can still tell who is doing the traveling, because $B$ will feel the acceleration at the turnaround. Constant velocity cannot be felt, but acceleration can be. (However, see Chapter 13 on General Relativity. Gravity complicates things.)

The above paragraphs show what is wrong with the "$A$ is younger" reasoning, but it doesn't show how to modify it quantitatively to obtain the correct answer. There are many different ways of doing this, and I'll let you tackle some of these in the problems. Also, Appendix K gives a list of all the possible resolutions to the twin paradox that I can think of.

---

**Example (Muon decay):** Elementary particles called *muons* (which are identical to electrons, except that they are about 200 times as massive) are created in the upper atmosphere when cosmic rays collide with air molecules. The muons have an average lifetime of about $2 \cdot 10^{-6}$ seconds[13] (then they decay into electrons, neutrinos, and the like), and move at nearly the speed of light.

Assume for simplicity that a certain muon is created at a height of 50 km, moves straight downward, has a speed $v = .99998\,c$, decays in exactly $T = 2 \cdot 10^{-6}$ seconds, and doesn't collide with anything on the way down.[14] Will the muon reach the earth before it decays?

**Solution:** The naive thing to say is that the distance traveled by the muon is $d = vT \approx (3 \cdot 10^8\,\text{m/s})(2 \cdot 10^{-6}\,\text{s}) = 600\,\text{m}$, and that this is less than 50 km, so the muon does not reach the earth. This reasoning is incorrect, because of the time-dilation effect. The muon lives longer in the earth frame, by a factor of $\gamma$, which is $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 160$ here. The correct distance traveled in the earth frame is therefore $v(\gamma T) \approx 100\,\text{km}$. Hence, the muon travels the 50 km, with room to spare.

The real-life fact that we actually do detect muons reaching the surface of the earth in the predicted abundances (while the naive $d = vT$ reasoning would predict that we shouldn't see any) is one of the many experimental tests that support the relativity theory.

---

### 10.2.3 Length contraction

Consider the following setup. Person $A$ stands on a train which he measures to have length $\ell'$, and person $B$ stands on the ground. A light source is located at the

[12]For the entire outward and return parts of the trip, $B$ *does* observe $A$'s clock running slow, but enough strangeness occurs during the turning-around period to make $A$ end up older. Note, however, that a discussion of acceleration is not required to quantitatively understand the paradox, as Problem 2 shows.

[13]This is the "proper" lifetime. That is, the lifetime as measured in the frame of the muon.

[14]In the real world, the muons are created at various heights, move in different directions, have different speeds, decay in lifetimes that vary according to a standard half-life formula, and may very well bump into air molecules. So technically we've got everything wrong here. But that's no matter. This example will work just fine for the present purpose.

back of the train, and a mirror is located at the front. The train moves at speed $v$ with respect to the ground. The source emits a flash of light which heads to the mirror, bounces off, then heads back to the source. By looking at how long this process takes in the two reference frames, we can determine the length of the train, as viewed by $B$.[15]

In $A$'s frame (see Fig. 10.9), the round-trip time for the light is simply

$$t_A = \frac{2\ell'}{c}. \tag{10.7}$$

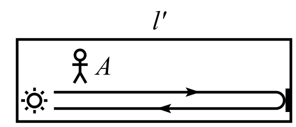


**Figure 10.9**

Things are a little more complicated in $B$'s frame (see Fig. 10.10). Let the length of the train, as viewed by $B$, be $\ell$. For all we know at this point, $\ell$ may equal $\ell'$, but we will soon find that it does not. The relative speed of the light and the mirror during the first part of the trip is $c - v$. The relative speed during the second part is $c+v$. During each part, the light must close a gap with initial length $\ell$. Therefore, the total round-trip time is

$$t_B = \frac{\ell}{c-v} + \frac{\ell}{c+v} = \frac{2\ell c}{c^2 - v^2} \equiv \frac{2\ell}{c}\gamma^2. \tag{10.8}$$

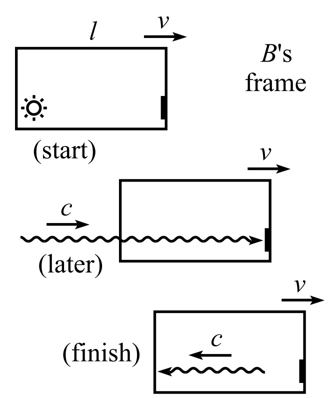


**Figure 10.10**

But we know from eq. (10.5) that

$$t_B = \gamma t_A. \tag{10.9}$$

Substituting the results for $t_A$ and $t_B$ from eqs. (10.7) and (10.8) into eq. (10.9), we find

$$\ell = \frac{\ell'}{\gamma}. \tag{10.10}$$

Since $\gamma \geq 1$, we see that $B$ measures the train to be shorter than $A$ measures.

Note that we could not have used this setup to find the length contraction if we had not already found the time dilation in eq. (10.5).

The term *proper length* is used to describe the length of an object in its rest frame. So $\ell'$ is the proper length of the above train.

Relativistic limericks have the attraction
Of being shrunk by a Lorentz contraction.
But for readers, unwary,
The results may be scary,
When a fraction...

Remarks:

1. The length-contraction result in eq. (10.10) is for lengths along the direction of the relative velocity. There is no length contraction in the perpendicular direction, as shown in Problem 1.

[15]The second remark below gives another (quicker) derivation of length contraction. But we'll go through with the present derivation since the calculation is instructive.

2. As with time dilation, this length contraction is a bit strange, but there doesn't seem to be anything actually paradoxical about it, until we look at things from $A$'s point of view. To make a nice symmetrical situation, let's say $B$ is standing on an identical train, which is motionless with respect to the ground. $A$ sees $B$ flying by at speed $v$ in the other direction. Neither train is any more fundamental than the other, so the same reasoning applies, and $A$ sees the same length contraction factor that $B$ sees. That is, $A$ measures $B$'s train to be short.

   But how can this be? Are we claiming that $A$'s train is shorter than $B$'s, and also that $B$'s train is shorter than $A$'s? Does the situation look like Fig. 10.11, or does it look like Fig. 10.12? Well...it depends.

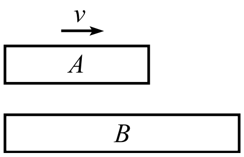


**Figure 10.11**

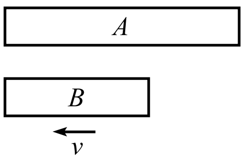


**Figure 10.12**

   The word "is" in the above paragraph is a very bad word to use, and is generally the cause of all the confusion. There is no such thing as "is-ness" when it comes to lengths. It makes no sense to say what the length of the train really *is*. It only makes sense to say what the length is in a given frame. The situation doesn't really *look like* one thing in particular. The look depends on the frame in which the looking is being done.

   Let's be a little more specific. How do you measure a length? You write down the coordinates of the ends of something *measured simultaneously*, and then you take the difference. But the word "simultaneous" here should send up all sorts of red flags. Simultaneous events in one frame are not simultaneous events in another.

   Stated more precisely, here is what we are really claiming: Let $B$ write down simultaneous coordinates of the ends of $A$'s train, and also simultaneous coordinates of the ends of her own train. Then the difference between the former is smaller than the difference between the latter. Likewise, let $A$ write down simultaneous coordinates of the ends of $B$'s train, and also simultaneous coordinates of the ends of his own train. Then the difference between the former is smaller than the difference between the latter. There is no contradiction here, because the times at which $A$ and $B$ are writing down the coordinates don't have much to do with each other, due to the loss of simultaneity. As with time dilation, we are comparing apples and oranges.

3. There is an easy argument to show that time dilation implies length contraction, and vice versa. Let $B$ stand on the ground, next to a stick of length $\ell$. Let $A$ fly past the stick at speed $v$. In $B$'s frame, it takes $A$ a time of $\ell/v$ to traverse the length of the stick. Therefore (assuming that we have demonstrated the time-dilation result), a watch on $A$'s wrist will advance by a time of only $\ell/\gamma v$ while he traverses the length of the stick.

   How does $A$ view the situation? He sees the ground and the stick fly by with speed $v$. The time between the two ends passing him is $\ell/\gamma v$ (since that is the time elapsed on his watch). To get the length of the stick in his frame, he simply multiplies the speed times the time. That is, he measures the length to be $(\ell/\gamma v)v = \ell/\gamma$, which is the desired contraction. The same argument also shows that length contraction implies time dilation.

4. As mentioned earlier, the length contraction factor $\gamma$ is independent of position. That is, all parts of the train are contracted by the same amount. This follows from the fact that all points in space are equivalent. Equivalently, we could put a number of small replicas of the above source-mirror system along the length of the train. They would all produce the same value for $\gamma$, independent of the position on the train. ♣

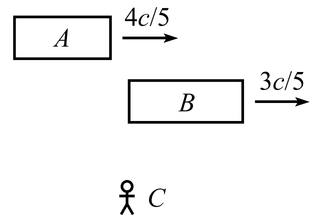

**Figure 10.13**

**Example:** Two trains, $A$ and $B$, each have proper length $L$ and move in the same direction. $A$'s speed is $4c/5$, and $B$'s speed is $3c/5$. $A$ starts behind $B$ (see Fig. 10.13). How long, as viewed by person $C$ on the ground, does it take for $A$ to overtake $B$? By this we mean the time between the front of $A$ passing the back of $B$, and the back of $A$ passing the front of $B$.

**Solution:** Relative to $C$ on the ground, the $\gamma$ factors associated with $A$ and $B$ are $5/3$ and $5/4$, respectively. Therefore, their lengths in the ground frame are $3L/5$ and $4L/5$. While overtaking $B$, $A$ must travel farther than $B$, by an excess distance equal to the sum of the lengths of the trains, which is $7L/5$. The relative speed of the two trains (as viewed by $C$ on the ground) is the difference of the speeds, which is $c/5$. The total time is therefore

$$t_C = \frac{7L/5}{c/5} = \frac{7L}{c}\,. \tag{10.11}$$

**Example (Muon decay, again):** Consider the "Muon decay" example from Section 10.2.2. From the muon's point of view, it lives for a time of $T = 2 \cdot 10^{-6}$ seconds, and the earth is speeding toward it at $v = .99998c$. How, then, does the earth (which will travel only $d = vT \approx 600\,\text{m}$ before the muon decays) reach the muon?

**Solution:** The important point here is that in the muon's frame, the distance to the earth is contracted by a factor $\gamma \approx 160$. Therefore, the earth starts only $50\,\text{km}/160 \approx 300\,\text{m}$ away. Since the earth can travel a distance of $600\,\text{m}$ during the muon's lifetime, the earth collides with the muon, with time to spare.

As stated in the third remark above, time dilation and length contraction are intimately related. We can't have one without the other. In the earth's frame, the muon's arrival on the earth is explained by time dilation. In the muon's frame, it is explained by length contraction.

Observe that for muons, created,
The dilation of time is related
To Einstein's insistence
Of shrunken-down distance
In the frame where decays aren't belated.

This concludes our treatment of the three fundamental effects. In the next section, we'll combine all the information we'll gained and use it to derive the Lorentz transformations.

In everything we've done so far, we've taken the route of having observers sitting in various frames, making various measurements. But as mentioned above, this can cause some ambiguity, because you might think that the time when someone actually sees something is important, whereas what we are generally concerned with is the time when something actually happens.

A way to avoid this ambiguity is to remove the observers and simply fill up space with a large rigid lattice of meter sticks and synchronized clocks. Different frames are defined by different lattices. All of the meter sticks in a given frame are at rest with respect to all the others, so there is no issue of length contraction within each frame. To measure the length of something, we simply need to determine where the ends are (at simultaneous times, as measured in that frame) with respect to the lattice.

As far as the synchronization of the clocks within each frame goes, this can be accomplished by putting a light source midway between any two of them and sending out a signal, and then setting the clocks to a certain value when the signal hits them. Alternatively, a more straightforward method of synchronization is to start with all the clocks synchronized right next to each other, and to then move them very slowly to their final positions. Any time-dilation effects can be made arbitrarily small by moving the clocks sufficiently slowly.

This lattice way of looking at things emphasizes that observers are not important, and that a frame is defined simply as a lattice of space and time coordinates. Anything that happens (an "event") is automatically assigned a space and time coordinate in every frame, independent of any observer. The concept of an "event" will be very important in the next section.

## 10.3 The Lorentz transformations

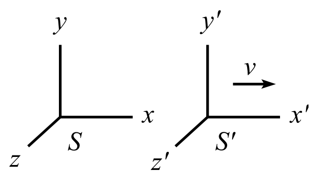

**Figure 10.14**

### 10.3.1 The derivation

Consider a coordinate system, $S'$, moving relative to another system, $S$ (see Fig. 10.14). Let the constant relative speed of the frames be $v$. Let the corresponding axes of $S$ and $S'$ point in the same direction, and let the origin of $S'$ move along the $x$-axis of $S$, in the positive direction. Nothing exciting happens in the $y$ and $z$ directions (see Problem 1), so we'll ignore them.

Our goal in this section is to look at two events (an event is simply anything that has space and time coordinates) in spacetime and relate the $\Delta x$ and $\Delta t$ of the coordinates in one frame to the $\Delta x'$ and $\Delta t'$ of the coordinates in another. We therefore want to find the constants $A$, $B$, $C$, and $D$ in the relations,

$$\begin{aligned} \Delta x &= A\,\Delta x' + B\,\Delta t', \\ \Delta t &= C\,\Delta t' + D\,\Delta x'. \end{aligned} \tag{10.12}$$

The four constants here will end up depending on $v$ (which is constant, given the two inertial frames). But we will not explicitly write this dependence, for ease of notation.

REMARKS:

1. We have assumed in eq. (10.12) that $\Delta x$ and $\Delta t$ are linear functions of $\Delta x'$ and $\Delta t'$. And we have also assumed that $A$, $B$, $C$, and $D$ are constants (that is, dependent only on $v$, and not on $x$,$t$,$x'$,$t'$).

   The first of these assumptions is justified by the fact that any finite interval can be built up from a series of many infinitesimal ones. But for an infinitesimal interval, any terms such as, for example, $(\Delta t')^2$, are negligible compared to the linear terms.

Therefore, if we add up all the infinitesimal intervals to obtain a finite one, we will be left with only the linear terms. Equivalently, it shouldn't matter whether we make a measurement with, say, meter sticks or half-meter sticks.

The second assumption can be justified in various ways. One is that all inertial frames should agree on what "non-accelerating" motion is. That is, if $\Delta x' = u'\,\Delta t'$, then we should also have $\Delta x = u\,\Delta t$, for some constant $u$. This is true only if the above coefficients are constants. Another justification comes from the second of our two relativity postulates, which says that all points in (empty) space are indistinguishable. With this in mind, let us assume that we have a transformation of the form, say, $\Delta x = A\,\Delta x' + B\,\Delta t' + Ex'\,\Delta x'$. The $x'$ in the last term implies that the absolute location in spacetime (and not just the relative position) is important. Therefore, this last term cannot exist.

2. If the relations in eq. (10.12) turned out to be the usual Galilean transformations (which are the ones that hold for everyday relative speeds, $v$) then we would have $\Delta x = \Delta x' + v\,\Delta t$, and $\Delta t = \Delta t'$ (that is, $A = C = 1$, $B = v$, and $D = 0$). We will find, however, under the assumptions of Special Relativity, that this is *not* the case. The Galilean transformations are not the correct transformations. But we will show below that the correct transformations do indeed reduce to the Galilean transformations in the limit of slow speeds, as they must. ♣

The constants $A$, $B$, $C$, and $D$ in eq. (10.12) are four unknowns, and we can solve for them by using four facts we found above in Section 10.2. These four facts we will use are:

| | effect | condition | result | eq. in text |
|---|---|---|---|---|
| 1 | Time dilation | $x' = 0$ | $t = \gamma t'$ | (10.5) |
| 2 | Length contraction | $t' = 0$ | $x' = x/\gamma$ | (10.10) |
| 3 | Relative $v$ of frames | $x = 0$ | $x' = -vt'$ | |
| 4 | "Head-start" | $t = 0$ | $t' = -vx'/c^2$ | (10.2) |

We have taken the liberty of dropping the $\Delta$'s in front of the coordinates, lest things get too messy. We will continue to omit the $\Delta$'s in what follows, but it should be understood that $x$ really means $\Delta x$, etc. We are always concerned with the *difference* between coordinates of two events in spacetime. The actual value of any coordinate is irrelevant, because there is no preferred origin in any frame.

You should pause for a moment and verify that the four "results" in the above table are in fact the proper mathematical expressions for the four effects, given the stated "conditions."[16] My advice is to continue pausing until you are comfortable with the conditions for time dilation and length contraction, discussed in various remarks above. Note also that the sign in the "Head-start" effect is indeed correct, because the front clock shows less time than the rear clock. So the clock with the higher $x'$ value is the one with the lower $t'$ value.

We can now use our four facts in the above table to quickly solve for the unknowns $A$, $B$, $C$, and $D$ in eq. (10.12).

[16]Of course, we can state the effects in other ways by switching the primes and unprimes. For example, time dilation can be written as "$t' = \gamma t$ when $x = 0$". But we've chosen the above ways of writing things because they will allow us to solve for the four unknowns in the quickest way.

Fact (1) gives $C = \gamma$.

Fact (2) gives $A = \gamma$.

Fact (3) gives $B/A = v \Longrightarrow B = \gamma v$.

Fact (4) gives $D/C = v/c^2 \Longrightarrow D = \gamma v/c^2$.

Eqs. (10.12), which are known as the *Lorentz transformations*, are therefore given by[17]

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v\,\Delta t'), \\ \Delta t &= \gamma(\Delta t' + v\,\Delta x'/c^2), \\ \Delta y &= \Delta y', \\ \Delta z &= \Delta z', \end{aligned} \tag{10.13}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{10.14}$$

We have tacked on the trivial transformations for $y$ and $z$, but we won't bother writing these in the future.

If we solve for $x'$ and $t'$ in terms of $x$ and $t$ in eq. (10.13), then we see that the inverse Lorentz transformations are given by

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx/c^2). \end{aligned} \tag{10.15}$$

Of course, which ones are the "inverse" transformations depends simply on your point of view. But it's intuitively clear that the only difference between the two sets of equations is the sign of $v$, because $S$ is simply moving backwards with respect to $S'$.

The reason why the derivation of eqs. (10.13) was so quick is that we already did most of the work in Section 10.2, when we derived the fundamental effects. If we wanted to derive the Lorentz transformations from scratch, that is, by starting with the two postulates in Section 10.1, then the derivation would be much longer. In Appendix I we give such a derivation, where it is clear what information comes from each of the postulates. The procedure there is somewhat cumbersome, but it's worth taking a look at, because we will invoke the results in a very cool way in Section 10.8.

Remarks:

1. In the limit $v \ll c$, eqs. (10.13) reduce to $x = x' + vt$ and $t = t'$, that is, simply the Galilean transformations. This must be the case, because we know from everyday experience (where $v \ll c$) that the Galilean transformations work just fine.

[17] This will be the last time we'll write the $\Delta$'s, but remember that they're always really there.

2. Eqs. (10.13) exhibit a nice symmetry between $x$ and $ct$. With $\beta \equiv v/c$, they become

$$\begin{aligned} x &= \gamma\big(x' + \beta(ct')\big), \\ ct &= \gamma\big((ct') + \beta x'\big). \end{aligned} \tag{10.16}$$

   Equivalently, in units where $c = 1$ (for example, where one unit of distance equals $3 \cdot 10^8$ meters, or where one unit of time equals $1/(3 \cdot 10^8)$ seconds), eqs. (10.13) take the symmetric form

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'). \end{aligned} \tag{10.17}$$

3. In matrix form, eqs. (10.16) are

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.18}$$

   This looks similar to a rotation matrix. More about this in Section 10.7, and in Problem 26.

4. The plus or minus sign on the right-hand side of the L.T.'s in eqs. (10.13) and (10.15) corresponds to which way the coordinate system on the left-hand side sees the coordinate system on the right-hand side moving. But if you get confused about the sign, simply write down $x_A = \gamma(x_B \pm vt_B)$, and then imagine sitting in system $A$ and looking at a fixed point in $B$, which satisfies (putting the $\Delta$'s back in to avoid any mixup) $\Delta x_B = 0$, which gives $\Delta x_A = \pm\gamma v \Delta t_B$. If the point moves to the right (that is, if it increases as time increases), then pick the "+". If it moves to the left, then pick the "−". In other words, the sign is determined by which way $A$ sees $B$ moving.

5. One very important thing we must check is that two successive Lorentz transformations (from $S_1$ to $S_2$ and then from $S_2$ to $S_3$) again yield a Lorentz transformation (from $S_1$ to $S_3$). This must be true because we showed that any two frames must be related by eq. (10.13). If we composed two L.T.'s and found that the transformation from $S_1$ to $S_3$ was not of the form of eqs. (10.13), for some new $v$, then the whole theory would be inconsistent, and we would have to drop one of our postulates. We'll let you show that the combination of an L.T. (with speed $v_1$) and an L.T. (with speed $v_2$) does indeed yield an L.T. (with speed $(v_1 + v_2)/(1 + v_1 v_2/c^2)$, which we'll see again in the velocity-addition formula in Section 10.3.3). This is the task of Exercise 14, and also Problem 26 (which is stated in terms of *rapidity*, introduced in Section 10.7). ♣

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**Example:** A train with proper length $L$ moves with speed $5c/13$ with respect to the ground. A ball is thrown from the back of the train to the front. The speed of the ball with respect to the train is $c/3$. As viewed by someone on the ground, how much time does the ball spend in the air, and how far does it travel?

**Solution:** The $\gamma$ factor associated with the speed $5c/13$ is $\gamma = 13/12$. The two events we are concerned with are "ball leaving back of train" and "ball arriving at front of train." The spacetime separation between these events is easy to calculate on the train. We have $\Delta x_T = L$, and $\Delta t_T = L/(c/3) = 3L/c$. The Lorentz transformations giving the coordinates on the ground are

$$\begin{aligned} x_G &= \gamma(x_T + vt_T), \\ t_G &= \gamma(t_T + vx_T/c^2). \end{aligned} \tag{10.19}$$

Therefore,

$$\begin{aligned} x_G &= \frac{13}{12}\left(L + \left(\frac{5c}{13}\right)\left(\frac{3L}{c}\right)\right) = \frac{7L}{3}, \qquad \text{and} \\ t_G &= \frac{13}{12}\left(\frac{3L}{c} + \frac{\frac{5c}{13}L}{c^2}\right) = \frac{11L}{3c}. \end{aligned} \tag{10.20}$$

---

In a given problem, such as the above example, one of the frames usually allows for an easy calculation of $\Delta x$ and $\Delta t$, so you simply have to mechanically plug these quantities into the L.T.'s to obtain $\Delta x'$ and $\Delta t'$ in the other frame, where they may not be as obvious.

Relativity is a subject where there are usually many ways to do a problem. If you are trying to find some $\Delta x$'s and $\Delta t$'s, then you can use the L.T.'s, or perhaps the invariant interval (introduced in Section 10.4), or maybe a velocity-addition approach (introduced in Section 10.3.3), or even the sending-of-light-signals strategy used in Section 10.2. Depending on the specific problem and what your personal preferences are, certain approaches will be more enjoyable than others. But no matter which method you choose, you should take advantage of the plethora of possibilities by picking a second method to double-check your answer. Personally, I find the L.T.'s to be the perfect option for this, because the other methods are generally more fun when solving a problem for the first time, while the L.T.'s are usually quick and easy to apply (perfect for a double-check).[18]

The excitement will build in your voice,
As you rise from your seat and rejoice,
"A Lorentz transformation
Provides confirmation
Of my alternate method of choice!"

### 10.3.2 The fundamental effects

Let us now see how the Lorentz transformations imply the three fundamental effects (namely, loss of simultaneity, time dilation, and length contraction) discussed in Section 10.2. Of course, we just used these effects to *derive* the Lorentz transformation, so we know everything will work out. We'll just be going in circles. But since these fundamental effects are, well, fundamental, let's belabor the point and discuss them one more time, with the starting point being the Lorentz transformations.

[18]I would, however, be very wary of solving a problem using only the L.T.'s, with no other check, because it's very easy to mess up a sign in the transformations. And since there's nothing to do except mechanically plug in numbers, there's not much opportunity for an intuitive check, either.

### Loss of Simultaneity

Let two events occur simultaneously in frame $S'$. Then the separation between them, as measured by $S'$, is $(x', t') = (x', 0)$. As usual, we are not bothering to write the $\Delta$'s in front of the coordinates. Using the second of eqs. (10.13), we see that the time between the events, as measured by $S$, is $t = \gamma v x'/c^2$. This is not equal to zero (unless $x' = 0$). Therefore, the events do not occur simultaneously in frame $S$.

### Time dilation

Consider two events that occur in the same place in $S'$. Then the separation between them is $(x', t') = (0, t')$. Using the second of eqs. (10.13), we see that the time between the events, as measured by $S$, is

$$t = \gamma t' \qquad (\text{if } x' = 0). \tag{10.21}$$

The factor $\gamma$ is greater than or equal to 1, so $t \geq t'$. The passing of one second on $S'$'s clock takes more than one second on $S$'s clock. $S$ sees $S'$ drinking his coffee very slowly.

The same strategy works if we interchange $S$ and $S'$. Consider two events that occur in the same place in $S$. The separation between them is $(x, t) = (0, t)$. Using the second of eqs. (10.15), we see that the time between the events, as measured by $S'$, is

$$t' = \gamma t \qquad (\text{if } x = 0). \tag{10.22}$$

Therefore, $t' \geq t$. Another way to derive this is to use the first of eqs. (10.13) to write $x' = -vt'$, and then substitute this into the second equation.

Remark: If we write down the two above equations by themselves, $t = \gamma t'$ and $t' = \gamma t$, they appear to contradict each other. This apparent contradiction arises from the omission of the conditions they are based on. The former equation is based on the assumption that $x' = 0$. The latter equation is based on the assumption that $x = 0$. They have nothing to do with each other. It would perhaps be better to write the equations as

$$\begin{aligned} (t &= \gamma t')_{x'=0}, \\ (t' &= \gamma t)_{x=0}, \end{aligned} \tag{10.23}$$

but this is somewhat cumbersome. ♣

### Length contraction

This proceeds just like the time dilation above, except that now we want to set certain time intervals equal to zero, instead of certain space intervals. We want to do this because to measure a length, we simply measure the distance between two points whose positions are measured *simultaneously*. That's what a length is.

Consider a stick at rest in $S'$, where it has length $\ell'$. We want to find the length $\ell$ in $S$. Simultaneous measurements of the coordinates of the ends of the stick in $S$ yield a separation of $(x, t) = (x, 0)$. Using the first of eqs. (10.15), we have

$$x' = \gamma x \qquad (\text{if } t = 0). \tag{10.24}$$

But $x$ is by definition the length in $S$. And $x'$ is the length in $S'$, because the stick is not moving in $S'$.[19] Therefore, $\ell = \ell'/\gamma$. Since $\gamma \geq 1$, we have $\ell \leq \ell'$, so $S$ sees the stick shorter than $S'$ sees it. Another way to derive eq. (10.24) is to use the second of eqs. (10.13) to write $t' = -(v/c^2)x'$, and then substitute this into the first equation.

Now interchange $S$ and $S'$. Consider a stick at rest in $S$, where it has length $\ell$. We want to find the length in $S'$. Measurements of the coordinates of the ends of the stick in $S'$ yield a separation of $(x', t') = (x', 0)$. Using the first of eqs. (10.13), we have

$$x = \gamma x' \qquad (\text{if } t' = 0). \tag{10.25}$$

But $x'$ is by definition the length in $S'$. And $x$ is the length in $S$, because the stick is not moving in $S$. Therefore, $\ell' = \ell/\gamma$, so $\ell' \leq \ell$.

Remark: As with time dilation, if we write down the two above equations by themselves, $\ell = \ell'/\gamma$ and $\ell' = \ell/\gamma$ they appear to contradict each other. But as before, this apparent contradiction arises from the omission of the conditions they are based on. The former equation is based on the assumptions that $t = 0$ and that the stick is at rest in $S'$. The latter equation is based on the assumptions that $t' = 0$ and that the stick is at rest in $S$. They have nothing to do with each other. We should really write,

$$\begin{aligned} (x &= x'/\gamma)_{t=0}, \\ (x' &= x/\gamma)_{t'=0}, \end{aligned} \tag{10.26}$$

and then identify $x'$ in the first equation with $\ell'$ only after invoking the further assumption that the stick is at rest in $S'$; likewise for the second equation. But this is a pain. ♣

### 10.3.3 Velocity addition

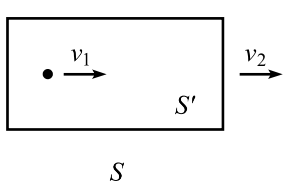


**Figure 10.15**

#### Longitudinal velocity addition

Consider the following setup. An object moves at speed $v_1$ with respect to frame $S'$. And frame $S'$ moves at speed $v_2$ with respect to frame $S$, in the same direction as the motion of the object; see Fig. 10.15. What is the speed, $u$, of the object with respect to frame $S$?

The Lorentz transformations may be used to easily answer this question. The relative speed of the frames is $v_2$. Consider two events along the object's path (for example, say it makes some beeps). We are given that $\Delta x'/\Delta t' = v_1$. Our goal is to find $u \equiv \Delta x/\Delta t$.

The Lorentz transformations from $S'$ to $S$, eqs. (10.13), are

$$\Delta x = \gamma_2(\Delta x' + v_2 \Delta t'), \qquad \text{and} \qquad \Delta t = \gamma_2(\Delta t' + v_2 \Delta x'/c^2), \tag{10.27}$$

[19] The measurements of the ends made by $S$ will *not* be simultaneous in the $S'$ frame. In the $S'$ frame, the separation between the events is $(x', t')$, where both $x'$ and $t'$ are nonzero. This does not satisfy our definition of a length measurement in $S'$ (because $t' \neq 0$), but the stick is not moving in $S'$, so $S'$ can measure the ends whenever he feels like it, and he will always get the same difference. So $x'$ is indeed the length in the $S'$ frame.

where $\gamma_2 \equiv 1/\sqrt{1-v_2^2/c^2}$. Therefore,

$$\begin{aligned} u \equiv \frac{\Delta x}{\Delta t} &= \frac{\Delta x' + v_2 \Delta t'}{\Delta t' + v_2 \Delta x'/c^2} \\ &= \frac{\Delta x'/\Delta t' + v_2}{1 + v_2(\Delta x'/\Delta t')/c^2} \\ &= \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \,. \end{aligned} \tag{10.28}$$

This is the *velocity-addition formula* (for adding velocities in the same direction). Let's look at some of its properties. (1) It is symmetric with respect to $v_1$ and $v_2$, as it should be, because we could switch the roles of the object and frame $S$. (2) For $v_1 v_2 \ll c^2$, it reduces to $u \approx v_1 + v_2$, which we know holds perfectly well for everyday speeds. (3) If $v_1 = c$ or $v_2 = c$, then we find $u = c$, as should be the case, because anything that moves with speed $c$ in one frame moves with speed $c$ in another. (4) The maximum (or minimum) of $u$ in the region $-c \le v_1, v_2 \le c$ equals $c$ (or $-c$), which can be seen by noting that $\partial u/\partial v_1$ and $\partial u/\partial v_2$ are never zero in the interior of the region.

If you take any two velocities that are less than $c$, and add them according to eq. (10.28), then you will obtain a velocity that is again less than $c$. This shows that no matter how much you keep accelerating an object (that is, no matter how many times you give the object a speed $v_1$ with respect to the frame moving at speed $v_2$ that it was just in), you can't bring the speed up to the speed of light. We'll give another argument for this result in Chapter 11 when we discuss energy.

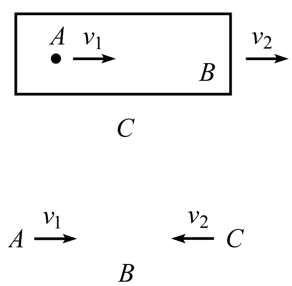


**Figure 10.16**

Remark: Consider the two scenarios shown in Fig. 10.16. If the goal is to find the speed of $A$ with respect to $C$, then the velocity-addition formula applies to both scenarios, because the second scenario is simply the first one, as observed in $B$'s frame.

The velocity-addition formula applies when we ask, "If $A$ moves at $v_1$ with respect to $B$, and $B$ moves at $v_2$ with respect to $C$ (which means, of course, that $C$ moves at speed $v_2$ with respect to $B$), then how fast does $A$ move with respect to $C$?" The formula does *not* apply if we ask the more mundane question, "What is the relative speed of $A$ and $C$, as viewed by $B$?" The answer to this is simply $v_1 + v_2$.

In short, if the two speeds are given with respect to $B$, and if you are asking for the relative speed as measured by $B$, then you simply add the speeds.[20] But if you are asking for the relative speed as measured by $A$ or $C$, then you have to use the velocity-addition formula. ♣

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**Example:** Consider again the scenario in the first example in Section 10.2.3.

(a) How long, as viewed by $A$ and as viewed by $B$, does it take for $A$ to overtake $B$?

[20] Note that the resulting speed can certainly be greater than $c$. If I see a ball heading toward me at $.9c$ from the right, and another one heading toward me at $.9c$ from the left, then the relative speed of the balls in my frame is $1.8c$. In the frame of one of the balls, however, the relative speed is $(1.8/1.81)c \approx (.9945)c$, from eq. (10.28).

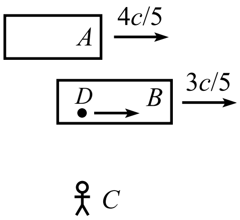


**Figure 10.17**

(b) Let event $E_1$ be "the front of $A$ passing the back of $B$", and let event $E_2$ be "the back of $A$ passing the front of $B$". Person $D$ walks at constant speed from the back of train $B$ to its front (see Fig. 10.17), such that he coincides with both events $E_1$ and $E_2$. How long does the "overtaking" process take, as viewed by $D$?

**Solution:**

(a) First consider $B$'s point of view. From the velocity-addition formula, $B$ sees $A$ move with speed

$$u = \frac{\frac{4c}{5} - \frac{3c}{5}}{1 - \frac{4}{5}\cdot\frac{3}{5}} = \frac{5c}{13}. \tag{10.29}$$

The $\gamma$ factor associated with this speed is $\gamma = 13/12$. Therefore, $B$ sees $A$'s train contracted to a length $12L/13$. During the overtaking, $A$ must travel a distance equal to the sum of the lengths of the trains in $B$'s frame, which is $L + 12L/13 = 25L/13$. Since $A$ moves at speed $5c/13$, the total time in $B$'s frame is

$$t_B = \frac{25L/13}{5c/13} = \frac{5L}{c}. \tag{10.30}$$

The exact same reasoning holds from $A$'s point of view, so we have $t_A = t_B = 5L/c$.

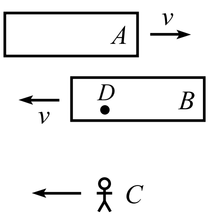


**Figure 10.18**

(b) Look at things from $D$'s point of view. $D$ is at rest, and the two trains move with equal and opposite speeds, $v$, as shown in Fig. 10.18. The relativistic addition of $v$ with itself must equal the $5c/13$ speed we found in part (a). Therefore,

$$\frac{2v}{1 + v^2/c^2} = \frac{5c}{13} \qquad \Longrightarrow \qquad v = \frac{c}{5}. \tag{10.31}$$

The $\gamma$ factor associated with this speed is $\gamma = 5/(2\sqrt{6})$. Therefore, $D$ sees both trains contracted to a length $2\sqrt{6}L/5$. During the overtaking, each train must travel a distance equal to its length, because both events, $E_1$ and $E_2$, take place right at $D$. The total time in $D$'s frame is therefore

$$t_D = \frac{2\sqrt{6}L/5}{c/5} = \frac{2\sqrt{6}L}{c}. \tag{10.32}$$

Remarks: There are a few double-checks we can perform. The speed of $D$ with respect to the ground can be obtained by either relativistically adding $3c/5$ and $c/5$, or subtracting $c/5$ from $4c/5$. These both give the same answer, namely $5c/7$, as they must. (The $c/5$ speed can in fact be determined by this reasoning, instead of using eq. (10.31).) The $\gamma$ factor between the ground and $D$ is therefore $7/2\sqrt{6}$. We can now use time dilation to say that someone on the ground sees the overtaking take a time of $(7/2\sqrt{6})t_D$. (We can say this because both events happen right at $D$.) Using eq. (10.32), this gives a ground-frame time of $7L/c$, in agreement with eq. (10.11).

Likewise, the $\gamma$ factor between $D$ and either train is $5/2\sqrt{6}$. So the time of the overtaking as viewed by either $A$ or $B$ is $(5/2\sqrt{6})t_D = 5L/c$, in agreement with eq. (10.30).

Note that we *cannot* use simple time dilation to relate the ground to $A$ or $B$, because the two events don't happen at the same place in the train frames. But since both events happen at the same place in $D$'s frame, namely right at $D$, we can indeed use time dilation to go from $D$'s frame to any other frame. ♣

**Transverse velocity addition**

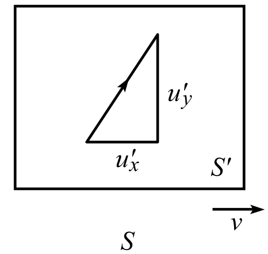


**Figure 10.19**

Consider the following general two-dimensional situation. An object moves with velocity $(u'_x, u'_y)$ with respect to frame $S'$. And frame $S'$ moves with speed $v$ with respect to frame $S$, in the $x$-direction (see Fig. 10.19). What is the velocity, $(u_x, u_y)$, of the object with respect to frame $S$?

The existence of motion in the $y$-direction doesn't affect the preceding derivation of the speed in the $x$-direction, so eq. (10.28) is still valid. In the present notation, it becomes

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,. \tag{10.33}$$

To find $u_y$, we may again make easy use of the Lorentz transformations. Consider two events along the object's path. We are given that $\Delta x'/\Delta t' = u'_x$, and $\Delta y'/\Delta t' = u'_y$. Our goal is to find $u_y \equiv \Delta y/\Delta t$. The relevant Lorentz transformations from $S'$ to $S$ in eqs. (10.13) are

$$\Delta y = \Delta y', \qquad \text{and} \qquad \Delta t = \gamma(\Delta t' + v\Delta x'/c^2). \tag{10.34}$$

Therefore,

$$\begin{aligned} u_y \equiv \frac{\Delta y}{\Delta t} &= \frac{\Delta y'}{\gamma(\Delta t' + v\Delta x'/c^2)} \\ &= \frac{\Delta y'/\Delta t'}{\gamma(1 + v(\Delta x'/\Delta t')/c^2)} \\ &= \frac{u'_y}{\gamma(1 + u'_x v/c^2)}\,. \end{aligned} \tag{10.35}$$

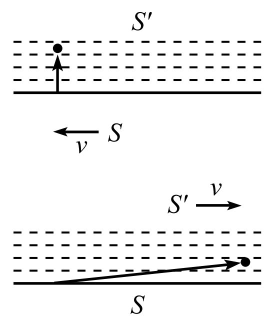


**Figure 10.20**

REMARK: In the special case where $u'_x = 0$, we have $u_y = u'_y/\gamma$. When $u'_y$ is small and $v$ is large, this result can be seen to be a special case of time dilation, in the following way. Consider a series of equally spaced lines parallel to the $x$-axis (see Fig. 10.20). Imagine that the object's clock ticks once every time it crosses a line. Since $u'_y$ is small, the object's frame is essentially frame $S'$, so the object is essentially moving at speed $v$ with respect to $S$. Therefore, $S$ sees the clock run slow by a factor $\gamma$. This means that $S$ sees the object cross the lines at a slower rate, by a factor $\gamma$ (because the clock still ticks once every time it crosses a line; this is a frame-independent statement). Since distances in the $y$-direction are the same in the two frames, we conclude that $u_y = u'_y/\gamma$. This $\gamma$ factor will be very important when we deal with momentum in Chapter 11.

To sum up: if you run in the $x$-direction past an object, then its $y$-speed slows down from your point of view (or speeds up, depending on the relative sign of $u'_x$ and $v$). Strange indeed, but no stranger than other effects we've seen. Problem 15 deals with the special case where $u'_x = 0$, but where $u'_y$ is not necessarily small. ♣

## 10.4 The invariant interval

Consider the quantity,

$$(\Delta s)^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2. \tag{10.36}$$

Technically, we should also subtract off $(\Delta y)^2$ and $(\Delta z)^2$, but nothing exciting happens in the transverse directions, so we'll ignore these. Using eq. (10.13), we can write $(\Delta s)^2$ in terms of the $S'$ coordinates, $\Delta x'$ and $\Delta t'$. The result is (dropping the $\Delta$'s)

$$\begin{aligned} c^2t^2 - x^2 &= \frac{c^2(t' + vx'/c^2)^2}{1 - v^2/c^2} - \frac{(x' + vt')^2}{1 - v^2/c^2} \\ &= \frac{t'^2(c^2 - v^2) - x'^2(1 - v^2/c^2)}{1 - v^2/c^2} \\ &= c^2t'^2 - x'^2 \\ &\equiv s'^2. \end{aligned} \tag{10.37}$$

We see that the Lorentz transformations imply that the quantity $c^2t^2 - x^2$ does not depend on the frame. This result is more than we bargained for, for the following reason. The speed-of-light postulate says that if $c^2t'^2 - x'^2 = 0$, then $c^2t^2 - x^2 = 0$. But eq. (10.37) says that if $c^2t'^2 - x'^2 = b$, then $c^2t^2 - x^2 = b$, for *any* $b$, not just zero. This is, as you might guess, very useful. There are enough things that change when we go from one frame to another, so it's nice to have a frame-independent quantity that we can hang on to. The fact that $s^2$ is invariant under Lorentz transformations of $x$ and $t$ is exactly analogous to the fact that $r^2$ is invariant under rotations in the $x$-$y$ plane. The coordinates themselves change under the transformation, but the special combination of $c^2t^2 - x^2$ for Lorentz transformations, of $x^2 + y^2$ for rotations, remains the same. All inertial observers agree on the value of $s^2$, independent of what they measure for the actual coordinates.

"Potato?! Pot*ah*to!" said she,
"And of *course* it's tom*ah*to, you see.
But the square of $ct$
Minus $x^2$ will be
Always something on which we agree."

A note on terminology: The separation in the coordinates, $(c\Delta t, \Delta x)$, is usually referred to as the *spacetime interval*, while the quantity $(\Delta s)^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2$ is referred to as the *invariant interval* (or technically the square of the invariant interval). At any rate, just call it $s^2$, and people will know what you mean.

The invariance of $s^2$ is actually just a special case of more general results involving inner products and 4-vectors, which we'll discuss in Chapter 12. Let's now look at the physical significance of $s^2 \equiv c^2t^2 - x^2$. There are three cases to consider.

### Case 1: $s^2 > 0$ (timelike separation)

In this case, we say that the two events are *timelike* separated. We have $c^2t^2 > x^2$, and so $|x/t| < c$. Consider a frame $S'$ moving at speed $v$ with respect to $S$. The Lorentz transformation for $x$ is

$$x' = \gamma(x - vt). \tag{10.38}$$

Since $|x/t| < c$, there exists a $v$ which is less than $c$ (namely $v = x/t$) that makes $x' = 0$. In other words, if two events are timelike separated, it is possible to find a frame $S'$ in which the two events happen at the same place. In short, the condition $|x/t| < c$ means that it is possible for a particle to travel from one event to the other. The invariance of $s^2$ then gives $s^2 = c^2t'^2 - x'^2 = c^2t'^2$. So we see that $s/c$ is simply the time between the events in the frame in which the events occur at the same place. This time is called the *proper time*.

**Case 2:** $s^2 < 0$ **(spacelike separation)**

In this case, we say that the two events are *spacelike* separated.[21] We have $c^2t^2 < x^2$, and so $|t/x| < 1/c$. Consider a frame $S'$ moving at speed $v$ with respect to $S$. The Lorentz transformation for $t'$ is

$$t' = \gamma(t - vx/c^2). \tag{10.39}$$

Since $|t/x| < 1/c$, there exists a $v$ which is less than $c$ (namely $v = c^2t/x$) that makes $t' = 0$. In other words, if two events are spacelike separated, it is possible to find a frame $S'$ in which the two events happen at the same time. (This statement is not as easy to see as the corresponding one in the timelike case above. But if you draw a Minkowski diagram, described in the next section, it becomes clear.) The invariance of $s^2$ then gives $s^2 = c^2t'^2 - x'^2 = -x'^2$. So we see that $|s|$ is simply the distance between the events in the frame in which the events occur at the same time. This distance is called the *proper distance*.

**Case 3:** $s^2 = 0$ **(lightlike separation)**

In this case, we say that the two events are *lightlike* separated. We have $c^2t^2 = x^2$, and so $|x/t| = c$. This holds in every frame, so in every frame a photon emitted at one of the events will arrive at the other. It is not possible to find a frame $S'$ in which the two events happen at the same place or the same time, because the frame would have to travel at the speed of light.

---

**Example (Time dilation):** An illustration of the usefulness of the invariance of $s^2$ is a derivation of time dilation. Let frame $S'$ move at speed $v$ with respect to frame $S$. Consider two events at the origin of $S'$, separated by time $t'$. The separation between the events is

$$\begin{aligned} &\text{in } S': \quad (x', t') = (0, t'), \\ &\text{in } S: \quad (x, t) = (vt, t). \end{aligned} \tag{10.40}$$

The invariance of $s^2$ implies $c^2t'^2 - 0 = c^2t^2 - v^2t^2$. Therefore,

$$t = \frac{t'}{\sqrt{1 - v^2/c^2}}\,. \tag{10.41}$$

Here it is clear that this time-dilation result rests on the assumption that $x' = 0$.

[21] It's fine that $s^2$ is negative in this case, which means that $s$ is imaginary. We can simply take the absolute value of $s$ if we want to obtain a real number.

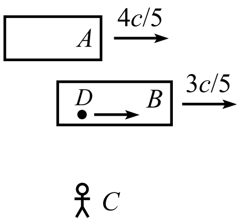


**Figure 10.21**

**Example:** Consider again the scenario in the examples in Sections 10.2.3 and 10.3.3. Verify that the $s^2$ between the events $E_1$ and $E_2$ is the same in all of the frames, $A$, $B$, $C$, and $D$ (see Fig. 10.21).

**Solution:** The only quantity that we'll need that we haven't already found in the two examples above is the distance between $E_1$ and $E_2$ in $C$'s frame (the ground frame). In this frame, train $A$ travels at a rate $4c/5$ for a time $t_C = 7L/c$, covering a distance of $28L/5$. But event $E_2$ occurs at the back of the train, which is a distance $3L/5$ behind the front end (this is the contracted length in the ground frame). Therefore, the distance between events $E_1$ and $E_2$ in the ground frame is $28L/5 - 3L/5 = 5L$. You can also apply the same line of reasoning using train $B$, in which the $5L$ takes the form $(3c/5)(7L/c) + 4L/5$.

Putting all of the previous results together, we have the following separations between events:

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $\Delta t$ | $5L/c$ | $5L/c$ | $7L/c$ | $2\sqrt{6}L/c$ |
| $\Delta x$ | $-L$ | $L$ | $5L$ | $0$ |

From the table, we see that $\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2 = 24L^2$ for all four frames, as desired. We could have, of course, worked backwards and used the $s^2 = 24L^2$ result from frames $A$, $B$, or $D$, to deduce that $\Delta x = 5L$ in frame $C$.

In Problem 11, you are asked to perform the mundane task of checking that the values in the above table satisfy the Lorentz transformations between the various pairs of frames.

## 10.5 Minkowski diagrams

Minkowski diagrams (or "spacetime" diagrams) are extremely useful in seeing how coordinates transform between different reference frames. If you want to produce exact numbers in a problem, you will probably have to use one of the strategies we've encountered so far. But as far as getting an overall intuitive picture in a problem goes (if there is in fact any such thing as intuition in relativity), there is no better tool than a Minkowski diagram. Here's how you make one.

Let frame $S'$ move at speed $v$ with respect to frame $S$ (along the $x$-axis, as usual; ignore the $y$ and $z$ components). Draw the $x$ and $ct$ axes of frame $S$.[22] What do the $x'$ and $ct'$ axes of $S'$ look like, superimposed on this diagram? That is, at what angles are the axes inclined, and what is the size of one unit on these axes? (There is no reason why one unit on the $x'$ and $ct'$ axes should have the same length on the paper as one unit on the $x$ and $ct$ axes.) We can answer these questions by using the Lorentz transformations, eqs. (10.13). We'll first look at the $ct'$ axis, and then the $x'$ axis.

[22] We choose to plot $ct$ instead of $t$ on the vertical axis, so that the trajectory of a light beam lies at a nice $45^\circ$ angle. Alternatively, we could choose units where $c = 1$.

### $ct'$-axis angle and unit size

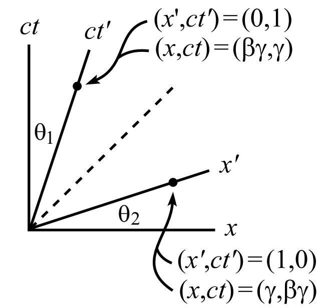

**Figure 10.22**

Look at the point $(x', ct') = (0, 1)$, which lies on the $ct'$ axis, one $ct'$ unit from the origin (see Fig. 10.22). Eqs. (10.13) tell us that this point is the point $(x, ct) = (\gamma v/c, \gamma)$. The angle between the $ct'$ and $ct$ axes is therefore given by $\tan\theta_1 = x/ct = v/c$. With $\beta \equiv v/c$, we have

$$\tan\theta_1 = \beta. \tag{10.42}$$

Alternatively, the $ct'$ axis is simply the "worldline" of the origin of $S'$. (A worldline is simply the path an object takes as it travels through spacetime.) The origin moves at speed $v$ with respect to $S$. Therefore, points on the $ct'$ axis satisfy $x/t = v$, or $x/ct = v/c$.

On the paper, the point $(x', ct') = (0, 1)$, which we just found to be the point $(x, ct) = (\gamma v/c, \gamma)$, is a distance $\gamma\sqrt{1 + v^2/c^2}$ from the origin. Using the definitions of $\beta$ and $\gamma$, we see that

$$\frac{\text{one } ct' \text{ unit}}{\text{one } ct \text{ unit}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\,, \tag{10.43}$$

as measured on a grid where the $x$ and $ct$ axes are orthogonal. Note that this ratio approaches infinity as $\beta \to 1$. And it of course equals 1 if $\beta = 0$.

### $x'$-axis angle and unit size

The same basic argument holds here. Look at the point $(x', ct') = (1, 0)$, which lies on the $x'$ axis, one $x'$ unit from the origin (see Fig. 10.22). Eqs. (10.13) tell us that this point is the point $(x, ct) = (\gamma, \gamma v/c)$. The angle between the $x'$ and $x$ axes is therefore given by $\tan\theta_2 = ct/x = v/c$. So, as in the $ct'$-axis case,

$$\tan\theta_2 = \beta. \tag{10.44}$$

On the paper, the point $(x', ct') = (1, 0)$, which we just found to be the point $(x, ct) = (\gamma, \gamma v/c)$, is a distance $\gamma\sqrt{1 + v^2/c^2}$ from the origin. So, as in the $ct'$-axis case,

$$\frac{\text{one } x' \text{ unit}}{\text{one } x \text{ unit}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\,, \tag{10.45}$$

as measured on a grid where the $x$ and $ct$ axes are orthogonal. Both the $x'$ and $ct'$ axes are therefore stretched by the same factor, and tilted by the same angle, relative to the $x$ and $ct$ axes.

REMARKS: If $v/c \equiv \beta = 0$, then $\theta_1 = \theta_2 = 0$, so the $ct'$ and $x'$ axes coincide with the $ct$ and $x$ axes, as they should. If $\beta$ is very close to 1, then the $x'$ and $ct'$ axes are both very close to the 45° light-ray line. Note that since $\theta_1 = \theta_2$, the light-ray line bisects the $x'$ and $ct'$ axes. Therefore (as we verified above), the scales on these axes must be the same, because a light ray must satisfy $x' = ct'$. ♣

We now know what the $x'$ and $ct'$ axes look like. Given any two points in a Minkowski diagram (that is, given any two events in spacetime), we can simply read off (if our graph is accurate enough) the $\Delta x$, $\Delta ct$, $\Delta x'$, and $\Delta ct'$ quantities that our two observers would measure. These quantities must of course be related by the Lorentz transformations, but the advantage of a Minkowski diagram is that you can actually see geometrically what's going on.

There are very useful physical interpretations of the $ct'$ and $x'$ axes. If you stand at the origin of $S'$, then the $ct'$ axis is the "here" axis, and the $x'$ axis is the "now" axis. That is, all events on the $ct'$ axis take place at your position (the $ct'$ axis is your worldline, after all), and all events on the $x'$ axis take place simultaneously (they all have $t' = 0$).

---

**Example (Length contraction):** For both parts of this problem, use a Minkowski diagram where the axes in frame $S$ are orthogonal.

(a) The relative speed of $S'$ and $S$ is $v$ (along the $x$ direction). A 1-meter stick (as measured by $S'$) lies along the $x'$ axis and is at rest in $S'$. $S$ measures its length. What is the result?

(b) Do the same problem, except with $S$ and $S'$ interchanged.

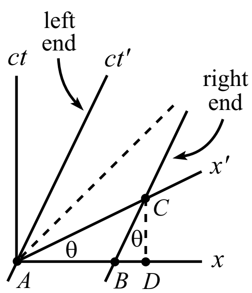

**Figure 10.23**

**Solution:**

(a) Without loss of generality, pick the left end of the stick to be at the origin in $S'$. Then the worldlines of the two ends are shown in Fig. 10.23. The distance $AC$ is 1 meter in $S'$'s frame (because $A$ and $C$ are the endpoints of the stick at simultaneous times in the $S'$ frame; this is how a length is measured). And since one unit on the $x'$ axis has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$, this is the length on the paper of the segment $AC$.

How does $S$ measure the length of the stick? He simply writes down the $x$ coordinates of the ends at simultaneous times (as measured by him, of course), and takes the difference. Let the time he makes the measurements be $t = 0$. Then he measures the ends to be at the points $A$ and $B$.[23]

Now it's time to do some geometry. We have to find the length of segment $AB$ in Fig. 10.23, given that segment $AC$ has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$. We know that the primed axes are tilted at an angle $\theta$, where $\tan\theta = \beta$. Therefore, $CD = (AC)\sin\theta$. And since $\angle BCD = \theta$, we have $BD = (CD)\tan\theta = (AC)\sin\theta\tan\theta$. Therefore (using $\tan\theta = \beta$),

$$\begin{aligned} AB &= AD - BD \\ &= (AC)\cos\theta - (AC)\sin\theta\tan\theta \\ &= (AC)\cos\theta(1-\tan^2\theta) \\ &= \sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{1}{\sqrt{1+\beta^2}}(1-\beta^2) \\ &= \sqrt{1-\beta^2}\,. \end{aligned} \tag{10.46}$$

[23]If $S$ measures the ends in a dramatic fashion by, say, blowing them up, then $S'$ will see the right end blow up first (the event at $B$ has a negative $t'$ coordinate, because it lies below the $x'$ axis), and then a little while later $S'$ will see the left end blow up (the event at $A$ has $t' = 0$). So $S$ measures the ends at different times in $S'$'s frame. This is part of the reason why $S'$ should not be at all surprised that $S$'s measurement is smaller than 1m.

Therefore, $S$ measures the meter stick to have length $\sqrt{1-\beta^2}$, which is the standard length-contraction result.

(b) The stick is now at rest in $S$, and we want to find the length that $S'$ measures. Pick the left end of the stick to be at the origin in $S$. Then the worldlines of the two ends are shown in Fig. 10.24. The distance $AB$ is 1 meter in $S$'s frame.

In measuring the length of the stick, $S'$ writes down the $x'$ coordinates of the ends at simultaneous times (as measured by him), and takes the difference. Let the time he makes the measurements be $t' = 0$. Then he measures the ends to be at the points $A$ and $E$.

Now we do the geometry, which is easy in this case. The length of $AE$ is simply $1/\cos\theta = \sqrt{1+\beta^2}$. But since one unit along the $x'$ axis has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$ on the paper, we see that $AE$ is $\sqrt{1-\beta^2}$ of one unit in $S'$'s frame. Therefore, $S'$ measures the meter stick to have length $\sqrt{1-\beta^2}$, which again is the standard length-contraction result.

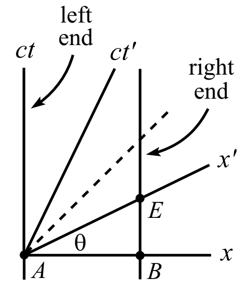

**Figure 10.24**

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The analysis used in the above example also works, of course, for time intervals. The derivation of time dilation, using a Minkowski diagram, is the task of Exercise 28.

## 10.6 The Doppler effect

### 10.6.1 Longitudinal Doppler effect

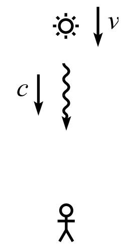

**Figure 10.25**

Consider a source that emits flashes at frequency $f'$ (in its own frame), while moving directly toward you at speed $v$, as shown in Fig. 10.25. With what frequency do the flashes hit your eye?

In these Doppler-effect problems, you must be careful to distinguish between the time at which an event *occurs* in your frame, and the time at which you *see* the event occur. This is one of the few situations where we will be concerned with the latter.

There are two effects contributing to the longitudinal Doppler effect. The first is relativistic time dilation; the flashes occur at a smaller frequency in your frame, that is, there is more time between them. The second is the everyday Doppler effect (as with sound), arising from the motion of the source; successive flashes have a smaller distance (or larger, if $v$ is negative) to travel to reach your eye. This effect increases the frequency at which the flashes hit your eye (or decreases it, if $v$ is negative).

Let's now be quantitative and find the observed frequency. The time between emissions in the source's frame is $\Delta t' = 1/f'$. The time between emissions in your frame is $\Delta t = \gamma\Delta t'$, by the usual time dilation. So the photons of one flash have traveled a distance (in your frame) of $c\Delta t = c\gamma\Delta t'$ by the time the next flash occurs. During this time between emissions, the source has traveled a distance $v\Delta t = v\gamma\Delta t'$ toward you in your frame. Hence, at the instant the next flash occurs, the photons of this next flash are a distance (in your frame) of $c\Delta t - v\Delta t = (c-v)\gamma\Delta t'$ behind the photons of the previous flash. This result holds for all adjacent flashes. The time, $\Delta T$, between the arrivals of the flashes at your eye is $1/c$ times this distance. Therefore,

$$\Delta T = \frac{1}{c}(c - v)\gamma \Delta t' = \frac{1-\beta}{\sqrt{1-\beta^2}}\,\Delta t' = \sqrt{\frac{1-\beta}{1+\beta}}\left(\frac{1}{f'}\right). \tag{10.47}$$

where $\beta = v/c$. Hence, the frequency you see is

$$f = \frac{1}{\Delta T} = \sqrt{\frac{1+\beta}{1-\beta}}\,f'. \tag{10.48}$$

If $\beta > 0$ (that is, the source is moving toward you), then $f > f'$. The everyday Doppler effect wins out over the time-dilation effect. In this case we say that the light is "blue-shifted," because blue light is at the high-frequency end of the visible spectrum. The light need not have anything to do with the color blue, of course; by "blue" we simply mean that the frequency is increased. If $\beta < 0$ (that is, the source is moving away from you), then $f < f'$. Both effects serve to decrease the frequency. In this case we say that the light is "red-shifted," because red light is at the low-frequency end of the visible spectrum.

### 10.6.2 Transverse Doppler effect

Consider a source that emits flashes at frequency $f'$ (in its own frame), while moving across your field of vision at speed $v$. There are two reasonable questions we may ask about the frequency you observe:

- **Case 1:**

  At the instant the source is at its closest approach to you, with what frequency do the flashes hit your eye?

- **Case 2:**

  When you *see* the source at its closest approach to you, with what frequency do the flashes hit your eye?

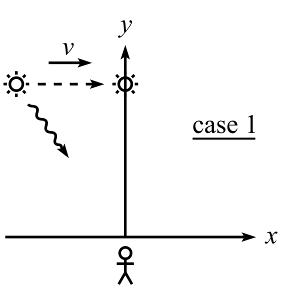


**Figure 10.26**

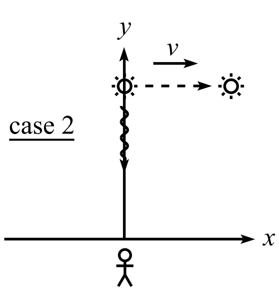


**Figure 10.27**

The difference between these two scenarios is shown in Fig. 10.26 and Fig. 10.27, where the source's motion is taken to be parallel to the $x$-axis.

In the first case, the photons you see must have been emitted at an earlier time, because the source will have moved during the non-zero time it takes the light to reach you. In this scenario, we are dealing with photons that hit your eye *when* the source crosses the $y$-axis. You will therefore see the photon come in at an angle with respect to the $y$-axis.

In the second case, you will see the photons come in along the $y$-axis (by the definition of this scenario). At the instant you observe such a photon, the source will be at a position past the $y$-axis.

Let's now find the observed frequencies in these two scenarios.

### Case 1

Let your frame be $S$, and let the source's frame be $S'$. Consider the situation from $S'$'s point of view. $S'$ sees you moving across his field of vision at speed $v$. The relevant photons hit your eye when you cross the $y'$-axis of $S'$'s frame. Because of time dilation, your clock ticks slowly (in $S'$'s frame) by a factor $\gamma$.

Now, $S'$ sees you get hit by a flash every $\Delta t' = 1/f'$ seconds in his frame. (This is true because when you are very close to the $y'$-axis, all points on your path are essentially equidistant from the source. So we don't have to worry about any longitudinal effects.) This means that you get hit by a flash every $\Delta T = \Delta t'/\gamma = 1/(f'\gamma)$ seconds in your frame. Therefore, the frequency in your frame is

$$f = \frac{1}{\Delta T} = \gamma f' = \frac{f'}{\sqrt{1-\beta^2}}. \tag{10.49}$$

Hence, $f$ is greater than $f'$; you see the flashes at a higher frequency than $S'$ emits them.

### Case 2

Again, let your frame be $S$, and let the source's frame be $S'$. Consider the situation from your point of view. Because of the time dilation, $S'$'s clock ticks slowly (in your frame) by a factor of $\gamma$. When you see the source cross the $y$-axis, you therefore observe a frequency of

$$f = \frac{1}{\Delta T} = \frac{1}{\gamma \Delta t'} = \frac{f'}{\gamma} = f'\sqrt{1-\beta^2}. \tag{10.50}$$

We have used the fact the relevant photons are emitted from points that are essentially equidistant from you. So they all travel the same distance, and we don't have to worry about any longitudinal effects.

We see that $f$ is smaller than $f'$; you see the flashes at a lower frequency than $S'$ emits them.

Remarks:

1. When people talk about the "transverse Doppler effect", they sometimes mean Case 1, and they sometimes mean Case 2. The title "transverse Doppler" is therefore ambiguous, so you should remember to state exactly which scenario you are talking about.

2. The two scenarios may alternatively be described, respectively (as you can convince yourself), in the following ways (see Fig. 10.28).

   - **Case 1:**
     A receiver moves with speed $v$ in a circle around a source. What frequency does the receiver register?
   - **Case 2:**
     A source moves with speed $v$ in a circle around a receiver. What frequency does the receiver register?

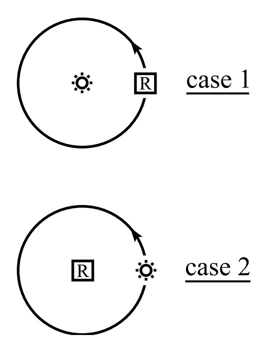


**Figure 10.28**

These setups involve accelerating objects. We must therefore invoke the fact (which has plenty of experimental verification) that if an inertial observer observes a moving clock, then only the instantaneous speed of the clock is important in computing the time dilation factor. The acceleration is irrelevant.[24]

3. Beware of the following incorrect reasoning for Case 1, leading to an incorrect version of eq. (10.49). "$S$ sees things in $S'$ slowed down by a factor $\gamma$ (that is, $\Delta t = \gamma \Delta t'$), by the usual time dilation effect. Hence, $S$ sees the light flashing at a slower pace. Therefore, $f = f'/\gamma$." This reasoning puts the $\gamma$ in the wrong place. Where is the error? The error lies in confusing the time at which an event *occurs* in $S$'s frame, with the time at which $S$ *sees* (with his eyes) the event occur. The flashes certainly *occur* at a lower frequency in $S$, but due to the motion of $S'$ relative to $S$, it turns out that the pulses meet $S$'s eye at a faster rate (because the source is moving slightly towards $S$ while it is emitting the relevant photons). We'll let you work out the details of the situation from $S$'s point of view.[25]

   Alternatively, the error can be stated as follows. The time dilation result, $\Delta t = \gamma \Delta t'$, rests on the assumption that the $\Delta x'$ between the two events is zero. This applies fine to two emissions of light from the source. However, the two events in question are the absorption of two light pulses by your eye (which is moving in $S'$), so $\Delta t = \gamma \Delta t'$ is not applicable. Instead, $\Delta t' = \gamma \Delta t$ is the relevant result, valid when $\Delta x = 0$.

4. Other cases that are "inbetween" the longitudinal and transverse cases can also be considered. But they get a little messy. ♣

## 10.7 Rapidity

### Definition

Let us define the *rapidity*, $\phi$, by

$$\tanh \phi \equiv \beta \equiv \frac{v}{c}\,. \tag{10.51}$$

A few properties of the hyperbolic trig functions are given in Appendix A. In particular, $\tanh \phi \equiv (e^{\phi} - e^{-\phi})/(e^{\phi} + e^{-\phi})$. The rapidity, $\phi$, defined in eq. (10.51) is very useful in relativity because many of our expressions take on a particularly nice form when written in terms of it.

Consider, for example, the velocity-addition formula. Let $\beta_1 = \tanh \phi_1$ and $\beta_2 = \tanh \phi_2$. Then if we add $\beta_1$ and $\beta_2$ using the velocity-addition formula, eq. (10.28), we obtain

$$\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} = \tanh(\phi_1 + \phi_2), \tag{10.52}$$

where we have used the addition formula for $\tanh \phi$ (which you can prove by writing things in terms of the exponentials, $e^{\pm \phi}$). Therefore, while the velocities add in the strange manner of eq. (10.28), the rapidities add by standard addition.

[24] The acceleration is, however, very important if things are considered from an accelerating object's point of view. But we'll wait until Chapter 13 on General Relativity to talk about this.

[25] This is a fun exercise (Exercise 31), but it should convince you that it is much easier to look at things in the frame in which there are no longitudinal effects, as we did in our solutions above.

The Lorentz transformations also take a nice form when written in terms of the rapidity. Our friendly $\gamma$ factor can be written as

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi. \tag{10.53}$$

Also,

$$\gamma\beta \equiv \frac{\beta}{\sqrt{1-\beta^2}} = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \sinh\phi. \tag{10.54}$$

Therefore, the Lorentz transformations in matrix form, eqs. (10.18), become

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.55}$$

This transformation looks similar to a rotation in a plane, which is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{10.56}$$

except that we now have hyperbolic trig functions instead of trig functions. The fact that the interval $s^2 \equiv c^2t^2 - x^2$ does not depend on the frame is clear from eq. (10.55); the cross terms in the squares cancel, and $\cosh^2\phi - \sinh^2\phi = 1$. (Compare with the invariance of $r^2 \equiv x^2 + y^2$ for rotations in a plane.)

The quantities associated with a Minkowski diagram also take a nice form when written in terms of the rapidity. The angle between the $S$ and $S'$ axes satisfies

$$\tan\theta = \beta = \tanh\phi. \tag{10.57}$$

And the size of one unit on the $x'$ or $ct'$ axes is, from eq. (10.43),

$$\sqrt{\frac{1+\beta^2}{1-\beta^2}} = \sqrt{\frac{1+\tanh^2\phi}{1-\tanh^2\phi}} = \sqrt{\cosh^2\phi + \sinh^2\phi} = \sqrt{\cosh 2\phi}. \tag{10.58}$$

For large $\phi$, this is approximately equal to $e^\phi/\sqrt{2}$.

### Physical meaning

The fact that the rapidity, $\phi$, makes many of our formulas look nice and pretty is reason enough to consider it. But in addition, it turns out to have a very meaningful physical interpretation.

Consider the following setup. A spaceship is initially at rest in the lab frame. At a given instant, it starts to accelerate. Let $a$ be the *proper acceleration*, which is defined as follows. Let $t$ be the time coordinate in the spaceship's frame.[26] If the

[26]This frame is of course changing as time goes by, because the spaceship is accelerating. The time $t$ is simply the spaceship's proper time. Normally, we would denote this by $t'$, but we don't want to have to keep writing the primes over and over in the following calculation.

proper acceleration is $a$, then at time $t + dt$, the spaceship is moving at speed $a\,dt$ relative to the frame it was in at time $t$. Equivalently, the astronaut feels a force of $ma$ applied to his body by the spaceship. If he is standing on a scale, then the scale shows a reading of $F = ma$.

What is the relative speed of the spaceship and the lab frame at (the spaceship's) time $t$?

We can answer this question by considering two nearby times and using the velocity-addition formula, eq. (10.28). From the definition of $a$, eq. (10.28) gives (with $v_1 \equiv a\,dt$ and $v_2 \equiv v(t)$)

$$v(t + dt) = \frac{v(t) + a\,dt}{1 + v(t)a\,dt/c^2}\,. \tag{10.59}$$

Expanding this to first order in $dt$ yields[27]

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right). \tag{10.60}$$

Separating variables and integrating gives, using $1/(1-z^2) = 1/2(1-z)+1/2(1+z)$,

$$\int_0^v \left(\frac{1}{1 - v/c} + \frac{1}{1 + v/c}\right) dv = 2a \int_0^t dt. \tag{10.61}$$

This yields $\ln\left[(1 + v/c)/(1 - v/c)\right] = 2at/c$. Solving for $v$, we find[28]

$$v(t) = c\left(\frac{e^{2at/c} - 1}{e^{2at/c} + 1}\right) = c\tanh(at/c). \tag{10.62}$$

Note that for small $a$ or small $t$ (more precisely, for $at/c \ll 1$), we obtain $v(t) \approx at$, as we should (because $\tanh z \approx z$ for small $z$, which follows from the exponential form of tanh). And for $at/c \gg 1$, we obtain $v(t) \approx c$, as we should.

If $a$ happens to be a function of time, $a(t)$, then we can't take the $a$ outside the integral in eq. (10.61), so we instead end up with the general formula,

$$v(t) = c\tanh\left(\frac{1}{c}\int_0^t a(t)\,dt\right). \tag{10.63}$$

We therefore see that the rapidity, $\phi$, as defined in eq. (10.51), is given by

$$\phi(t) \equiv \frac{1}{c}\int_0^t a(t)\,dt. \tag{10.64}$$

Note that whereas $v$ has $c$ as a limiting value, $\phi$ can become arbitrarily large. Looking at eq. (10.63) we see that the $\phi$ associated with a given $v$ is simply $1/mc$

[27]Equivalently, just take the derivative of $(v + w)/(1 + vw/c^2)$ with respect to $w$, and then set $w = 0$.

[28]Alternatively, you can just use $\int 1/(1 - z^2) = \tanh^{-1} z$. You can also use the result of Problem 12 to find $v(t)$. See the remark in the solution to that problem (after trying to solve it, of course!).

times the time integral of the force (felt by the astronaut) needed to bring the astronaut up to speed $v$. By applying a force for an arbitrarily long time, we can make $\phi$ arbitrarily large.

The integral $\int a(t)\,dt$ may be described as the naive, incorrect speed. That is, it is the speed that the astronaut might *think* he has, if he has his eyes closed and knows nothing about the theory of relativity. And indeed, his thinking would be essentially correct for small speeds. This quantity $\int a(t)\,dt$ seems like a reasonably physical thing, so if there is any justice in the world, $\int a(t)\,dt = \int F(t)\,dt/m$ should have *some* meaning. And indeed, although it doesn't equal $v$, all you have to do to get $v$ is take a tanh and throw in some factors of $c$.

The fact that rapidities add via simple addition when using the velocity-addition formula, as we saw in eq. (10.52), is evident from eq. (10.63). There is really nothing more going on here than the fact that

$$\int_{t_0}^{t_2} a(t)\,dt = \int_{t_0}^{t_1} a(t)\,dt + \int_{t_1}^{t_2} a(t)\,dt. \tag{10.65}$$

To be explicit, let a force be applied from $t_0$ to $t_1$ that brings a mass up to speed $\beta_1 = \tanh\phi_1 = \tanh(\int_{t_0}^{t_1} a\,dt)$, and then let an additional force be applied from $t_1$ to $t_2$ that adds on an additional speed of $\beta_2 = \tanh\phi_2 = \tanh(\int_{t_1}^{t_2} a\,dt)$, relative to the speed at $t_1$. Then the resulting speed may be looked at in two ways: (1) it is the result of relativistically adding the speeds $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$, and (2) it is the result of applying the force from $t_0$ to $t_2$ (you get the same final speed, of course, whether or not you bother to record the speed along the way at $t_1$), which is $\beta = \tanh(\int_{t_0}^{t_2} a\,dt) = \tanh(\phi_1 + \phi_2)$, where the second equality comes from the statement, eq. (10.65), that integrals simply add. Therefore, the relativistic addition of $\tanh\phi_1$ and $\tanh\phi_2$ gives $\tanh(\phi_1 + \phi_2)$, as we wanted to show.

## 10.8 Relativity without $c$

In Section 10.1, we introduced the two postulates of Special Relativity, namely the speed-of-light postulate and the relativity postulate. In Appendix I, we show that together these imply that the coordinate intervals in two frames must be related by the Lorentz transformations, eqs. (10.13).

It is interesting to see what happens if we relax these postulates. It is hard to imagine a "reasonable" universe where the relativity postulate does not hold (geocentric theories aside), but it is easy to imagine a universe where the speed of light depends on the frame of reference. Light could behave like a baseball, for example. So let's drop the speed-of-light postulate and see what we can say about the coordinate transformations between frames, using only the relativity postulate.

In Appendix I, the form of the transformations, just prior to invoking the speed-of-light postulate, is given in eq. (14.83) as

$$\begin{aligned} x &= A_v(x' + vt'), \\ t &= A_v\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A_v^2}\Big)x'\right). \end{aligned} \tag{10.66}$$

We'll put a subscript on $A$ in this section, to remind us of the $v$ dependence. Can we say anything about $A_v$ without invoking the speed-of-light postulate? Indeed we can. Define $V_v$ by

$$\frac{1}{V_v^2} \equiv \frac{1}{v^2}\left(1 - \frac{1}{A_v^2}\right), \qquad \text{so that} \qquad A_v = \frac{1}{\sqrt{1 - v^2/V_v^2}}\,. \tag{10.67}$$

Eqs. (10.66) then become

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/V_v^2}}(x' + vt'), \\ t &= \frac{1}{\sqrt{1 - v^2/V_v^2}}\left(\frac{v}{V_v^2}x' + t'\right). \end{aligned} \tag{10.68}$$

All we've done so far is make a change of variables. But we now make the following claim.

**Claim 10.1** $V_v^2$ *is independent of* $v$.

**Proof:** As stated in the last remark in Section 10.3.1, we know that two successive applications of the transformations in eq. (10.68) must again yield a transformation of the same form.

Consider a transformation characterized by velocity $v_1$, and another one characterized by velocity $v_2$. For simplicity, define

$$\begin{aligned} V_1 &\equiv V_{v_1}, & V_2 &\equiv V_{v_2}, \\ \gamma_1 &\equiv \frac{1}{\sqrt{1 - v_1^2/V_1^2}}\,, & \gamma_2 &\equiv \frac{1}{\sqrt{1 - v_2^2/V_2^2}}\,. \end{aligned} \tag{10.69}$$

To calculate the composite transformation, it is easiest to use matrix notation. Looking at eq. (10.68), we see that the composite transformation is given by the matrix

$$\begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2\frac{v_2}{V_2^2} & \gamma_2 \end{pmatrix}\begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1\frac{v_1}{V_1^2} & \gamma_1 \end{pmatrix} = \gamma_1\gamma_2\begin{pmatrix} 1 + \frac{v_1 v_2}{V_1^2} & v_2 + v_2 \\ \frac{v_1}{V_1^2} + \frac{v_2}{V_2^2} & 1 + \frac{v_1 v_2}{V_2^2} \end{pmatrix}. \tag{10.70}$$

The composite transformation must still be of the form of eq. (10.68). But this implies that the upper-left and lower-right entries of the composite matrix must be equal. Therefore, $V_1^2 = V_2^2$. Since this holds for arbitrary $v_1$ and $v_2$, we see that $V_v^2$ must be a constant, that is, independent of $v$. ∎

Denote the constant value of $V_v^2$ by $V^2$. Then the coordinate transformations in eq. (10.68) become

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/V^2}}(x' + vt'), \\ t &= \frac{1}{\sqrt{1 - v^2/V^2}}\left(t' + \frac{v}{V^2}x'\right). \end{aligned} \tag{10.71}$$

We have obtained this result using only the relativity postulate. These transformations have the same form as the Lorentz transformations, eqs. (10.13). The only extra information in eqs. (10.13) is that $V$ is equal to the speed of light, $c$. It is remarkable that we were able to prove so much by using only the relativity postulate.

We can say a few more things. There are four possibilities for the value of $V^2$. However, two of these are not physical.

- $V^2 = \infty$: This gives the Galilean transformations, $x = x' + vt'$ and $t = t'$.
- $0 < V^2 < \infty$: This gives transformations of the Lorentz type. $V$ is the limiting speed of an object.
- $V^2 = 0$: This case is not physical, because any nonzero value of $v$ will make the $\gamma$ factor imaginary (and infinite). Nothing could ever move.
- $V^2 < 0$: It turns out that this case is also not physical. You might be concerned that the square of $V$ is less than zero, but this is fine because $V$ appears in the transformations (10.71) only through its square. There's no need for $V$ to actually be the speed of anything. The trouble is that the nature of eqs. (10.71) implies the possibility of time reversal. This opens the door for causality violation and all the other problems associated with time reversal. We therefore reject this case.

  To be a little more explicit, define $b^2 \equiv -V^2$, where $b$ is a positive number. Then eqs. (10.71) may be written in the form,

$$\begin{aligned} x &= x' \cos\theta + (bt') \sin\theta, \\ bt &= -x' \sin\theta + (bt') \cos\theta, \end{aligned} \tag{10.72}$$

  where $\tan\theta = v/b$. This transformation is simply a rotation in the plane, through an angle of $-\theta$. We have the usual trig functions here, instead of the hyperbolic trig functions in the Lorentz transformations in eq. (10.55).

  Eqs. (10.72) satisfy the requirement that the composition of two transformations is again a transformation of the same form. Rotation by $\theta_1$, and then by $\theta_2$, yields a rotation by $\theta_1 + \theta_2$. However, if the resulting rotation is through an angle, $\theta$, that is greater than 90°, then we have a problem. The tangent of such an angle is negative. Therefore, $\tan\theta = v/b$ implies that $v$ is negative.

  The situation is shown in Fig. 10.29. Frame $S''$ moves at velocity $v_2 > 0$ with respect to frame $S'$, which moves at velocity $v_1 > 0$ with respect to frame $S$. From the figure, we see that someone standing at rest in frame $S''$ (that is, someone whose worldline is the $bt''$-axis) is going to have some serious issues in frame $S$. For one, the $bt''$-axis has a negative slope in frame $S$, which means that as $t$ increases, $x$ decreases. The person will therefore be moving at a *negative* velocity with respect to $S$. Adding two positive velocities and obtaining a negative one is clearly absurd. But even worse, someone standing at rest in $S''$ will be moving in the positive direction along the $bt''$-axis, which

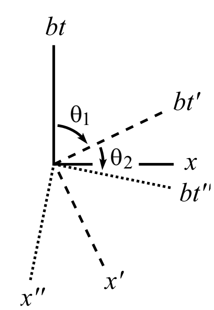


**Figure 10.29**

means that he will be traveling *backwards* in time in $S$. That is, he will die before he is born. This is not good.

Note that all of the finite $0 < V^2 < \infty$ possibilities are essentially the same. Any difference in the numerical definition of $V$ can be absorbed into the definitions of the unit sizes for $x$ and $t$. Given that $V$ is finite, it has to be *something*, so it doesn't make sense to put much importance on its numerical value.

There is therefore only one decision to be made when constructing the spacetime structure of an (empty) universe. You just have to say whether $V$ is finite or infinite, that is, whether the universe is Lorentzian or Galilean. Equivalently, all you have to say is whether or not there is an upper limit for the speed of any object. If there is, then you can simply postulate the existence of something that moves with this limiting speed. In other words, to create your universe, you simply have to say, "Let there be light."

## 10.9 Exercises

*Section 10.2: The fundamental effects*

1. **Effectively speed $c$** $*$

   A rocket flies between two planets that are one light-year apart. What should the rocket's speed be so that the time elapsed on the captain's watch is one year?

2. **A passing train**

   A train of length $15\,c$s moves at speed $3c/5$.[29] How much time does it take to pass a person standing on the ground (as measured by that person)? Solve this by working in the frame of the person, and then again by working in the frame of the train.

3. **Passing trains** $*$

   Train $A$ has length $L$. Train $B$ moves past $A$ with relative speed $4c/5$, in the same direction. The length of $B$ is such that $A$ says that the fronts of the trains coincide at exactly the same time as the backs coincide. What is the time difference between the fronts coinciding and the backs coinciding, as measured by $B$?

4. **Seeing the light** $*$

   $A$ and $B$ leave from a common point and travel in opposite directions with relative speed $4c/5$. When $B$'s clock shows that a time $T$ has elapsed, he ($B$) sends out a light signal. When $A$ receives the signal, what time does his ($A$'s) clock show? Answer this question by doing the calculation entirely in (a) $A$'s frame, and then (b) $B$'s frame.

5. **Walking on a train** $*$

   A train of proper length $L$ and speed $3c/5$ approaches a tunnel of length $L$. At the moment when the front of the train enters the tunnel, a person leaves the front of the train and walks (briskly) toward the back. He arrives at the back of the train right when it leaves the tunnel.

   (a) How much time does this take in the ground frame?

   (b) What is the person's speed with respect to the ground?

   (c) How much time elapses on the person's watch?

6. **Simultaneous waves** $*$

   Alice flies past Bob at speed $v$. Right when she passes, they both set their watches to zero. When Alice's watch shows a time $T$, she waves to Bob. Bob then waves to Alice simultaneously (as measured by him) with Alice's wave. Alice then waves to Bob simultaneously (as measured by her) with

[29] $1\,c$s is one "light-second." It equals $(1)(3\cdot 10^8\,\text{m/s})(1\,\text{s}) = 3\cdot 10^8\,\text{m}$.

Bob's wave. Bob then waves to Alice simultaneously (as measured by him) with Alice's second wave. And so on. What are the readings on Alice's watch for all the times she waves? And likewise for Bob?

7. **Triplets** $*$

   Triplet $A$ stays on the earth. Triplet $B$ travels at speed $4c/5$ to a planet (a distance $L$ away) and back. Triplet $C$ travels out to the planet at speed $3c/4$, and then returns at the necessary speed to arrive back exactly when $B$ does. How much does each triplet age during this process? Who is youngest?

8. **People clapping** $**$

   Two people stand a distance $L$ apart along an east-west road. They both clap their hands at precisely noon in the ground frame. You are driving eastward down this road at speed $4c/5$. You notice that you encounter the western person at the same instant (as measured in your frame) that the eastern person claps. Where are you along the road at the instant (as measured in your frame) that the western person claps?

9. **Coinciding runner** $**$

   A train of length $L$ moves at speed $4c/5$ eastward, and a train of length $3L$ moves at speed $3c/5$ westward. How fast must someone run along the ground if he is to coincide with both the fronts-passing-each-other and backs-passing-each-other events?

10. **Photon on a train** $**$

    (a) A train of proper length $L$ moves with speed $v$ with respect to the ground. At the instant the back of the train passes a certain tree, someone at the back of the train shines a photon toward the front. The photon happens to hit the front of the train at the instant the front passes a certain house. As measured in the ground frame, how far apart are the tree and the house? Solve this by working in the ground frame.

    (b) Now look at the setup from the point of view of the train frame. Using your result for the tree-house distance from part (a), verify that the house meets the front of the train at the same instant the photon meets it.

11. **Another train in a tunnel** $***$

    Consider the scenario of Problem 6, with the only change being that the train now has length $r\ell$, where $r$ is some numerical factor.

    What is the largest value of $r$, in terms of $v$, for which it is possible for the bomb to not explode? (Verify that you obtain the same answer working in the frame of the train and working in the frame of the tunnel.)

12. **Falling stick** $**$

    A horizontal falling stick bounces off the ground. What does this look like in the frame of someone running by at speed $v$?

13. **Through the hole?** $**$

A stick of proper length $L$ moves at speed $v$ in the direction of its length. It passes over a infinitesimally thin sheet that has a hole of diameter $L$ cut out of it. As the stick passes over the hole, the sheet is raised, and the stick then ends up underneath the sheet.

In the lab frame, the stick's length is contracted to $L/\gamma$, so it appears to easily makes it through the hole. But in the stick frame, the hole is contracted to $L/\gamma$, so it seems like the stick does *not* make it through the hole. So the question is: Does the stick make it through the hole or not?

*Section 10.3: The Lorentz transformations*

14. **Successive L.T.'s** $*$

Show that the combination of an L.T. (with speed $v_1$) and an L.T. (with speed $v_2$) yields an L.T. with speed $(v_1 + v_2)/(1 + v_1 v_2/c^2)$.

15. **Loss of simultaneity** $*$

A train moves at speed $v$ with respect to the ground. Two events occur simultaneously, a distance $L$ apart, in the frame of the train. What is the time separation in the frame of the ground? Solve this by using the Lorentz transformations, and then again by using only the results in Section 10.2.

16. **Pythagorean triples** $*$

Let $(a, b, h)$ be a pythagorean triplet. (We'll use $h$ to denote the hypotenuse, instead of $c$, for obvious reasons.) Consider the relativistic addition or subtraction of the two speeds, $\beta_1 = a/h$ and $\beta_2 = b/h$. Show that the numerator and denominator of the result are members of another pythagorean triplet, and find the third member. What is the associated $\gamma$ factor?

17. **Running on a train** $*$

A train of length $L$ moves at speed $v_1$ with respect to the ground. A passenger runs from the back of the train to the front, at speed $v_2$ with respect to the train. How much time does this take, as viewed by someone on the ground? Solve this problem in two different ways:

(a) Find the relative speed of the passenger and the train (as viewed by someone on the ground), and then find the time it takes for the passenger to erase the initial "head start" that the front of the train had.

(b) Find the time it takes on the passenger's clock, and then use time dilation to get the time elapsed on a ground clock.

18. **Running away** $*$

$A$ and $B$ both start at the origin and simultaneously head off in opposite directions at speed $3c/5$. $A$ moves to the right, and $B$ moves to the left. Consider a mark on the ground at $x = L$. As viewed in the ground frame, $A$

and $B$ are of course a distance $2L$ apart when $A$ passes this mark. As viewed by $A$, how far away is $B$ when $A$ coincides with the mark?

19. **Velocity addition** $**$

Derive the velocity-addition formula by using the following setup: A train of length $L$ moves at speed $a$ with respect to the ground, and a ball is thrown at speed $b$ with respect to the train, from the back to the front. Let the speed of the ball with respect to the ground be $V$.

Calculate the time of the ball's journey, as measured by an observer on the ground, in the following two different ways, and then set them equal to solve for $V$ in terms of $a$ and $b$.

(a) First way: Find the relative speed of the ball and the train (as viewed by someone on the ground), and then find the time it takes for the ball to erase the initial "head start" that the front of the train had.

(b) Second way: Find the time elapsed on the ball's clock, and then use time dilation to get the time elapsed on a ground clock.

20. **Bullets on a train** $**$

A train moves at speed $v$. Bullets are successively fired at speed $u$ (relative to the train) from the back of a train to the front. A new bullet is fired at the instant (as measured in the train frame) the previous bullet hits the front. In the frame of the ground, what fraction of the way along the train is a given bullet, at the instant (as measured in the ground frame) the next bullet is fired? What is the maximum number of bullets an observer on the ground can see in flight at any given instant?

21. **Some $\gamma$'s**

Show that the relativistic addition (or subtraction) of the velocities $u$ and $v$ has a $\gamma$ factor given by $\gamma = \gamma_u \gamma_v (1 \pm uv)$.

22. **Time dilation and $Lv/c^2$** $**$

A person walks very slowly at speed $u$ from the back of a train of length $L$ to the front. The time-dilation effect in the train frame can be made arbitrarily small by picking $u$ to be sufficiently small (because the effect is of second-order in $u$). Therefore, if the person's watch agrees with a clock at the back of the train when he starts, then it will also agree with a clock at the front when he finishes.

Now consider this situation from the ground frame, where the train moves at speed $v$. The rear clock reads $Lv/c^2$ more than the front, so in view of the preceding paragraph, the person's watch must gain a time that is $Lv/c^2$ less than the time gained by the front clock during the process. By working in the ground frame, explain why this is the case.

23. **Angled photon** $*$

    A photon moves at an angle $\theta$ with respect to the $x'$-axis in frame $S'$. Frame $S'$ moves at speed $v$ with respect to frame $S$ (along the $x'$ axis). Calculate the components of the photon's velocity in $S$, and verify that the speed is $c$.

*Section 10.4: The invariant interval*

24. **Head start** $*$

    Derive the $Lv/c^2$ "head-start" result (given in eq. 10.2) by making use of the invariant spacetime interval.

25. **Passing trains** $***$

    Train $A$ of length $L$ moves eastward at speed $v$, and train $B$ of length $2L$ moves westward also at speed $v$. How much time does it take for the trains to pass each other (defined as the time between the front of $B$ coinciding with the front of $A$, and the back of $B$ coinciding with the back of $A$):

    (a) As viewed by $A$?

    (b) As viewed by $B$?

    (c) As viewed by the ground?

    (d) Verify that the invariant interval is indeed the same in all three frames.

26. **Passing a train** $*$

    Person $C$ stands on the ground. Train $B$ (with proper length $L$) moves to the right at speed $3c/5$. And person $A$ runs to the right at speed $4c/5$. $A$ starts behind the train and eventually passes it.

    Let event $E_1$ be "$A$ passes the back of the train," and let event $E_2$ be "$A$ passes the front of the train." Find $\Delta t$ and $\Delta x$ between events $E_1$ and $E_2$ in the frames of $A$, $B$, and $C$. And show that $c^2\Delta t^2 - \Delta x^2$ is the same in all frames.

27. **Throwing on a train** $**$

    A train with proper length $L$ moves at speed $3c/5$ with respect to the ground. A ball is thrown from the back to the front, at speed $c/2$ with respect to the train. How much time does this take, and what distance does the ball cover, in:

    (a) The train frame?

    (b) The ground frame? Solve this by

    i. Using a velocity-addition argument.

    ii. Using the Lorentz transformations to go from the train frame to the ground frame.

    (c) The ball frame?

    (d) Verify that the invariant interval is indeed the same in all three frames.

(e) Show that the times in the ball frame and ground frame are related by the relevant $\gamma$-factor.

(f) Ditto for the ball frame and train frame.

(g) Show that the times in the train frame and ground frame are *not* related by the relevant $\gamma$-factor. Why not?

*Section 10.5: Minkowski diagrams*

28. **Time dilation via Minkowski** $*$

In the spirit of the example in Section 10.5, use a Minkowski diagram to derive the time-dilation result between frames $S$ and $S'$ (in both directions, as in the example).

29. **Simultaneous claps** $***$

With respect to the ground, $A$ moves to the right at speed $c/\sqrt{3}$, and $B$ moves to the left, also at speed $c/\sqrt{3}$. At the instant they are a distance $d$ apart (as measured in the ground frame), $A$ claps his hands. $B$ then claps his hands simultaneously (as measured by $B$) with $A$'s clap. $A$ then claps his hands simultaneously (as measured by $A$) with $B$'s clap. $B$ then claps his hands simultaneously (as measured by $B$) with $A$'s second clap, and so on. As measured in the ground frame, how far apart are $A$ and $B$ when $A$ makes his $n$th clap? What is the answer if $c/\sqrt{3}$ is replaced by a general speed $v$?

30. **Train in tunnel** $**$

Repeat Exercise 11, but now solve it by using a Minkowski diagram. (Do this from the point of view of the train, and also of the tunnel.)

*Section 10.6: The Doppler effect*

31. **Transverse Doppler** $**$

As mentioned in Remark 3 of Section 10.6.2, it is possible to solve the transverse Doppler effect for Case 1 by working in the frame of the observer, provided that you account for the longitudinal component of the source's motion. Solve the problem this way and reproduce eq. (10.49).

32. **Twin paradox via Doppler** $**$

Twin $A$ stays on earth, and twin $B$ flies at speed $v$ to a distant star and back. The star is a distance $L$ from the earth. Use the Doppler effect to show that $B$ is younger by a factor $\gamma$ when she returns. Do this in two ways:

(a) $A$ sends out flashes at 1-second intervals. Let $T_A$ and $T_B$ be the total times on $A$'s and $B$'s clocks. In terms of $T_B$, how many red- and blue-shifted flashes does $B$ receive? Equate the total number of flashes $B$ receives with with the total number of flashed $A$ emits, to show that $T_B = T_A/\gamma$.

(b) $B$ sends out flashes at 1-second intervals. In terms of $L$ and $v$, how many red- and blue-shifted flashes does $A$ receive? The total number of flashes $A$ receives is simply the total time on $B$'s clock. Use this to show that $T_B = T_A/\gamma$.

## 10.10 Problems

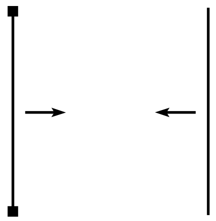

**Figure 10.30**

*Section 10.2: The fundamental effects*

1. **No transverse length contraction** $*$

   Two meter sticks, $A$ and $B$, move past each other as shown in Fig. 10.30. Stick $A$ has paint brushes at its ends. Use this setup to show that in the frame of one stick, the other stick still has a length of one meter.

2. **Explaining time dilation** $**$

   Two planets, $A$ and $B$, are at rest with respect to each other, a distance $L$ apart, with synchronized clocks. A spaceship flies at speed $v$ past planet $A$ and synchronizes its clock with $A$'s (they both set their clocks to zero). It then flies past planet $B$ and compares its clock to $B$'s. We know that when the spaceship reaches $B$, $B$'s clock will simply read $L/v$. And the spaceship's clock will read $L/\gamma v$, since it runs slow by a factor of $\gamma$, compared to the planets' clocks.

   How would someone on the spaceship quantitatively explain to you why $B$'s clock reads $L/v$ (which is *more* than its own $L/\gamma v$), considering that the spaceship sees $B$'s clock running *slow*?

3. **Explaining Length contraction** $**$

   Two bombs lie on a train platform, a distance $L$ apart. As a train passes by at speed $v$, the bombs explode simultaneously (in the platform frame) and leave marks on the train. Due to the length contraction of the train, we know that the marks on the train will be a distance $\gamma L$ apart when viewed in the train's frame (since this distance is what is length-contracted down to the given distance $L$ in the platform frame).

   How would someone on the train quantitatively explain to you why the marks are $\gamma L$ apart, considering that the bombs are only a distance $L/\gamma$ apart in the train frame?

4. **A passing stick** $**$

   A stick of length $L$ moves past you at speed $v$. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in

   (a) your frame? (Calculate this by working in your frame.)
   (b) your frame? (Work in the stick's frame.)
   (c) the stick's frame? (Work in your frame. This is the tricky one.)
   (d) the stick's frame? (Work in the stick's frame.)

5. **Rotated square** $*$

   A square with side $L$ flies past you at speed $v$, in a direction parallel to two of its sides. You stand in the plane of the square. When you see the square at its nearest point to you, show that it *looks* to you like it is simply rotated, instead of contracted. (Assume that $L$ is small compared to the distance between you and the square.)

6. **Train in a tunnel** $**$

A train and a tunnel both have proper lengths $L$. The train speeds toward the tunnel, with speed $v$. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, this sensor tells the bomb to disarm itself. Does the bomb explode?

7. **Seeing behind the stick** $**$

A ruler is positioned perpendicular to a wall. A stick of length $L$ flies by at speed $v$. It travels in front of the ruler, so that it obscures part of the ruler from your view. When the stick hits the wall it stops.

In your reference frame, the stick is shorter than $L$. Therefore, right before it hits the wall, you will be able to see a mark on the ruler that is less than $L$ units from the wall (see Fig. 10.31).

But in the stick's frame, the marks on the ruler are closer together. Therefore, when the wall hits the stick, the closest mark to the wall that you can see on the ruler is greater than $L$ units (see Fig. 10.31).

Which view is correct (and what is wrong with the incorrect one)?

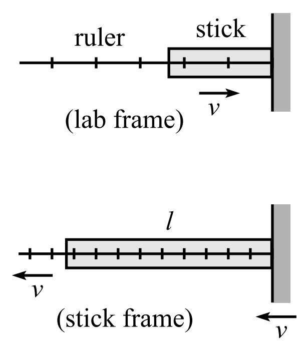


**Figure 10.31**

8. **Cookie cutter** $**$

Cookie dough (chocolate chip, of course) lies on a conveyor belt which moves along at speed $v$. A circular stamp stamps out cookies as the dough rushes by beneath it. When you buy these cookies in a store, what shape are they? That is, are they squashed in the direction of the belt, stretched in that direction, or circular?

9. **The twin paradox** $**$

Consider the usual twin paradox: Person $A$ stays on the earth, while person $B$ flies quickly to a distant star and back. $B$ is younger than $A$ when they meet up again. The paradox is that one might argue that although $A$ will see $B$'s clock moving slowly, $B$ will also see $A$'s clock moving slowly, so $A$ should be younger than $B$.

There are many resolutions to this "paradox". Perform the following one: Let $B$'s path to the distant star be lined with a wire that periodically zaps $B$ as he flies along (see Fig. 10.32). Let this be accomplished by having every point in the wire emit a 'zap' simultaneously in $A$'s frame. Let $t_A$ be the time between zaps in $A$'s frame. Find the time between zaps in $B$'s frame, and then use the fact that both $A$ and $B$ agree on the total number of times $B$ gets zapped.

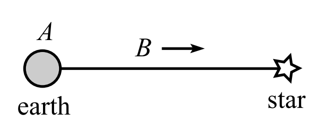


**Figure 10.32**

10. **Velocity addition from scratch** ∗∗∗

A ball moves at speed $v_1$ with respect to a train. The train moves at speed $v_2$ with respect to the ground. What is the speed of the ball with respect to the ground? Solve this problem (that is, derive the velocity addition formula, eq. (10.28)) in the following way. (Do not use time dilation, length contraction, etc. Use only the fact that the speed of light is the same in any inertial frame.)

Let the ball be thrown from the back of the train. At this instant, a photon is released next to it (see Fig. 10.33). The photon heads to the front of the train, bounces off a mirror, heads back, and eventually runs into the ball. In both frames, find the fraction of the way along the train the meeting occurs, and then equate these fractions.

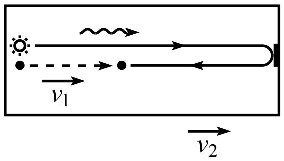


**Figure 10.33**

*Section 10.3: The Lorentz transformations*

11. **A bunch of L.T.'s** ∗

Verify that the values of $\Delta x$ and $\Delta t$ in the table in the example in Section 10.4 satisfy the Lorentz transformations between the six pairs of frames, namely $AB$, $AC$, $AD$, $BC$, $BD$, and $CD$. (See Fig. 10.34.)

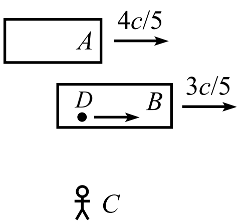


**Figure 10.34**

12. **Many velocity additions** ∗∗

An object moves at speed $v_1/c \equiv \beta_1$ with respect to $S_1$, which moves at speed $\beta_2$ with respect to $S_2$, which moves at speed $\beta_3$ with respect to $S_3$, and so on, until finally $S_{N-1}$ moves at speed $\beta_N$ with respect to $S_N$ (see Fig. 10.35). Show that the speed, $\beta_{(N)}$, of the object with respect to $S_N$ can be written as

$$\beta_{(N)} = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}\,, \tag{10.73}$$

where

$$P_N^+ \equiv \prod_{i=1}^{N}(1+\beta_i), \qquad \text{and} \qquad P_N^- \equiv \prod_{i=1}^{N}(1-\beta_i). \tag{10.74}$$

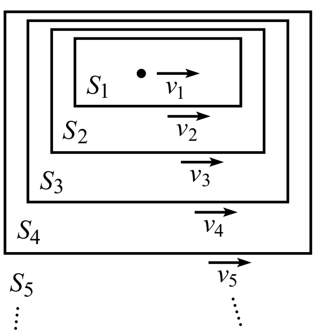


**Figure 10.35**

13. **Equal speeds**

$A$ and $B$ travel at $4c/5$ and $3c/5$, respectively, as shown in Fig. 10.36. How fast should $C$ travel between them, so that he sees $A$ and $B$ approaching him at the same speed? What is this speed?

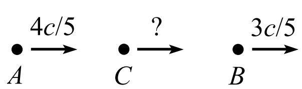


**Figure 10.36**

14. **More equal speeds** ∗

$A$ moves at speed $v$, and $B$ is at rest, as shown in Fig. 10.37. How fast must $C$ travel, so that she sees $A$ and $B$ approaching her at the same rate?

In the lab frame ($B$'s frame), what is the ratio of the distances $CB$ and $AC$? The answer to this is very nice and clean. Can you think of a simple intuitive explanation for the result?

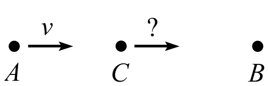


**Figure 10.37**

15. **Transverse velocity addition** $**$

For the special case $u'_x = 0$, the transverse velocity addition formula, eq. (10.35), yields $u_y = u'_y/\gamma$. Derive this in the following way.

In frame $S'$, a particle moves with speed $u'$ in the $y'$-direction. Frame $S$ moves to the left with speed $v$, so that the situation in $S$ looks like that in Fig. 10.38, with the $y$-speed now $u$. Consider a series of equally spaced dotted lines, as shown. The ratio of times between passes of the dotted lines in frames $S$ and $S'$ is $T_S/T_{S'} = (1/u)/(1/u') = u'/u$. Derive another expression for this ratio by using time dilation arguments, and then equate the two expressions to solve for $u$ in terms of $u'$ and $v$.

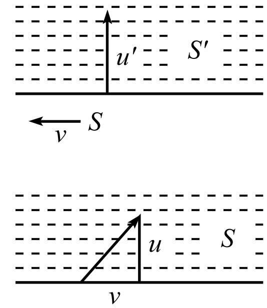

**Figure 10.38**

16. **Equal transverse speeds** $*$

In the lab frame, an object moves with velocity $(u_x, u_y)$, and you move with speed $v$ in the $x$-direction. What must $v$ be so that you also see the object moving with speed $u_y$ in your $y$-direction? (One solution is of course $v = 0$. Find the other one.)

17. **Relative speed** $*$

In the lab frame, two particles move with speed $v$ along the paths shown in Fig. 10.39. The angle between the trajectories is $2\theta$. What is the speed of one particle, as viewed by the other? (This problem is posed again in Chapter 12, where it can be solved in a much simpler way, using 4-vectors.)

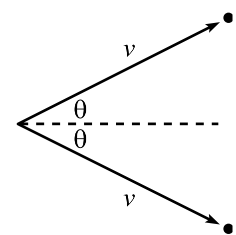

**Figure 10.39**

18. **Another relative speed** $**$

In the lab frame, particles $A$ and $B$ move with speeds $u$ and $v$ along the paths shown in Fig. 10.40. The angle between the trajectories is $\theta$. What is the speed of one particle, as viewed by the other? (This problem is posed again in Chapter 12, where it can be solved in a much simpler way, using 4-vectors.)

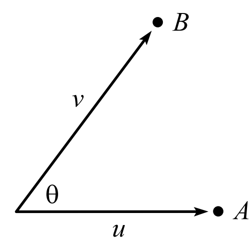

**Figure 10.40**

19. **Modified twin paradox** $***$

Consider the following variation of the twin paradox. $A$, $B$, and $C$ each have a clock. In $A$'s reference frame, $B$ flies past $A$ with speed $v$ to the right. When $B$ passes $A$, they both set their clocks to zero. Also, in $A$'s reference frame, $C$ starts far to the right and moves to the left with speed $v$. When $B$ and $C$ pass each other, $C$ sets his clock to read the same as $B$'s. Finally, when $C$ passes $A$, they compare the readings on their clocks. At this event, let $A$'s clock read $T_A$, and let $C$'s clock read $T_C$.

(a) Working in $A$'s frame, show that $T_C = T_A/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(b) Working in $B$'s frame, show again that $T_C = T_A/\gamma$.

(c) Working in $C$'s frame, show again that $T_C = T_A/\gamma$.

*Section 10.4: The invariant interval*

20. **Throwing on a train** $**$

A train with proper length $L$ moves at speed $c/2$ with respect to the ground. A ball is thrown from the back to the front, at speed $c/3$ with respect to the train. How much time does this take, and what distance does the ball cover, in:

(a) The train frame?

(b) The ground frame? Solve this by

i. Using a velocity-addition argument.

ii. Using the Lorentz transformations to go from the train frame to the ground frame.

(c) The ball frame?

(d) Verify that the invariant interval is indeed the same in all three frames.

(e) Show that the times in the ball frame and ground frame are related by the relevant $\gamma$-factor.

(f) Ditto for the ball frame and train frame.

(g) Show that the times in the train frame and ground frame are *not* related by the relevant $\gamma$-factor. Why not?

*Section 10.5: Minkowski diagrams*

21. **A new frame**

In one reference frame, Event 1 happens at $x = 0$, $ct = 0$, and Event 2 happens at $x = 2$, $ct = 1$. Find a frame where the two events are simultaneous.

22. **Minkowski diagram units** $*$

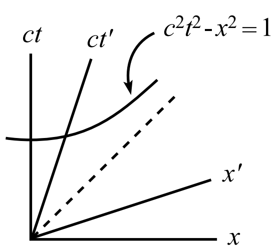

**Figure 10.41**

Consider the Minkowski diagram in Fig. 10.41. In frame $S$, the hyperbola $c^2t^2 - x^2 = 1$ is drawn. Also drawn are the axes of frame $S'$, which moves past $S$ with speed $v$. Use the invariance of the interval $s^2 = c^2t^2 - x^2$ to derive the ratio of the unit sizes on the $ct'$ and $ct$ axes (and check the result with eq. (10.43)).

23. **Velocity addition via Minkowski** $*$

An object moves at speed $v_1$ with respect to frame $S'$. Frame $S'$ moves at speed $v_2$ with respect to frame $S$. (in the same direction as the motion of the object). What is the speed, $u$, of the object with respect to frame $S$?

Solve this problem (i.e., derive the velocity addition formula) by drawing a Minkowski diagram with frames $S$ and $S'$, drawing the worldline of the object, and doing a little geometry.

24. **Acceleration and redshift** ***

Use a Minkowski diagram to do the following problem:

Two people stand a distance $d$ apart. They simultaneously start accelerating in the same direction (along the line between them), each with proper acceleration $a$. At the instant they start to move, how fast does each person see the other person's clock tick?

25. **Break or not break?** ***

**Figure 10.42**

Two spaceships float in space and are at rest relative to each other. They are connected by a string (see Fig. 10.42). The string is strong, but it cannot withstand an arbitrary amount of stretching. At a given instant, the spaceships simultaneously start accelerating (along the direction of the line between them) with the same acceleration. (Assume they bought identical engines from the same store, and they put them on the same setting.)

Will the string eventually break?

*Section 10.7: Rapidity*

26. **Successive Lorentz transformations**

The Lorentz transformation in eq. (10.55) may be written in matrix form as

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.75}$$

Show that by applying an L.T. with $v_1 = \tanh\phi_1$, and then another L.T. with $v_2 = \tanh\phi_2$, you do indeed obtain the L.T. with $v = \tanh(\phi_1 + \phi_2)$.

27. **Accelerator's time** *

A spaceship is initially at rest in the lab frame. At a given instant, it starts to accelerate. Let this happen when the lab clock reads $t = 0$ and the spaceship clock reads $t' = 0$. The proper acceleration is $a$. (That is, at time $t' + dt'$, the spaceship is moving at a speed $a\,dt'$ relative to the frame it was in at time $t'$.) Later on, a person in the lab measures $t$ and $t'$. What is the relation between them?

## 10.11 Solutions

1. **No transverse length contraction**

   Assume that the paint brushes are capable of leaving marks on stick $B$ if $B$ is long enough, or if $A$ short enough. The key fact we need here is the second postulate of relativity, which says that the frames of the sticks are equivalent. That is, if $A$ sees $B$ shorter than (or longer than) (or equal to) itself, then $B$ also sees $A$ shorter than (or longer than) (or equal to) itself. The contraction factor must be the same in going each way between the frames.

   Assume that $A$ sees $B$ shortened. Then $B$ won't reach out to the ends of $A$, so there will be no marks on $B$. But in this case, $B$ must *also* see $A$ shortened, so there *will* be marks on $B$ (see Fig. 10.43). This is a contradiction.

   Likewise, if we assume that $A$ sees $B$ lengthened, we also reach a contradiction. Hence, they each must see the other stick as one meter long.

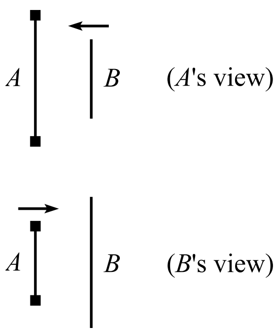


**Figure 10.43**

2. **Explaining time dilation**

   The resolution to the apparent paradox is the "head start" that $B$'s clock has over $A$'s clock (in the spaceship frame). From eq. (10.2), we know that in the spaceship frame, $B$'s clock reads $Lv/c^2$ more than $A$'s. (The two stars may be considered to be at the ends of the train in the example in Section 10.2.1.)

   Therefore, what a person on the spaceship says is this: "My clock advances by $L/\gamma v$ during the whole trip. I see $B$'s clock running slow by a factor $\gamma$, so I see $B$'s clock advance by only $(L/\gamma v)/\gamma = L/\gamma^2 v$. However, $B$'s clock started not at zero but at $Lv/c^2$. Therefore, the final reading on $B$'s clock when I get there is

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}\,, \tag{10.76}$$

   as it should be."

3. **Explaining Length contraction**

   The resolution to the apparent paradox is that the explosions do not occur simultaneously in the train frame. As the platform rushes past the train, the rear bomb explodes before the front bomb explodes. The front bomb then gets to travel farther by the time it explodes and leaves its mark. The distance between the marks is therefore larger than you might naively expect. Let's be quantitative about this.

   Let the two bombs contain clocks that read a time $t$ when they explode (they are synchronized in the ground frame). Then in the frame of the train, the front bomb's clock reads only $t - Lv/c^2$ when the rear bomb explodes when showing a time $t$. (This is the "head start" result from eq. (10.2).) The front bomb's clock must therefore advance by a time of $Lv/c^2$ before it explodes. But since the train sees the bombs' clocks running slow by a factor $\gamma$, we conclude that in the frame of the train, the front bomb explodes a time $\gamma Lv/c^2$ after the rear bomb explodes. During this time of $\gamma Lv/c^2$, the platform moves a distance $(\gamma Lv/c^2)v$ relative to the train.

   Therefore, what a person on the train says is this: "Due to length contraction, the distance between the bombs is $L/\gamma$. The front bomb is therefore a distance $L/\gamma$ ahead of the rear bomb when the latter explodes. The front bomb then travels an additional distance of $L\gamma v^2/c^2$ by the time it explodes, at which point it is a distance of

$$\frac{L}{\gamma} + \frac{\gamma L v^2}{c^2} = L\gamma\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = L\gamma\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = L\gamma \tag{10.77}$$

   ahead of the rear bomb's mark, as we wanted to show."

4. **A passing stick**

   (a) The stick has length $L/\gamma$ in your frame. It moves with speed $v$. Therefore, the time taken in your frame to cover the distance of $L/\gamma$ is $L/\gamma v$.

   (b) The stick sees you fly by at speed $v$. The stick has length $L$ in its own frame, so the time elapsed in the stick frame is simply $L/v$. During this time, the stick will see the watch on your wrist run slow, by a factor $\gamma$. Therefore, a time of $L/\gamma v$ elapses on your watch, in agreement with part (a).

   REMARK: Logically, the two solutions (a) and (b) differ in that one uses length contraction and the other uses time dilation. Mathematically, they differ simply in the order in which the divisions by $\gamma$ and $v$ occur. ♣

   (c) You see the rear clock on the train showing a time of $Lv/c^2$ more than the front clock. In addition to this head start, more time will of course elapse on the rear clock by the time it reaches you. The time in your frame is $L/\gamma v$ (since the train has length $L/\gamma$ in your frame). But the train's clocks run slow, so a time of only $L/\gamma^2 v$ will elapse on the rear clock by the time it reaches you. The total time that the rear clock shows is therefore

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}, \qquad (10.78)$$

   in agreement with the quick calculation below in part (d).

   (d) The stick sees you fly by at speed $v$. The stick has length $L$ in its own frame, so the time elapsed in the stick frame is simply $L/v$.

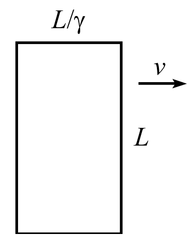


**Figure 10.44**

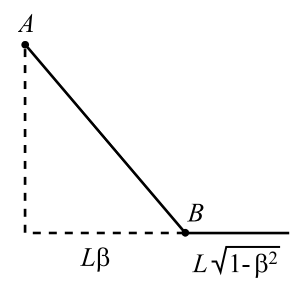


**Figure 10.45**

5. **Rotated square**

   Fig. 10.44 shows a top view of the square at the instant (in your frame) when it is closest to you. Its length is contracted along the direction of motion, so it takes the shape of a rectangle with sides $L$ and $L/\gamma$. This is what the shape *is* in your frame (where *is*-ness is defined by where all the points of an object are at simultaneous times). But what does the square *look* like to you? That is, what is the nature of the photons hitting your eye at a given instant?[30]

   Photons from the far side of the square have to travel an extra distance $L$ to get to your eye, compared to ones from the near side. So they need an extra time $L/c$ of flight. During this time $L/c$, the square moves a distance $Lv/c = L\beta$ sideways. Therefore, referring to Fig. 10.45, a photon emitted at point $A$ reaches your eye at the same time as a photon emitted from point $B$.

   This means that the trailing side of length $L$ spans a distance $L\beta$ across your field of vision, while the near side spans a distance $L/\gamma = L\sqrt{1-\beta^2}$ across your field of vision. But this is exactly what a rotated square of side $L$ looks like, as shown in Fig. 10.46 (where the angle of rotation satisfies $\sin\theta = \beta$).

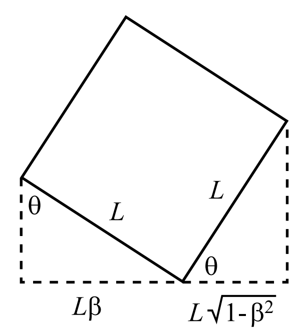


**Figure 10.46**

6. **Train in a tunnel**

   Yes, the bomb explodes. This is clear in the frame of the train (see Fig. 10.47). In this frame, the train has length $L$, and the tunnel speeds past it. The tunnel is

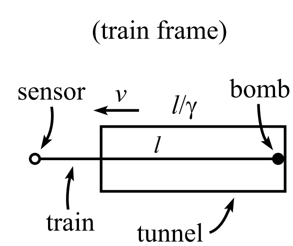


**Figure 10.47**

[30]In relativity problems, we virtually always subtract off the time it takes light to travel from the object to your eye (that is, we find out what really *is*). Along with the Doppler effect discussed in Section 10.6, this problem is one of the few exceptions where we actually want to determine what your eye registers.

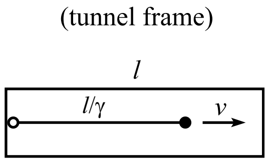


**Figure 10.48**

length-contracted down to $L/\gamma$. Therefore, the far end of the tunnel passes the front of the train before the near end passes the back, and so the bomb explodes.

We may, however, look at things in the frame of the tunnel (see Fig. 10.48). Here the tunnel has length $L$, and the train is length-contracted down to $L/\gamma$. Therefore, the deactivation device gets triggered *before* the front of the train passes the far end of the tunnel, so you might think that the bomb does *not* explode. We appear to have a paradox.

The resolution to this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself. It takes a finite time for the signal to travel the length of the train from the sensor to the bomb. It turns out that this transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving. Let's prove this.

The signal has the best chance of winning this "race" if it has speed $c$. So let us assume this is the case. Now, the signal gets to the bomb before the bomb gets to the far end of the tunnel if and only if a light pulse emitted from the near end of the tunnel (at the instant the back of the train goes by) reaches the far end of the tunnel before the front of the train does. (That sentence was a mouthful.) The former takes a time $L/c$. The latter takes a time $L(1-1/\gamma)/v$ (since the front of the train is already a distance $L/\gamma$ through the tunnel). So if the bomb is to not explode, we must have

$$\begin{aligned} L/c &< L(1-1/\gamma)/v \\ \Longleftrightarrow \quad \beta &< 1-\sqrt{1-\beta^2} \\ \Longleftrightarrow \quad \sqrt{1-\beta^2} &< 1-\beta \\ \Longleftrightarrow \quad \sqrt{1+\beta} &< \sqrt{1-\beta}. \end{aligned} \tag{10.79}$$

This can never be true. Therefore, the signal always arrives too late, and the bomb always explodes.

7. **Seeing behind the stick**

The first reasoning is correct. You will be able to see a mark on the ruler that is less than $L$ units from the wall. (You will actually be able to see a mark even closer to the wall than $L/\gamma$, as we'll show below). The main point of this problem (and many other ones) is that signals do not travel instantaneously. The back of the stick does not know that the front of the stick has come into contact with the wall until a finite time has passed. Let's be quantitative about this and calculate (in both frames) the closest mark to the wall that you can see.

Consider your reference frame. The stick has length $L/\gamma$. Therefore, when the stick hits the wall, you can see mark a distance $L/\gamma$ from the wall. You will, however, be able to see a mark even closer to the wall, because the back end of the stick will keep moving forward, since it doesn't yet know that the front end has hit the wall. The stopping signal (shock wave, etc.) takes time to travel.

Let's assume that the stopping signal travels along the stick at speed $c$. (You can work with a general speed $u$. But the speed $c$ is simpler, and it yields an upper bound on the closest mark you can see.) Where will the signal reach the back end? Starting from the time the stick hits the wall, the signal travels backwards from the wall at speed $c$, and the back end of the stick travels forwards at speed $v$ (from a point $L/\gamma$ away from the wall). So the relative speed (as viewed by you) of the signal and the back end is $c+v$. Therefore, the signal hits the back end after a time $(L/\gamma)/(c+v)$.

During this time, the signal has traveled a distance $c(L/\gamma)/(c+v)$ from the wall. The closest point to the wall that you can see is therefore the

$$\frac{L}{\gamma(1+\beta)} = L\sqrt{\frac{1-\beta}{1+\beta}} \tag{10.80}$$

mark on the ruler.

Now consider the stick's reference frame. The wall is moving to the left towards it at speed $v$. After the wall hits the right end of the stick, the signal moves to the left with speed $c$, and the wall keeps moving to the left with speed $v$. Where is the wall when the signal reaches the left end? The wall travels $v/c$ as fast as the signal, so it travels a distance $Lv/c$ in the time that the signal travels the distance $L$. This means that it is $L(1-v/c)$ away from the left end of the stick. In the stick's frame, this corresponds to a distance $\gamma L(1-v/c)$ on the ruler (because the ruler is length-contracted). So the left end of the stick is at the

$$L\gamma(1-\beta) = L\sqrt{\frac{1-\beta}{1+\beta}} \tag{10.81}$$

mark on the ruler, in agreement with eq. (10.80).

8. **Cookie cutter**

Let the diameter of the cookie cutter be $L$, and consider the two following reasonings.

- In the lab frame, the dough is length-contracted, so the diameter $L$ corresponds to a distance larger than $L$ (namely $\gamma L$) in the dough's frame. Therefore, when you buy a cookie, it is stretched out by a factor $\gamma$ in the direction of the belt.[31]
- In the frame of the dough, the cookie cutter is length-contracted in the direction of motion. It has length $L/\gamma$. So in the frame of the dough, the cookies have a length of only $L/\gamma$. Therefore, when you buy a cookie, it is squashed by a factor $\gamma$ in the direction of the belt.

Which reasoning is correct? The first one is. The cookies are stretched out. The fallacy in the second reasoning is that the various parts of the cookie cutter do *not* strike the dough simultaneously in the dough frame. What the dough sees is this: The cutter moves to, say, the left. The right side of the cutter stamps the dough, then nearby parts of the cutter stamp it, and so on, until finally the left side of the cutter stamps the dough. But by this time the front (that is, the left) of the cutter has moved farther to the left. So the cookie turns out to be longer than $L$. It takes a little work to demonstrate that the length is actually $\gamma L$, but let's do that now (by working in the dough frame).

Consider the moment when the the rightmost point of the cutter strikes the dough. In the dough frame, a clock at the rear (the right side) of the cutter reads $Lv/c^2$ more than a clock at the front (the left side). The front clock must therefore advance by $Lv/c^2$ by the time it strikes the dough. (This is true because all points on the cutter strike the dough simultaneously in the cutter frame. Hence, all cutter clocks read the same when they strike.) But due to time dilation, this takes a time $\gamma(Lv/c^2)$ in the dough frame. During this time, the cutter travels a distance $v(\gamma Lv/c^2)$. Since the

[31] The shape is an ellipse, since that's what a stretched-out circle is. The eccentricity of an ellipse is the focal distance divided by the semi-major axis length. As an exercise, you can show that this equals $\beta \equiv v/c$ here.

front of the cutter was initially a distance $L/\gamma$ (due to length contraction) ahead of the back, the total length of the cookie in the dough frame equals

$$\begin{aligned}\ell &= \frac{L}{\gamma} + v\left(\frac{\gamma L v}{c^2}\right) \\ &= \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) \\ &= \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) \\ &= \gamma L,\end{aligned}$$

as we wanted to show.

9. **The twin paradox**

The main point is that because the zaps occur simultaneously in $A$'s frame, they do *not* occur simultaneously in $B$'s frame. The zaps further ahead of $B$ occur earlier (as you can show), so there is less time between zaps in $B$'s frame than one might think (since $B$ is constantly moving toward the zaps ahead, which happen earlier).

Consider two successive zapping events. The $\Delta x_B$ between them is zero, so the Lorentz transformation $\Delta t_A = \gamma(\Delta t_B - v\Delta x_B/c^2)$ gives

$$\Delta t_B = \Delta t_A/\gamma. \tag{10.82}$$

So if $t_A$ is the time between zaps in $A$'s frame, then $t_B = t_A/\gamma$ is the time between zaps in $B$'s frame. (This is the usual time dilation result.)

Let $N$ be the total number of zaps $B$ gets. Then the total time in $A$'s frame is $T_A = Nt_A$, while the total time in $B$'s frame is $T_B = Nt_B = N(t_A/\gamma)$. Therefore,

$$T_B = \frac{T_A}{\gamma}. \tag{10.83}$$

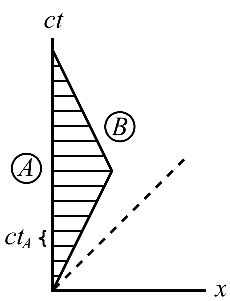

**Figure 10.49**

So $B$ is younger.

This can all be seen quite clearly if we draw a Minkowski diagram. Fig. 10.49 shows our situation where the zaps occur simultaneously in $A$'s frame. We know from eq. (10.43) that the unit size on $B$'s $ct$ axis on the paper is $\sqrt{(1+\beta^2)/(1-\beta^2)}$ times the unit size of $A$'s $ct$ axis. Since the two pieces of $B$'s $ct$ axis are $\sqrt{1+\beta^2}$ times as long as the corresponding piece of $A$'s $ct$ axis, we see that only $\sqrt{1-\beta^2}$ as many time units fit on $B$'s worldline as fit on $A$'s worldline. So $B$ is younger.

Remark: Eq. (10.83) is the result we wanted to show. But the obvious question is: Why doesn't it work the other way around? That is, if we let $A$ be zapped by zaps along a wire that occur simultaneously in $B$'s frame, why don't we conclude that $A$ is younger? The answer is that there is no *one* $B$ frame; $B$ has a different frame going out and coming in.

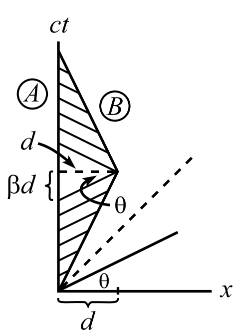

**Figure 10.50**

As usual, the best way to see what is going on is to draw a Minkowski diagram. Fig. 10.50 shows the situation where the zaps occur simultaneously in $B$'s frame. The lines of simultaneity (as viewed by $B$) are tilted one way on the trip outward, and the other way on the trip back. The result is that $A$ gets zapped frequently for a while, then no zaps occur for a while, then he gets zapped frequently again. The overall result, as we will now show, is that more time elapses in $A$'s frame than in $B$'s frame.

Let the distant star be a distance $d$ from the earth, in $A$'s frame. Let the zaps occur at intervals $\Delta t_B$ in $B$'s frame. Then they occur at intervals $\Delta t_B/\gamma$ in $A$'a frame. (The same factor applies to both parts of the journey.) Let there be $N$ total zaps during the journey. Then the total times registered by $B$ and $A$ are, respectively, $T_B = N\Delta t_B$, and $T_A = N(\Delta t_B/\gamma) + t$, where $t$ is the time where no zaps occur in $A$'s frame (in the middle of the journey). The distance between the earth and star in $B$'s frame is $vT_B/2$. But we know that this distance also equals $d/\gamma$. Therefore, $N\Delta t_B = T_B = 2d/(\gamma v)$. So we have

$$\begin{aligned} T_B &= \frac{2d}{\gamma v}, \\ T_A &= \frac{2d}{\gamma^2 v} + t. \end{aligned} \tag{10.84}$$

We must now calculate $t$. Since the slopes of $B$'s lines of simultaneity in the figure are $\pm\beta$, we see that $ct = 2d\tan\theta = 2d\beta$. Therefore,

$$\begin{aligned} T_B &= \frac{2d}{\gamma v}, \\ T_A &= \frac{2d}{\gamma^2 v} + \frac{2d\beta}{c} = \frac{2d}{v}. \end{aligned} \tag{10.85}$$

Hence, $T_B = T_A/\gamma$.

(We know of course, without doing any calculations, that $T_A = 2d/v$. But it is reassuring to add up the times when $A$ is getting zapped and when he is not getting zapped, to show that we still get the same answer.) ♣

10. **Velocity addition from scratch**

The important point here is that the meeting of the photon and the ball occurs at the same fraction of the way along the train, independent of the frame. This is true because although distances may change depending on the frame, fractions remain the same (because length contraction doesn't depend on position). Let's compute the desired fraction in the train frame $S'$, and then in the ground frame $S$.

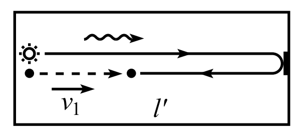


**Figure 10.51**

**Train frame**: Let the train have length $L'$. We'll first need to find the time at which the photon meets the ball (see Fig. 10.51). Light takes a time $L'/c$ to reach the mirror. At this time, the ball has traveled a distance $v_1(L'/c)$. The separation between the light and the ball at this time is therefore $L'(1 - v_1/c)$. From this point on, the relative speed of the light and the ball is $c + v_1$. Hence, it takes them an additional time $L'(1 - v_1/c)/(c + v_1)$ to meet. The total time before the meeting is therefore

$$t' = \frac{L'}{c} + \frac{L'(1 - v_1/c)}{c + v_1} = \frac{2L}{c + v_1}\,. \tag{10.86}$$

The distance the ball has traveled is thus $v_1 t' = 2v_1 L'/(c + v_1)$, and the desired fraction $F'$ is

$$F' = \frac{2v_1}{c + v_1}\,. \tag{10.87}$$

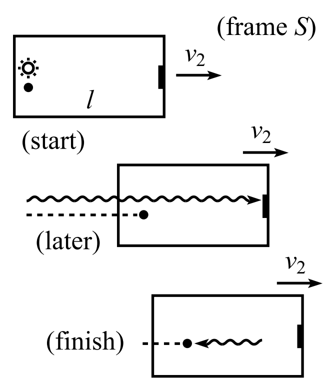


**Figure 10.52**

**Ground frame**: Let the speed of the ball with respect to the ground be $v$. Let the train have length $L$. Again, we'll first need to find the time at which the photon meets the ball (see Fig. 10.52). Light takes a time $L/(c - v_2)$ to reach the mirror (since the mirror is receding at a speed $v_2$). At this time, the ball has traveled a distance $vL/(c-v_2)$, and the light has traveled a distance $cL/(c-v_2)$. The separation between the light and the ball at this time is therefore $L(c - v)/(c - v_2)$. From this point on, the relative speed of the light and ball is $c + v$. Hence, it takes them an additional time $L(c-v)/[(c-v_2)(c+v)]$ to meet. The total time before the meeting is therefore

$$t = \frac{L}{c - v_2} + \frac{L(c - v)}{(c - v_2)(c + v)} = \frac{2cL}{(c - v_2)(c + v)}\,. \tag{10.88}$$

The distance the ball has traveled is thus $2vcL/[(c-v_2)(c+v)]$. But the distance the back of the train has traveled in this time is $2v_2cL/[(c-v_2)(c+v)]$. So the distance between the ball and the back of the train is $2(v-v_2)cL/[(c-v_2)(c+v)]$. The desired fraction $F$ is therefore

$$F = \frac{2(v-v_2)c}{(c-v_2)(c+v)} \,. \tag{10.89}$$

We can now equate the above expressions for $F'$ and $F$. For convenience, define $\beta \equiv v/c$, $\beta_1 \equiv v_1/c$, and $\beta_2 \equiv v_2/c$. Then $F' = F$ yields

$$\frac{\beta_1}{1+\beta_1} = \frac{\beta-\beta_2}{(1-\beta_2)(1+\beta)} \,. \tag{10.90}$$

Solving for $\beta$ in terms of $\beta_1$ and $\beta_2$ gives[32]

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \,, \tag{10.91}$$

as desired.

11. **Throwing on a train**

(a) In the train frame, the distance is simply $d = L$. And the time is $t = L/(c/3) = 3L/c$.

(b) i. The velocity of the ball with respect to the ground is (with $u = c/3$ and $v = c/2$)

$$V_g = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{\frac{c}{3}+\frac{c}{2}}{1+\frac{1}{3}\cdot\frac{1}{2}} = \frac{5c}{7} \,. \tag{10.92}$$

The length of the train with respect to the ground is $L/\gamma_{1/2} = \sqrt{3}L/2$. Therefore, at time $t$, the position of the front of the train is $\sqrt{3}L/2 + vt$. And the position of the ball is $V_g t$. These two positions are equal when

$$(V_g - v)t = \frac{\sqrt{3}L}{2} \qquad \Longrightarrow \qquad t = \frac{\frac{\sqrt{3}L}{2}}{\frac{5c}{7}-\frac{c}{2}} = \frac{7L}{\sqrt{3}c} \,. \tag{10.93}$$

Equivalently, this time is obtained by noting that the ball closes the initial head-start of $\sqrt{3}L/2$ that the front of the train had, at a relative speed of $V_g - v$.[33]

The distance the ball travels is then $d = V_g t = (5c/7)(7L/\sqrt{3}c) = 5L/\sqrt{3}$.

ii. In the train frame, the space and time intervals are $x' = L$ and $t' = 3L/c$, from part (a). The $\gamma$-factor between the frames is $\gamma_{1/2} = 2/\sqrt{3}$, so the Lorentz transformation give the coordinates in the ground frame as

$$\begin{aligned} x &= \gamma(x'+vt') = \frac{2}{\sqrt{3}}\left(L + \frac{c}{2}\left(\frac{3L}{c}\right)\right) = \frac{5L}{\sqrt{3}} \,, \\ t &= \gamma(t'+vx'/c^2) = \frac{2}{\sqrt{3}}\left(\frac{3L}{c} + \frac{\frac{c}{2}(L)}{c^2}\right) = \frac{7L}{\sqrt{3}c} \,, \end{aligned} \tag{10.94}$$

in agreement with above.

[32]N. David Mermin does this problem in Am. J. Phys., **51**, 1130 (1983), and then takes things one step further in Am. J. Phys., **52**, 119 (1984).

[33]Yes, it's legal to simply subtract these velocities, because they're both measured with respect to the same frame, namely the ground.

(c) In the ball frame, the train has length, $L/\gamma_{1/3} = \sqrt{8}L/3$. Therefore, the time it takes the train to fly (at speed $c/3$) past the ball is $t = (\sqrt{8}L/3)/(c/3) = 2\sqrt{2}L/c$. And the distance is $d = 0$, of course, because the ball doesn't move in the ball frame.

(d) The values of $c^2t^2 - x^2$ in the three frames are:

Train frame: $c^2t^2 - x^2 = c^2(3L/c)^2 - L^2 = 8L^2$.

Ground frame: $c^2t^2 - x^2 = c^2(7L/\sqrt{3}c)^2 - (5L/\sqrt{3})^2 = 8L^2$.

Ball frame: $c^2t^2 - x^2 = c^2(2\sqrt{2}L/c)^2 - (0)^2 = 8L^2$.

These are all equal, as they should be.

(e) The relative speed between the ball frame and the ground frame is $5c/7$. Therefore, $\gamma_{5/7} = 7/2\sqrt{6}$, and the times are indeed related by

$$t_g = \gamma t_b \qquad \Longleftrightarrow \qquad \frac{7L}{\sqrt{3}c} = \frac{7}{2\sqrt{6}}\left(\frac{2\sqrt{2}L}{c}\right), \qquad \text{which is true.} \tag{10.95}$$

(f) The relative speed between the ball frame and the train frame is $c/3$. Therefore, $\gamma_{1/3} = 3/2\sqrt{2}$, and the times are indeed related by

$$t_t = \gamma t_b \qquad \Longleftrightarrow \qquad \frac{3L}{c} = \frac{3}{2\sqrt{2}}\left(\frac{2\sqrt{2}L}{c}\right), \qquad \text{which is true.} \tag{10.96}$$

(g) The relative speed between the train frame and the ground frame is $c/2$. Therefore, $\gamma_{1/2} = 2/\sqrt{3}$, and the times are *not* related by a simple time-dilation factor.

$$t_g \neq \gamma t_t \qquad \Longleftrightarrow \qquad \frac{7L}{\sqrt{3}c} \neq \frac{2}{\sqrt{3}}\left(\frac{3L}{c}\right). \tag{10.97}$$

We don't obtain an equality because we can use time dilation only if the two events happen at the *same place* in one of the frames. In this problem, the "ball leaving the rear" and the "ball hitting the front" events happen at the same place in the ball frame, but not in the train frame or the ground frame.

12. **A bunch of L.T.'s**

The relative speeds, and the associated $\gamma$ factors, for the six pairs of frames are (using the results from the examples in Sections 10.2.3, 10.3.3, and 10.4)

| | $AB$ | $AC$ | $AD$ | $BC$ | $BD$ | $CD$ |
|---|---|---|---|---|---|---|
| $v$ | $5c/13$ | $4c/5$ | $c/5$ | $3c/5$ | $c/5$ | $5c/7$ |
| $\gamma$ | $13/12$ | $5/3$ | $5/2\sqrt{6}$ | $5/4$ | $5/2\sqrt{6}$ | $7/2\sqrt{6}$ |

The separations between the two events in the four frames are (from the example in Section 10.4))

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $\Delta x$ | $-L$ | $L$ | $5L$ | $0$ |
| $\Delta t$ | $5L/c$ | $5L/c$ | $7L/c$ | $2\sqrt{6}L/c$ |

The Lorentz transformations are

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2). \end{aligned} \tag{10.98}$$

For each of the six pairs, we'll transform from the faster frame to the slower frame. This means that the coordinates of the faster frame will be on the right-hand side of the L.T.'s. The sign on the right-hand side of the L.T.'s will therefore always be a "+". In the $AB$ case, for example, we will write, "Frames $B$ and $A$," in that order, to signify that the $B$ coordinates are on the left-hand side, and the $A$ coordinates are on the right-hand side. We'll simply list the L.T.'s for the six cases, and you can check that they do indeed all work out.

$$\begin{aligned}
\text{Frames } B \text{ and } A: \qquad L &= \frac{13}{12}\left(-L + \left(\frac{5c}{13}\right)\left(\frac{5L}{c}\right)\right), \\
\frac{5L}{c} &= \frac{13}{12}\left(\frac{5L}{c} + \frac{\frac{5c}{13}(-L)}{c^2}\right). \\
\text{Frames } C \text{ and } A: \qquad 5L &= \frac{5}{3}\left(-L + \left(\frac{4c}{5}\right)\left(\frac{5L}{c}\right)\right), \\
\frac{7L}{c} &= \frac{5}{3}\left(\frac{5L}{c} + \frac{\frac{4c}{5}(-L)}{c^2}\right). \\
\text{Frames } D \text{ and } A: \qquad 0 &= \frac{5}{2\sqrt{6}}\left(-L + \left(\frac{c}{5}\right)\left(\frac{5L}{c}\right)\right), \\
\frac{2\sqrt{6}L}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{5L}{c} + \frac{\frac{c}{5}(-L)}{c^2}\right). \\
\text{Frames } C \text{ and } B: \qquad 5L &= \frac{5}{4}\left(L + \left(\frac{3c}{5}\right)\left(\frac{5L}{c}\right)\right), \\
\frac{7L}{c} &= \frac{5}{4}\left(\frac{5L}{c} + \frac{\frac{3c}{5}L}{c^2}\right). \\
\text{Frames } B \text{ and } D: \qquad L &= \frac{5}{2\sqrt{6}}\left(0 + \left(\frac{c}{5}\right)\left(\frac{2\sqrt{6}L}{c}\right)\right), \\
\frac{5L}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{2\sqrt{6}L}{c} + \frac{\frac{c}{5}(0)}{c^2}\right). \\
\text{Frames } C \text{ and } D: \qquad 5L &= \frac{7}{2\sqrt{6}}\left(0 + \left(\frac{5c}{7}\right)\left(\frac{2\sqrt{6}L}{c}\right)\right), \\
\frac{7L}{c} &= \frac{7}{2\sqrt{6}}\left(\frac{2\sqrt{6}L}{c} + \frac{\frac{5c}{7}(0)}{c^2}\right).
\end{aligned} \tag{10.99}$$

13. **Many velocity additions**

Let's first check the formula for $N = 1$ and $N = 2$. When $N = 1$, the formula gives

$$\beta_{(1)} = \frac{P_1^+ - P_1^-}{P_1^+ + P_1^-} = \frac{(1+\beta_1) - (1-\beta_1)}{(1+\beta_1) + (1-\beta_1)} = \beta_1, \tag{10.100}$$

as it should. When $N = 2$, the formula gives

$$\beta_{(2)} = \frac{P_2^+ - P_2^-}{P_2^+ + P_2^-} = \frac{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)}{(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\,, \qquad (10.101)$$

which agrees with the velocity addition formula.

Let's now prove the formula for general $N$. We will use induction. That is, we will assume that the result holds for $N$ and then show that it holds for $N+1$. To find the speed, $\beta_{(N+1)}$, of the object with respect to $S_{N+1}$, we can relativistically add the speed of the object with respect to $S_N$ (which is $\beta_{(N)}$) with the speed of $S_N$ with respect to $S_{N+1}$ (which is $\beta_{N+1}$). This gives

$$\beta_{(N+1)} = \frac{\beta_{N+1} + \beta_{(N)}}{1 + \beta_{N+1}\beta_{(N)}}\,. \qquad (10.102)$$

Under the assumption that our formula holds for $N$, this becomes

$$\begin{aligned}
\beta_{(N+1)} &= \frac{\beta_{N+1} + \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}}{1 + \beta_{N+1}\frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}} \\
&= \frac{\beta_{N+1}(P_N^+ + P_N^-) + (P_N^+ - P_N^-)}{(P_N^+ + P_N^-) + \beta_{N+1}(P_N^+ - P_N^-)} \\
&= \frac{P_N^+(1+\beta_{N+1}) - P_N^-(1-\beta_{N+1})}{P_N^+(1+\beta_{N+1}) + P_N^-(1-\beta_{N+1})} \\
&\equiv \frac{P_{N+1}^+ - P_{N+1}^-}{P_{N+1}^+ + P_{N+1}^-}\,, \qquad (10.103)
\end{aligned}$$

as we wanted to show. We have therefore shown that if the result holds for $N$, then it holds for $N+1$. Since know that the result holds for $N = 1$, it thus holds for all $N$.

The expression for $\beta_{(N)}$ has some expected properties. It is symmetric in the $\beta_i$. And if at least one of the $\beta_i$ equals 1, then $P_N^- = 0$. This yields $\beta_{(N)} = 1$, as it should. And if at least one of the $\beta_i$ equals $-1$, then $P_N^+ = 0$. This yields $\beta_{(N)} = -1$, as it should.

Remark: We can use the result of this problem to derive the $v(t)$ given in eq. (10.62). First, note that if all the $\beta_i$ in this problem are equal, and if their common value is much less than 1, then

$$\beta_{(N)} = \frac{(1+\beta)^N - (1-\beta)^N}{(1+\beta)^N + (1-\beta)^N} \approx \frac{e^{\beta N} - e^{-\beta N}}{e^{\beta N} + e^{-\beta N}} = \tanh(\beta N). \qquad (10.104)$$

Let $\beta$ equal $a\,dt/c$, which is the relative speed of two frames at nearby times in the spaceship scenario leading up to eq. (10.62). If we let $N = t/dt$ be the number of frames here (and if we take the limit $dt \to 0$), then we have reproduced the spaceship scenario. Therefore, the $\beta_{(N)}$ in eq. (10.104) equals the $v(t)$ in eq. (10.62). With the present values of $N$ and $\beta$, eq. (10.104) gives $\beta_{(N)} = \tanh(at/c)$, as desired. ♣

14. **Equal speeds**

**First Solution:** Let $C$ move at speed $v$. If $C$ sees both $A$ and $B$ approaching him at the same speed, then the relativistic subtraction of $v$ from $4c/5$ must equal the relativistic subtraction of $3c/5$ from $v$. Therefore, (dropping the $c$'s),

$$\frac{\frac{4}{5}-v}{1-\frac{4}{5}v}=\frac{v-\frac{3}{5}}{1-\frac{3}{5}v}\,. \tag{10.105}$$

This yields $0 = 35v^2 - 74v + 35 = (5v - 7)(7v - 5)$. Since the $v = 7/5$ root represents a speed larger than $c$, our answer is

$$v=\frac{5}{7}c. \tag{10.106}$$

Plugging this back into eq. (10.105) gives the relative speed of $C$ and both $A$ and $B$ as $u = c/5$.

**Second Solution:** Let the relative speed of $C$ and both $A$ and $B$ be $u$. Then the relativistic subtraction of $u$ from $4c/5$ must equal the relativistic addition of $u$ to $3c/5$ (because both of the results equal the speed of $C$ with respect to the ground). Therefore,

$$\frac{\frac{4}{5}-u}{1-\frac{4}{5}u}=\frac{\frac{3}{5}+u}{1+\frac{3}{5}u}\,. \tag{10.107}$$

This yields $0 = 5u^2 - 26u + 5 = (5u - 1)(u - 5)$. Since the $u = 5$ root represents a speed larger than $c$, our answer is

$$u=\frac{c}{5}\,. \tag{10.108}$$

Plugging this back into eq. (10.107) gives the speed of $C$ with respect to the ground as $v = 5c/7$.

**Third Solution:** The relative speed of $A$ and $B$ is (dropping the $c$'s)

$$\frac{\frac{4}{5}-\frac{3}{5}}{1-\frac{4}{5}\cdot\frac{3}{5}}=\frac{5}{13}\,. \tag{10.109}$$

From $C$'s point of view, this $5/13$ is the result of relativistically adding $u$ with another $u$. Therefore,

$$\frac{5}{13}=\frac{2u}{1+u^2}\qquad\Longrightarrow\qquad 5u^2-26u+5=0, \tag{10.110}$$

as in the second solution.

15. **More equal speeds**

Let $u$ be the speed at which $C$ sees $A$ and $B$ approaching her. From $C$'s point of view, $v$ is the result of relativistically adding $u$ with another $u$. Therefore (dropping the $c$'s),

$$v=\frac{2u}{1+u^2}\qquad\Longrightarrow\qquad u=\frac{1-\sqrt{1-v^2}}{v}\,. \tag{10.111}$$

Note that the quadratic equation for $u$ has another solution with a plus sign in front of the square root, but this solution cannot be correct, because it goes to infinity as $v$ goes to zero. The above solution for $u$ has the proper limit as $v$ goes to zero, namely $u \to v/2$, which is obtained by using the Taylor expansion for the square root.

The ratio of the distances $CB$ and $AC$ in the lab frame is the same as the ratio of the differences in velocity,

$$\frac{CB}{AC} = \frac{V_C - V_B}{V_A - V_C} = \frac{\frac{1-\sqrt{1-v^2}}{v} - 0}{v - \frac{1-\sqrt{1-v^2}}{v}} = \frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2} - (1-v^2)} = \frac{1}{\sqrt{1-v^2}} \equiv \gamma. \tag{10.112}$$

We see that $C$ is $\gamma$ times as far from $B$ as she is from $A$. Note that for nonrelativistic speeds, we have $\gamma \approx 1$, and $C$ is of course midway between $A$ and $B$.

An intuitive reason for the simple factor of $\gamma$ is the following. Imagine that $A$ and $B$ are carrying identical jousting sticks as they run toward $C$. Consider what the situation looks like when the tips of the sticks reach $C$. In the lab frame (in which $B$ is at rest), $B$'s stick is uncontracted, but $A$'s stick is length-contracted by a factor $\gamma$. Therefore, $A$ is closer to $C$ than $B$ is, by a factor $\gamma$.

16. **Transverse velocity addition**

Assume that a clock on the particle shows a time $T$ between successive passes of the dotted lines. In frame $S'$, the speed of the particle is $u'$, so the time dilation factor is $\gamma' = 1/\sqrt{1-u'^2}$. The time between successive passes of the dotted lines is therefore $T_{S'} = \gamma' T$.

In frame $S$, the speed of the particle is $\sqrt{v^2+u^2}$. (Yes, the Pythagorean theorem holds for the speeds, because both speeds are measured with respect to the same frame.) Hence, the time dilation factor is $\gamma = 1/\sqrt{1-v^2-u^2}$. The time between successive passes of the dotted lines is therefore $T_S = \gamma T$.

Equating our two expressions for $T_S/T_{S'}$ gives

$$\frac{u'}{u} = \frac{T_S}{T_{S'}} = \frac{\sqrt{1-u'^2}}{\sqrt{1-v^2-u^2}}\,. \tag{10.113}$$

Solving for $u$ gives

$$u = u'\sqrt{1-v^2} \equiv \frac{u'}{\gamma_v}\,, \tag{10.114}$$

as desired.

17. **Equal transverse speeds**

From your point of view, the lab frame is moving with speed $v$ in the negative $x$-direction. The transverse velocity addition formula, eq. (10.35), therefore gives the $y$-speed in your frame as $u_y/\gamma(1-u_x v)$. Demanding that this equals $u_y$ gives

$$\gamma(1-u_x v) = 1 \quad \Longrightarrow \quad \sqrt{1-v^2} = (1-u_x v) \quad \Longrightarrow \quad v = \frac{2u_x}{1+u_x^2}\,, \tag{10.115}$$

or $v = 0$, of course.

Remark: This answer makes sense. The fact that $v$ is simply the relativistic addition of $u_x$ with itself means that both your frame and the original lab frame move at speed $u_x$ (but in opposite directions) relative to the frame in which the object has no speed in the $x$-direction. By symmetry, therefore, the $y$-speed of the object must be the same in your frame and in the lab frame. ♣

18. **Relative speed**

Consider the frame, $S'$, traveling along with the point $P$ midway between the particles. $S'$ moves at speed $v\cos\theta$, so the $\gamma$ factor relating it to the lab frame is

$$\gamma = \frac{1}{\sqrt{1-v^2\cos^2\theta}}\,. \tag{10.116}$$

Let's find the vertical speeds of the particles in $S'$. Since the particles have $u'_x = 0$, the transverse velocity-addition formula, eq. (10.35), gives $v\sin\theta = u'_y/\gamma$. Therefore, each particle moves along the vertical axis away from $P$ with speed

$$u'_y = \gamma v \sin\theta. \tag{10.117}$$

The speed of one particle as viewed by the other can now be found via the velocity-addition formula,

$$V = \frac{2u'_y}{1+{u'_y}^2} = \frac{\frac{2v\sin\theta}{\sqrt{1-v^2\cos^2\theta}}}{1+\frac{v^2\sin^2\theta}{1-v^2\cos^2\theta}} = \frac{2v\sin\theta\sqrt{1-v^2\cos^2\theta}}{1-v^2\cos 2\theta}\,. \tag{10.118}$$

If desired, this can be written as

$$V = \sqrt{1-\frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2}}\,. \tag{10.119}$$

REMARK: If $2\theta = 180^\circ$, then $V = 2v/(1+v^2)$, as it should. And if $\theta = 0^\circ$, then $V = 0$, as it should. If $\theta$ is very small, then the result reduces to $V \approx 2v\sin\theta/\sqrt{1-v^2}$. You can convince yourself that this makes sense. ♣

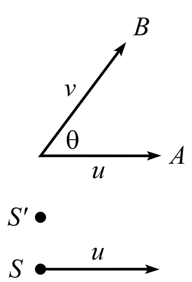


**Figure 10.53**

19. **Another relative speed**

Let the velocity **u** point in the $x$-direction, as shown in Fig. 10.53. Let $S'$ be the lab frame, and let $S$ be $A$'s frame (so frame $S'$ moves at speed $-u$ with respect to $S$). The $x$- and $y$-speeds of $B$ in frame $S'$ are $v\cos\theta$ and $v\sin\theta$. Therefore the longitudinal and transverse velocity-addition formulas, eqs. (10.28) and (10.35), give the components of $B$'s speed in $S$ as

$$\begin{aligned} V_x &= \frac{v\cos\theta - u}{1-uv\cos\theta}\,, \\ V_y &= \frac{v\sin\theta}{\gamma_u(1-uv\cos\theta)} = \frac{\sqrt{1-u^2}\;v\sin\theta}{1-uv\cos\theta}\,. \end{aligned} \tag{10.120}$$

The total speed of $B$ in frame $S$ (that is, from the point of view of $A$) is therefore

$$\begin{aligned} V &= \sqrt{V_x^2+V_y^2} \\ &= \sqrt{\left(\frac{v\cos\theta-u}{1-uv\cos\theta}\right)^2+\left(\frac{\sqrt{1-u^2}\;v\sin\theta}{1-uv\cos\theta}\right)^2} \\ &= \frac{\sqrt{u^2+v^2-2uv\cos\theta-u^2v^2\sin^2\theta}}{1-uv\cos\theta}\,. \end{aligned} \tag{10.121}$$

If desired, this can be written as

$$V = \sqrt{1-\frac{(1-u^2)(1-v^2)}{(1-uv\cos\theta)^2}}\,. \tag{10.122}$$

The reason why this can be written in such an organized form will become clear in Chapter 12.

REMARK: If $u = v$, this reduces to the result of the previous problem (if we replace $\theta$ by $2\theta$). If $\theta = 180^\circ$, then $V = (u+v)/(1+uv)$, as it should. And if $\theta = 0^\circ$, then $V = |v-u|/(1-uv)$, as it should. ♣

20. **Modified twin paradox**

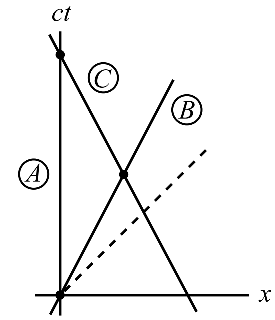

**Figure 10.54**

(a) In $A$'s reference frame, the worldlines of $A$, $B$, $C$ are shown in Fig. 10.54. $B$'s clock runs slow by a factor $1/\gamma$. Therefore, if $A$'s clock reads $t$ when $B$ meets $C$, then $B$'s clock will read $t/\gamma$ when he meets $C$. So the time he gives to $C$ is $t/\gamma$.

In $A$'s reference frame, the time between this event and the event where $C$ meets $A$ is again $t$ (since $B$ and $C$ travel at the same speed). But $A$ sees $C$'s clock run slow by a factor $1/\gamma$, so $A$ sees $C$'s clock increase by $t/\gamma$.

Therefore, when $A$ and $C$ meet, $A$'s clock reads $2t$, and $C$'s clock reads $2t/\gamma$. That is, $T_C = T_A/\gamma$.

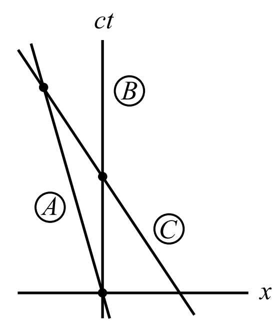

**Figure 10.55**

(b) Now let's looks at things in $B$'s frame. The worldlines of $A$, $B$, and $C$ are shown in Fig. 10.55. At first glance, you might think that $C$'s clock should read *more* than $A$'s clock when they meet (which would contradict the result in part (a)), for the following reason. From $B$'s point of view, he sees $A$'s clock run slow, so at the moment he transfers his time to $C$, he sees $C$'s clock read *more* than $A$'s. The resolution of this apparent paradox is that during the remainder of the journeys, $B$ sees $C$'s clock run much slower than $A$'s (because the relative speed of $C$ and $B$ is greater than the relative speed of $A$ and $B$). So in the end, $C$'s clock shows less time than $A$'s. Let's be quantitative about this.

Let $B$'s clock read $t_B$ when he meets $C$. Then at this time, $B$ sees $A$'s clock read $t_B/\gamma$, and he hands off the time $t_B$ to $C$.

We must now determine how much additional time elapses on $A$'s clock and $C$'s clock, by the time they meet. From the velocity-addition formula, $B$ sees $C$ flying by to the left at speed $2v/(1+v^2)$. He also sees $A$ fly by to the left at speed $v$, but $A$ had a head-start of $vt_B$ in front of $C$. Therefore, if $t$ is the time between the meeting of $B$ and $C$ and the meeting of $A$ and $C$ (as viewed from $B$), then

$$\frac{2vt}{1+v^2} = vt + vt_B. \tag{10.123}$$

This gives

$$t = t_B\left(\frac{1+v^2}{1-v^2}\right). \tag{10.124}$$

During this time, $B$ sees $A$'s and $C$'s clocks increase by $t$ divided by the relevant time-dilation factor. For $A$, this factor is $\gamma = 1/\sqrt{1-v^2/c^2}$, and for $C$ it is

$$\frac{1}{\sqrt{1-\left(\frac{2v}{1+v^2}\right)^2}} = \frac{1+v^2}{1-v^2}. \tag{10.125}$$

Therefore, the total time shown on $A$'s clock when $A$ and $C$ meet is

$$T_A = \frac{t_B}{\gamma} + t\sqrt{1-v^2}$$

$$= t_B\sqrt{1-v^2} + t_B\left(\frac{1+v^2}{1-v^2}\right)\sqrt{1-v^2}$$
$$= \frac{2t_B}{\sqrt{1-v^2}}\,. \qquad (10.126)$$

And the total time shown on $C$'s clock when $A$ and $C$ meet is

$$T_C = t_B + t\left(\frac{1-v^2}{1+v^2}\right)$$
$$= t_B + t_B\left(\frac{1+v^2}{1-v^2}\right)\left(\frac{1-v^2}{1+v^2}\right)$$
$$= 2t_B. \qquad (10.127)$$

Therefore, $T_C = T_A\sqrt{1-v^2} \equiv T_A/\gamma$.

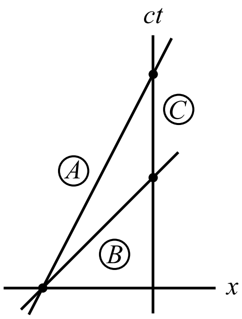


**Figure 10.56**

(c) Now let's work in $C$'s frame. The worldlines of $A$, $B$, and $C$ are shown Fig. 10.56. As in part (b), the relative speed of $B$ and $C$ is $2v/(1+v^2)$, and the time-dilation factor between $B$ and $C$ is $(1+v^2)/(1-v^2)$. Also, as is part (b), let $B$ and $C$ meet when $B$'s clock reads $t_B$. So this is the time that $B$ hands off to $C$. We'll find all relevant times below in terms of $t_B$.

$C$ sees $B$'s clock running slow, so from $C$'s point of view, $B$ travels for a time $t_B(1+v^2)/(1-v^2)$. In this time, $B$ covers a distance (from where he passed $A$) equal to

$$d = t_B\left(\frac{1+v^2}{1-v^2}\right)\frac{2v}{1+v^2} = \frac{2vt_B}{1-v^2}\,, \qquad (10.128)$$

in $C$'s frame. $A$ must of course travel this same distance (from where he met $B$) to meet up with $C$.

We can now find $T_A$. The time (as viewed by $C$) that it takes $A$ to travel the distance $d$ to reach $C$ is $d/v = 2t_B/(1-v^2)$. But since $C$ sees $A$'s clock run slow, $A$'s clock will read only

$$T_A = \frac{2t_B}{\sqrt{1-v^2}}\,. \qquad (10.129)$$

Now let's find $T_C$. To find $T_C$, we must take $t_B$ and add to it the extra time it takes $A$ to reach $C$, compared to the time it takes $B$ to reach $C$. From above, this extra time is $2t_B/(1-v^2) - t_B(1+v^2)/(1-v^2) = t_B$. Therefore, $C$'s clock reads

$$T_C = 2t_B. \qquad (10.130)$$

Hence, $T_C = T_A\sqrt{1-v^2} \equiv T_A/\gamma$.

21. **A new frame**

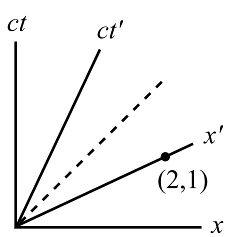


**Figure 10.57**

**First Solution:** Consider the Minkowski diagram in Fig. 10.57. Event 1 is at the origin, and Event 2 is at the point $(2, 1)$, in the frame $S$.

Consider a frame $S'$ whose $x'$ axis passes through the point $(2, 1)$. Since all points on the $x'$ axis are simultaneous in the $S'$ frame (they all have $t' = 0$), we see that $S'$ is the desired frame. From Section 10.5, the slope of the $x'$ axis is equal to $\beta \equiv v/c$. Since the slope is $1/2$, we have $v = c/2$.

(Looking at our Minkowski diagram, it is clear that if $v > c/2$, then Event 2 occurs before Event 1 in the new frame. And if $v < c/2$, then Event 2 occurs after Event 1 in the new frame.)

**Second Solution:** Let the original frame be $S$, and let the desired frame be $S'$. Let $S'$ move at speed $v$ with respect to $S$. Our goal is to find $v$.

The Lorentz transformations are

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1-v^2/c^2}}, \qquad \Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1-v^2/c^2}}. \tag{10.131}$$

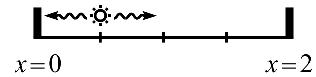

**Figure 10.58**

We want to make $\Delta t'$ equal to zero, so the second of these equations yields $\Delta t - v\Delta x/c^2 = 0$, or $v = c^2\Delta t/\Delta x$. We are given $\Delta x = 2$, and $\Delta t = 1/c$, so the desired $v$ is

$$v = c^2\Delta t/\Delta x = c/2. \tag{10.132}$$

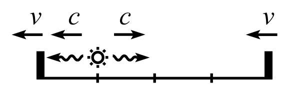

**Figure 10.59**

**Third Solution:** Consider the setup in Fig. 10.58, which explicitly constructs two such given events. Receivers are located at $x = 0$ and $x = 2$. A light source is located at $x = 1/2$. This source emits a flash of light, and when the light hits a receiver we will say an event has occurred. So the left event happens at $x = 0$, $ct = 1/2$; and the right event happens at $x = 2$, $ct = 3/2$. (We may shift our clocks by $-1/(2c)$ seconds in order to make the events happen at $ct = 0$ and $ct = 1$, but this shift will be irrelevant since all we are concerned with is differences in time.)

Now consider an observer flying by to the right at speed $v$. She sees the apparatus flying by to the left at speed $v$ (see Fig. 10.59). Our goal is to find the $v$ for which she sees the photons hit the receivers at the same time.

Consider the photons moving to the left. She sees them moving at speed $c$, but the left-hand receiver is retreating at speed $v$. So the relative speed of the photons and the left-hand receiver is $c - v$. By similar reasoning, the relative speed of the photons and the right-hand receiver is $c + v$.

The light source is three times as far from the right-hand receiver as it is from the left-hand receiver. Therefore, if the light is to reach the two receivers at the same time, we must have $c + v = 3(c - v)$. This gives $v = c/2$.

22. **Minkowski diagram units**

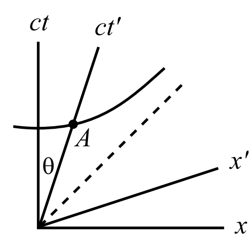

**Figure 10.60**

All points on the $ct'$-axis have the property that $x' = 0$. All points on the hyperbola have the property that $c^2t'^2 - x'^2 = 1$, due to the invariance of $s^2$. So the $ct'$ value at the intersection point, $A$, equals 1. Therefore, we simply have to determine the distance from $A$ to the origin (see Fig. 10.60).

We'll do this by finding the $(x, ct)$ coordinates of $A$. We know that $\tan\theta = \beta \equiv v/c$. Therefore, $x = \beta(ct)$ (i.e., $x = vt$). Plugging this into the given information, $c^2t^2 - x^2 = 1$, we find $ct = 1/\sqrt{1-\beta^2}$. So the distance from $A$ to the origin is $\sqrt{c^2t^2 + x^2} = ct\sqrt{1+\beta^2} = \sqrt{(1+\beta^2)/(1-\beta^2)}$. The ratio of the unit sizes on the $ct'$ and $ct$ axes is therefore

$$\sqrt{\frac{1+\beta^2}{1-\beta^2}}, \tag{10.133}$$

which agrees with eq. (10.43).

(Exactly the same analysis holds for the $x$-axis unit size ratio, of course.)

23. **Velocity Addition via Minkowski**

Pick a point $P$ on the object's worldline. Let the coordinates of $P$ in frame $S$ be $(x, ct)$. Our goal is to find the speed $u = x/t$. Throughout this problem, it will be easier to work with the quantities $\beta \equiv v/c$; so our goal is to find $\beta_u \equiv x/(ct)$.

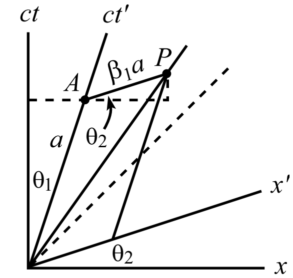

**Figure 10.61**

The coordinates of $P$ in $S'$, namely $(x', ct')$, are shown in Fig. 10.61. For convenience, let $ct'$ have length $a$ on the paper. Then from the given information, we have $x' = v_1t' \equiv \beta_1(ct') = \beta_1 a$. In terms of $a$, we can easily determine the coordinates $(x, ct)$ of $P$. The coordinates of point $A$ (shown in the figure) are simply

$$(x, ct)_A = (a\sin\theta_2, a\cos\theta_2). \tag{10.134}$$

The coordinates of $P$, relative to $A$, are

$$(x, ct)_{P-A} = (\beta_1 a\cos\theta_2, \beta_1 a\sin\theta_2). \tag{10.135}$$

So the coordinates of point $P$ are

$$(x, ct)_P = (a\sin\theta_2 + \beta_1 a\cos\theta_2, a\cos\theta_2 + \beta_1 a\sin\theta_2). \tag{10.136}$$

The ratio of $x$ to $ct$ at the point $P$ is therefore

$$\begin{aligned}\beta_u \equiv \frac{x}{ct} &= \frac{\sin\theta_2 + \beta_1\cos\theta_2}{\cos\theta_2 + \beta_1\sin\theta_2} \\ &= \frac{\tan\theta_2 + \beta_1}{1 + \beta_1\tan\theta_2} \\ &= \frac{\beta_2 + \beta_1}{1 + \beta_1\beta_2},\end{aligned} \tag{10.137}$$

where we have used $\tan\theta_2 = v_2/c \equiv \beta_2$, because $S'$ moves at speed $v_2$ with respect to $S$. If we change from the $\beta$'s back to the $v$'s, the result is $u = (v_2 + v_1)/(1 + v_1v_2/c^2)$.

24. **Acceleration and redshift**

There are various ways to do this problem (for example, by sending photons between the people, or by invoking the gravitational equivalence principle in GR, etc.). We'll do it here by using a Minkowski diagram, to demonstrate that it can be solved perfectly fine using only basic Special Relativity.

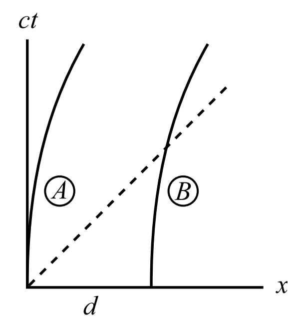

**Figure 10.62**

Draw the world lines of the two people, $A$ and $B$, as seen by an observer, $C$, in the frame where they were both initially at rest. We have the situation shown in Fig. 10.62.

Consider an infinitesimal time $\Delta t$, as measured by $C$. At this time (in $C$'s frame), $A$ and $B$ are both moving at speed $a\Delta t$. The axes of the $A$ frame are shown in Fig. 10.63. Both $A$ and $B$ have moved a distance $a(\Delta t)^2/2$, which can be neglected since $\Delta t$ is small.[34] Also, the special-relativity time-dilation factor between any of the $A$,$B$,$C$ frames can be neglected. (Any relative speeds are no greater than $v = a\Delta t$, so the time-dilation factors differ from 1 by at most order $(\Delta t)^2$.) Let $A$ make a little explosion, $E_1$, at this time. Then $\Delta t$ (which was defined to be the time as measured by $C$) is also the time of the explosion, as measured by $A$ (up to an error of order $(\Delta t)^2$).

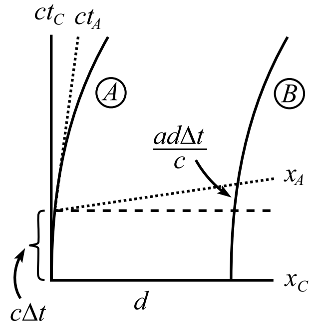

**Figure 10.63**

Let's figure out where $A$'s $x$-axis (i.e., the 'now' axis in $A$'s frame) meets $B$'s worldline. The slope of $A$'s $x$-axis in the figure is $v/c = a\Delta t/c$. So the axis starts at a height $c\Delta t$, and then climbs up by the amount $ad\Delta t/c$, over the distance $d$. Therefore, the axis meets $B$'s worldline at a height $c\Delta t + ad\Delta t/c$, as viewed by $C$; i.e, at the time $\Delta t + ad\Delta t/c^2$, as viewed by $C$. But $C$'s time is the same as $B$'s time (up to order

[34] It will turn out that the leading-order terms in the result below are of order $\Delta t$. Any $(\Delta t)^2$ terms can therefore be ignored.

$(\Delta t)^2$). So $B$'s clock reads $\Delta t(1 + ad/c^2)$. Let's say that $B$ makes a little explosion, $E_2$, at this time.

$A$ sees both $E_1$ and $E_2$ occur at the same time (they both lie along a line of constant time in $A$'s frame). In other words, $A$ sees $B$'s clock read $\Delta t(1+ad/c^2)$ when he sees his own clock read $\Delta t$. Therefore, $A$ sees $B$'s clock sped up by a factor

$$\frac{\Delta t_B}{\Delta t_A} = 1 + \frac{ad}{c^2}. \tag{10.138}$$

We can perform the same procedure to see how $B$ views $A$'s clocks. Drawing $B$'s $x$-axis at time $\Delta t$, we easily find that $B$ sees $A$'s clock slowed down by a factor

$$\frac{\Delta t_A}{\Delta t_B} = 1 - \frac{ad}{c^2}. \tag{10.139}$$

Remark: In the usual special-relativity situation where two observers fly past each other with relative speed $v$, they *both* see the other person's time slowed down by the same factor. This had better be the case, since the situation is symmetric between the observers. But in this problem, $A$ sees $B$'s clock sped up, and $B$ sees $A$'s clock slowed down. This difference is possible because the situation is *not* symmetric between $A$ and $B$. The acceleration vector determines a direction in space, and one person (namely $B$) is further along this direction than the other person ($A$). ♣

25. **Break or not break?**

There are two possible reasonings.

(1) To an observer in the original rest frame, the spaceships stay the same distance, $d$, apart. Therefore, in the frame of the spaceships, the distance between them, $d'$, must be greater than $d$. This is the case because $d$ equals $d'/\gamma$, by the usual length contraction. After a long enough time, $\gamma$ will differ appreciably from 1, and the string will be stretched by a large factor. Therefore, it *will* break.

(2) Let $A$ be the back spaceship, and let $B$ be the front spaceship. From the point of view of $A$ ($B$'s point of view would work just as well), it looks like $B$ is doing exactly what $A$ is doing. It looks like $B$ undergoes the same acceleration as $A$, so $B$ should stay the same distance ahead of $A$. Therefore, the string should *not* break.

The second reasoning is incorrect. The first reasoning is (mostly) correct. The trouble with the second reasoning is that the two spaceships are in different frames. $A$ in fact sees $B$'s clock sped up, and $B$ sees $A$'s clock slowed down (from Problem 24). $A$ sees $B$'s engine working faster, and $B$ therefore pulls away from $A$. So the string eventually breaks.

The first reasoning is mostly correct. The only trouble with it is that there is no one "frame of the spaceships". Their frames differ. It is not clear exactly what is meant by the "length of the string", because it is not clear what frame the measurement should take place in.

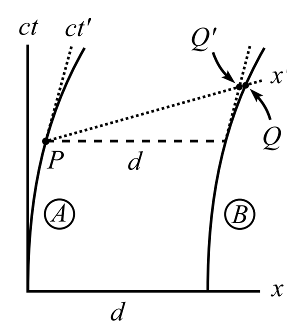


**Figure 10.64**

Everything becomes more clear once we draw a Minkowski diagram. Fig. 10.64 shows the $x'$ and $ct'$ axes of $A$'s frame. The $x'$-axis is tilted up, so it meets $B$'s worldline further to the right than one might think. The distance $PQ$ along the $x'$-axis is the distance that $A$ measures the string to be. Although it is not obvious that this distance in $A$'s frame is larger than $d$ (because the unit size on the $x'$ axis is larger than that in $C$'s frame), we can easily demonstrate this. In $A$'s frame, the distance $PQ$ is greater than the distance $PQ'$. But $PQ'$ is simply the length of something in $A$'s frame which has length $d$ in $C$'s frame. So $PQ'$ is $\gamma d$ in $A$'s frame. Since $PQ > \gamma d > d$ in $A$'s frame, the string breaks.

REMARKS:

(a) If you want there to eventually be a well-defined "frame of the spaceships", you can modify the problem by stating that after a while, the spaceships both stop accelerating simultaneously (as measured by $C$). Equivalently, both $A$ and $B$ turn off their engines after equal proper times.

What $A$ sees is the following. $B$ pulls away from $A$. Then $B$ turns off his engine. The gap continues to widen. But $A$ continues firing his engine until be reaches $B$'s speed. Then they sail onward, in a common frame, keeping a constant separation (which is greater than the original separation.)

(b) The main issue in this problem is that it depends exactly how you choose to accelerate an extended object. If you accelerate a stick by pushing on the back end (or by pulling on the front end), the length will remain essentially the same in its own frame, and it will become shorter in the original frame. But if you arrange for each end (or perhaps a number of points on the stick) to speed up in such a way that they always move at the same speed with respect to the original frame, then the stick will get torn apart. ♣

26. **Successive Lorentz transformations**

It is not necessary, of course, to use matrices in this problem; but things look nicer if you do. The desired composite L.T. is obtained by multiplying the matrices for the individual L.T.'s. So we have

$$\begin{aligned} L &= \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 & \sinh\phi_1\cosh\phi_2 + \cosh\phi_1\sinh\phi_2 \\ \cosh\phi_1\sinh\phi_2 + \sinh\phi_1\cosh\phi_2 & \sinh\phi_1\sinh\phi_2 + \cosh\phi_1\cosh\phi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}. \end{aligned} \tag{10.140}$$

This is the L.T. with $v = \tanh(\phi_1 + \phi_2)$, as desired. This proof is just like the one for successive rotations in the plane (except for a few minus signs).

27. **Accelerator's time**

Eq. (10.62) gives the speed as a function of the spaceship's time,

$$\beta(t') \equiv \frac{v(t')}{c} = \tanh(at'/c). \tag{10.141}$$

The person in the lab sees the spaceship's clock slowed down by a factor $1/\gamma = \sqrt{1-\beta^2}$, i.e., $dt = dt'/\sqrt{1-\beta^2}$. So we have

$$\begin{aligned} t = \int_0^t dt &= \int_0^{t'} \frac{dt'}{\sqrt{1-\beta(t')^2}} \\ &= \int_0^{t'} \cosh(at'/c)\,dt' \\ &= \frac{c}{a}\sinh(at'/c). \end{aligned} \tag{10.142}$$

Note that for small $a$ or $t'$ (more precisely, if $at'/c \ll 1$), we obtain $t \approx t'$, as we should. For very large times, we essentially have

$$t \approx \frac{c}{2a} e^{at'/c}, \qquad \text{or} \qquad t' = \frac{c}{a} \ln(2at/c). \tag{10.143}$$

The lab frame will see the astronaut read all of "Moby Dick", but it will take an exponentially long time.

# Chapter 11

# Relativity (Dynamics)



In the previous chapter, we dealt only with abstract particles flying through space and time. We didn't concern ourselves with the nature of the particles, how they got to be moving the way they were moving, or what would happen if various particles interacted. In this chapter we will deal with these issues. That is, we will discuss masses, forces, energy, momentum, etc.

The two main results of this chapter are that the momentum and energy of a particle are given by

$$\mathbf{p} = \gamma m \mathbf{v}, \qquad \text{and} \qquad E = \gamma m c^2, \tag{11.1}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and $m$ is the mass of the particle.[1] When $v \ll c$, the expression for $\mathbf{p}$ reduces to $\mathbf{p} = m\mathbf{v}$, as it should for a non-relativistic particle. When $v = 0$, the expression for $E$ reduces to the well-known $E = mc^2$.

## 11.1 Energy and momentum

In this section, we'll give some justification for eqs. (11.1). The reasoning here should convince you of their truth. An alternative, and perhaps more convincing, motivation comes from the 4-vector formalism in Chapter 12. In the end, however, the justification for eqs. (11.1) is obtained through experiments. Every day, experiments in high-energy accelerators are verifying the truth of these expressions. (More precisely, they are verifying that these energy and momenta are *conserved* in any type of collision.) We therefore conclude, with reasonable certainty, that eqs. (11.1) are the correct expressions for energy and momentum.

But actual experiments aside, let's consider a few thought-experiments that motivate the above expressions.

[1] People use the word "mass" in different ways in relativity. They talk about "rest mass" and "relativistic mass". These terms, however, are misleading. There is only one thing that can reasonably be called "mass" in relativity. It is the same thing that we call "mass" in Newtonian physics, and it is what some people would call "rest mass", although the qualifier "rest" is redundant. See Section 11.8 for a discussion of this issue.

### 11.1.1 Momentum

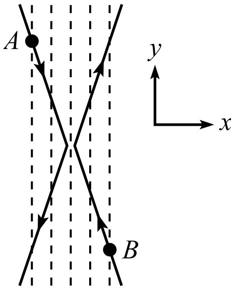


**Figure 11.1**

Consider the following system. In the lab frame, identical particles $A$ and $B$ move as shown in Fig. 11.1. They move with equal and opposite small speeds in the $x$-direction, and with equal and opposite large speeds in the $y$-direction. Their paths are arranged so that they glance off each other and reverse their motion in the $x$-direction.

For clarity, imagine a series of equally spaced vertical lines for reference. Assume that both $A$ and $B$ have identical clocks that tick every time they cross one of the lines.

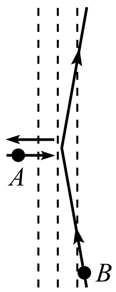


**Figure 11.2**

Consider now the reference frame that moves in the $y$-direction, with the same $v_y$ as $A$. In this frame, the situation looks like Fig. 11.2. The collision simply changes the sign of the $x$-velocities of the particles. Therefore, the $x$-momenta of the two particles must be the same.[2]

However, the $x$-speeds of the two particles are *not* the same in this frame. $A$ is essentially at rest in this frame, and $B$ is moving with a very large speed, $v$. Therefore, $B$'s clock is running slower than $A$'s, by a factor essentially equal to $1/\gamma \equiv \sqrt{1-v^2/c^2}$. And since $B$'s clock ticks once for every vertical line it crosses (this fact is independent of the frame), $B$ must therefore be moving slower in the $x$-direction, by a factor of $1/\gamma$.

Therefore, the Newtonian expression, $p_x = mv_x$, cannot be the correct one for momentum, because $B$'s momentum would be smaller than $A$'s (by a factor of $1/\gamma$), due to their different $v_x$'s. But the $\gamma$ factor in

$$p_x = \gamma m v_x \equiv \frac{mv_x}{\sqrt{1-v^2/c^2}} \tag{11.2}$$

precisely takes care of this problem, because $\gamma \approx 1$ for $A$, and $\gamma = 1/\sqrt{1-v^2/c^2}$ for $B$, which precisely cancels the effect of $B$'s smaller $v_x$.

To obtain the three-dimensional form for $\mathbf{p}$, we now note that the vector $\mathbf{p}$ must point in the same direction as the vector $\mathbf{v}$ points.[3] Therefore, eq. (11.2) implies that the momentum vector must be

$$\mathbf{p} = \gamma m \mathbf{v} \equiv \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\,, \tag{11.3}$$

in agreement with eq. (11.1). Note that that all the components of $\mathbf{p}$ have the same denominator, which involves the whole speed, $v^2 = v_x^2 + v_y^2 + v_z^2$. The denominator of, say, $p_x$, is *not* $\sqrt{1-v_x^2/c^2}$.

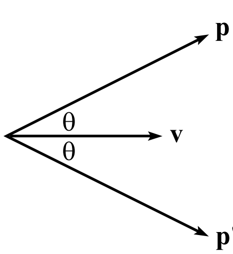


**Figure 11.3**

Remark: The above setup is only one specific type of collision, among an infinite number of possible types of collisions. What we've shown with this setup is that the only

[2] This is true because if, say, $A$'s $p_x$ were larger than $B$'s $p_x$, then the total $p_x$ would point to the right before the collision, and to the left after the collision. Since momentum is something we want to be conserved, this cannot be the case.

[3] This is true because any other direction for $\mathbf{p}$ would violate rotation invariance. If someone claims that $\mathbf{p}$ points in the direction shown in Fig. 11.3, then he would be hard-pressed to explain why it doesn't instead point along the direction $\mathbf{p}'$ shown. In short, the direction of $\mathbf{v}$ is the only preferred direction in space.

possible vector of the form $f(v)m\mathbf{v}$ (where $f$ is some function) that has any chance at being conserved in all collisions is $\gamma m\mathbf{v}$ (or some constant multiple of this). We haven't proved that it actually *is* conserved in all collisions. This is where the gathering of data from experiments comes in. But we've shown above that it would be a waste of time to consider, for example, the vector $\gamma^5 m\mathbf{v}$. ♣

### 11.1.2 Energy

Having given some justification for the momentum expression, $\mathbf{p} = \gamma m\mathbf{v}$, let us now try to justify the energy expression,

$$E = \gamma mc^2. \tag{11.4}$$

More precisely, we will show that $\gamma mc^2$ is *conserved* in interactions (or at least in the specific interaction below). There are various ways to do this. The best way, perhaps, is to use the 4-vector formalism in Chapter 12. But we'll study one simple setup here that should do the job.

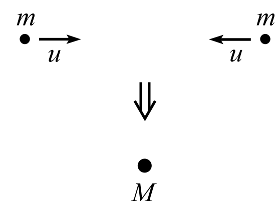


**Figure 11.4**

Consider the following system. Two identical particles of mass $m$ head toward each other, both with speed $u$, as shown in Fig. 11.4. They stick together and form a particle of mass $M$. $M$ is at rest, due to the symmetry of the situation. At the moment we cannot assume anything about the size of $M$. We will find below that it does *not* equal the naive value of $2m$.

This is a fairly uninteresting setup (conservation of momentum gives $0 = 0$), but now consider the less trivial view from a frame moving to the left at speed $u$. This situation is shown in Fig. 11.5. The right mass is at rest, $M$ moves to the right at speed $u$, and the left mass moves to the right at speed $v = 2u/(1+u^2)$, from the velocity addition formula.[4] Note that the $\gamma$-factor associated with this speed $v$ is

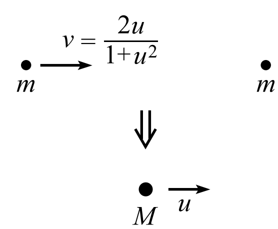


**Figure 11.5**

$$\gamma_v \equiv \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2u}{1+u^2}\right)^2}} = \frac{1+u^2}{1-u^2}\,. \tag{11.5}$$

Conservation of momentum in this collision then gives

$$\begin{aligned}
\gamma_v mv + 0 &= \gamma_u Mu \\
\Longrightarrow \quad m\left(\frac{1+u^2}{1-u^2}\right)\left(\frac{2u}{1+u^2}\right) &= \frac{Mu}{\sqrt{1-u^2}} \\
\Longrightarrow \quad M = \frac{2m}{\sqrt{1-u^2}}\,. & \qquad (11.6)
\end{aligned}$$

Conservation of momentum therefore tells us that $M$ does *not* equal $2m$. But if $u$ is very small, then $M$ is approximately equal to $2m$, as we know from everyday experience.

Using the value of $M$ from eq. (11.6), let's now check that our candidate for energy, $E = \gamma mc^2$, is conserved in this collision. There is no freedom left in any of

[4] We're going to set $c = 1$ for a little while here, because this calculation would get a bit messy if we kept in the $c$'s. We'll discuss the issue of setting $c = 1$ in more detail later in this section.

the parameters, so $\gamma mc^2$ is either conserved or it isn't. In the original frame where $M$ is at rest, $E$ is conserved if

$$\gamma_0 Mc^2 = 2(\gamma_u mc^2) \qquad \Longleftrightarrow \qquad \frac{2m}{\sqrt{1-u^2}} = 2\left(\frac{1}{\sqrt{1-u^2}}\right)m\,, \tag{11.7}$$

which is indeed true.

Let's also check that $E$ is conserved in the frame where the right mass is at rest. $E$ is conserved if

$$\begin{aligned} \gamma_v mc^2 + \gamma_0 mc^2 &= \gamma_u Mc^2, \qquad \text{or} \\ \left(\frac{1+u^2}{1-u^2}\right)m + m &= \frac{M}{\sqrt{1-u^2}}\,, \qquad \text{or} \\ \frac{2m}{1-u^2} &= \left(\frac{2m}{\sqrt{1-u^2}}\right)\frac{1}{\sqrt{1-u^2}}\,, \end{aligned} \tag{11.8}$$

which is indeed true. So $E$ is also conserved in this frame.

Hopefully at this point you're convinced that $\gamma mc^2$ is a believable expression for the energy of a particle. But just as in the case of momentum, we haven't proved that $\gamma mc^2$ actually *is* conserved in all collisions. This is the duty of experiments. But we've shown that it would be a waste of time to consider, for example, the quantity $\gamma^4 mc^2$.

One thing that we certainly need to check is that if $E$ and $p$ are conserved in one reference frame, then they are conserved in any other. We'll demonstrate this in Section 11.2. A conservation law shouldn't depend on what frame you're in, after all.

REMARKS:

1. To be precise, we should say that technically we're not trying to justify eqs. (11.1) here. These two equations by themselves are devoid of any meaning. All they do is define the letters $\mathbf{p}$ and $E$. Our goal is to make a meaningful physical statement, not just a definition.

   The meaningful physical statement we want to make is that the quantities $\gamma m\mathbf{v}$ and $\gamma mc^2$ are *conserved* in an interaction among particles (and this is what we tried to justify above). This fact then makes these quantities worthy of special attention, because conserved quantities are very helpful in understanding what is happening in a given physical situation. And anything worthy of special attention certainly deserves a label, so we may then attach the names "momentum" and "energy" to $\gamma m\mathbf{v}$ and $\gamma mc^2$. Any other names would work just as well, of course, but we choose these because in the limit of small speeds, $\gamma m\mathbf{v}$ and $\gamma mc^2$ reduce (as we will soon show) to some other nicely conserved quantities, which someone already tagged with the labels "momentum" and "energy" long ago.

2. As mentioned above, the fact of the matter is that we can't *prove* that $\gamma m\mathbf{v}$ and $\gamma mc^2$ are conserved. In Newtonian physics, conservation of $\mathbf{p} \equiv m\mathbf{v}$ is basically postulated by Newton's third law, and we're not going to be able to do any better than that here. All we can hope to do as physicists is provide some motivation for considering $\gamma m\mathbf{v}$ and $\gamma mc^2$, show that it is consistent for $\gamma m\mathbf{v}$ and $\gamma mc^2$ to be conserved during an interaction, and gather a large amount of experimental evidence, all of which is

consistent with $\gamma m\mathbf{v}$ and $\gamma mc^2$ being conserved. That's how physics works. You can't prove anything. So you learn to settle for the things you can't disprove.

Consider, when seeking gestalts,
The theories that physics exalts.
It's not that they're known
To be written in stone.
It's just that we can't say they're false.

As far as the experimental evidence goes, suffice it to say that high-energy accelerators, cosmological observations, and many other forums are continually verifying everything that we think is true about relativistic dynamics. If the theory is not correct, then we know that it must be the limiting theory of a more complete one (just as Newtonian physics is a limiting theory of relativity). But all this experimental induction has to count for something...

"To three, five, and seven, assign
A name," the prof said, "We'll define."
But he botched the instruction
With woeful induction
And told us the next prime was nine.

3. Conservation of energy in relativistic mechanics is actually a much simpler concept than it is in nonrelativistic mechanics, because $E = \gamma m$ is conserved, period. We don't have to worry about the generation of heat, which ruins conservation of the nonrelativistic $E = mv^2/2$. The heat is simply built into the energy. In the example above, the two $m$'s collide and generate heat in the resulting mass $M$. This heat shows up as an increase in mass, which makes $M$ larger than $2m$. The energy that corresponds to the increase in mass is due to the initial kinetic energy of the two $m$'s.

4. Problem 1 gives an alternate derivation of the energy and momentum expressions in eq. (11.1). This derivation uses additional facts, namely that the energy and momentum of a photon are given by $E = h\nu$ and $p = h\nu/c$, where $\nu$ is the frequency of the light wave, and $h$ is Planck's constant. ♣

Any multiple of $\gamma mc^2$ is also conserved, of course. Why did we pick $\gamma mc^2$ to label as "$E$" instead of, say, $5\gamma mc^3$? Consider the approximate form $\gamma mc^2$ takes in the Newtonian limit, that is, in the limit $v \ll c$. We have, using the Taylor series expansion for $(1-x)^{-1/2}$,

$$\begin{aligned} E \equiv \gamma mc^2 &= \frac{mc^2}{\sqrt{1-v^2/c^2}} \\ &= mc^2\left(1+\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}+\cdots\right) \\ &= mc^2+\frac{1}{2}mv^2+\cdots. \end{aligned} \tag{11.9}$$

The dots represent higher-order terms in $v^2/c^2$, which may be neglected if $v \ll c$. In an elastic collision in Newtonian physics, no heat is generated, so mass is conserved. That is, the quantity $mc^2$ has a fixed value. We therefore see that conservation of

$E \equiv \gamma mc^2$ reduces to the familiar conservation of Newtonian kinetic energy, $mv^2/2$, for elastic collisions in the limit of slow speeds.

Likewise, we picked $\mathbf{p} \equiv \gamma m\mathbf{v}$, instead of, say, $6\gamma mc^4\mathbf{v}$, because the former reduces to the familiar Newtonian momentum, $m\mathbf{v}$, in the limit of slow speeds.

> Whether abstract, profound, or just mystic,
> Or boring, or somewhat simplistic,
> A theory must lead
> To results that we need
> In limits, nonrelativistic.

Whenever we use the term "energy", we will mean the total energy, $\gamma mc^2$. If we use the term "kinetic energy", we will mean a particle's excess energy over its energy when it is motionless, that is, $\gamma mc^2 - mc^2$. Note that kinetic energy is *not* necessarily conserved in a collision, because mass is not necessarily conserved, as we saw in eq. (11.6) in the above scenario, where $M = 2m/\sqrt{1-u^2}$. In the CM frame, there was kinetic energy before the collision, but none after. Kinetic energy is a rather artificial concept in relativity. You virtually always want to use the total energy, $\gamma mc^2$, when solving a problem.

Note the following extremely important relation,

$$\begin{aligned} E^2 - |\mathbf{p}|^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 |\mathbf{v}|^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m^2 c^4. \end{aligned} \tag{11.10}$$

This is a primary ingredient in solving relativistic collision problems, as we will soon see. It replaces the $\text{KE} = p^2/2m$ relation between kinetic energy and momentum in Newtonian physics. It can be derived in more profound ways, as we will see in Chapter 12. Let's put it in a box, since it's so important,

$$\boxed{E^2 = p^2c^2 + m^2c^4}. \tag{11.11}$$

In the case where $m = 0$ (as with photons), eq. (11.11) says that $E = pc$. This is the key equation for massless objects. For photons, the two equations, $\mathbf{p} = \gamma m\mathbf{v}$ and $E = \gamma mc^2$, don't tell us much, because $m = 0$ and $\gamma = \infty$, so their product is undetermined. But $E^2 - |\mathbf{p}|^2c^2 = m^2c^4$ still holds, and we conclude that $E = pc$.

Note that any massless particle must have $\gamma = \infty$. That is, it must travel at speed $c$. If this weren't the case, then $E = \gamma mc^2$ would equal zero, in which case the particle isn't much of a particle. We'd have a hard time observing something with no energy.

Another nice relation, which holds for particles of any mass, is

$$\frac{\mathbf{p}}{E} = \frac{\mathbf{v}}{c^2}\,. \tag{11.12}$$

Given $p$ and $E$, this is definitely the quickest way to get $v$.

**Setting $c = 1$**

For the remainder of our treatment of relativity, we will invariably work in units where $c = 1$. For example, instead of one meter being the unit of distance, we can make $3 \cdot 10^8$ meters equal to one unit. Or, we can keep the meter as is, and make $1/(3 \cdot 10^8)$ seconds the unit of time. In such units, our various expressions become

$$\mathbf{p} = \gamma m \mathbf{v}, \qquad E = \gamma m, \qquad E^2 = p^2 + m^2, \qquad \frac{\mathbf{p}}{E} = \mathbf{v}. \tag{11.13}$$

Said in another way, you can simply ignore all the $c$'s in your calculations (which will generally save you a lot of strife), and then put them back into your final answer to make the units correct. For example, let's say the goal of a certain problem is to find the time of some event. If your answer comes out to be $\ell$, where $\ell$ is a given length, then you know that the correct answer (in terms of the usual mks units) has to be $\ell/c$, because this has units of time. In order for this procedure to work, there must be only one way to put the $c$'s back in at the end. This is always the case, because if there were two ways, then we would have $c^a = c^b$, for some numbers $a \neq b$. But this is impossible, because $c$ has units.

**The general size of $mc^2$**

What is the general size of $mc^2$? If we let $m = 1\,\text{kg}$, then we have $mc^2 = (1\,\text{kg})(3 \cdot 10^8\,\text{m/s})^2 \approx 10^{17}\,\text{J}$. How big is this? A typical household electric bill might around \$50 per month, or \$600 per year. At about 10 cents per kilowatt-hour, this translates to 6000 kilowatt-hours per year. Since there are 3600 seconds in an hour, this converts to $(6000)(10^3)(3600) \approx 2 \cdot 10^{10}$ watt-seconds. That is, $2 \cdot 10^{10}$ Joules per year. We therefore see that if one kilogram were converted completely into usable energy, it would be enough to provide electricity to $10^{17}/(2 \cdot 10^{10})$, or 5 million, homes for a year. That's a lot.

In a nuclear reactor, only a small fraction of the mass energy is converted into usable energy. Most of the mass remains in the final products, which doesn't help in lighting up your home. If a particle were to combine with its antiparticle, then it would be possible for all of the mass energy to be converted into usable energy. But we're still a few years away from this.

However, even a small fraction of the very large quantity, $E = mc^2$, can still be large, as evidenced by the use of nuclear power and nuclear weapons. Any quantity with a few factors of $c$ is bound to change the face of the world.

## 11.2 Transformations of $E$ and $\vec{p}$

Consider the following one-dimensional situation, where all the motion is along the $x$-axis. A particle has energy $E'$ and momentum $p'$ in frame $S'$. Frame $S'$ moves at speed $v$ with respect to frame $S$, in the positive $x$-direction (see Fig. 11.6). What are $E$ and $p$ in $S$?

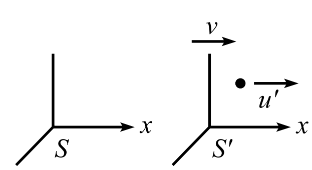


**Figure 11.6**

Let $u'$ be the particle's speed in $S'$. From the velocity addition formula, the particle's speed in $S$ is (dropping the factors of $c$)

$$u = \frac{u' + v}{1 + u'v} \,. \tag{11.14}$$

This is all we need to know, because a particle's velocity determines its energy and momentum. But we'll need to go through a little algebra to make things look pretty. The $\gamma$-factor associated with the speed $u$ is

$$\gamma_u = \frac{1}{\sqrt{1 - \left(\frac{u'+v}{1+u'v}\right)^2}} = \frac{1 + u'v}{\sqrt{(1 - u'^2)(1 - v^2)}} \equiv \gamma_{u'}\gamma_v(1 + u'v). \tag{11.15}$$

The energy and momentum in $S'$ are

$$E' = \gamma_{u'} m, \qquad \text{and} \qquad p' = \gamma_{u'} m u', \tag{11.16}$$

while the energy and momentum in $S$ are, using eq. (11.15),

$$\begin{aligned} E &= \gamma_u m = \gamma_{u'}\gamma_v(1 + u'v)m, \\ p &= \gamma_u m u = \gamma_{u'}\gamma_v(1 + u'v)m\left(\frac{u' + v}{1 + u'v}\right) = \gamma_{u'}\gamma_v(u' + v)m. \end{aligned} \tag{11.17}$$

Using the $E'$ and $p'$ from eq. (11.16), we can rewrite $E$ and $p$ as (with $\gamma \equiv \gamma_v$)

$$\begin{aligned} E &= \gamma(E' + vp'), \\ p &= \gamma(p' + vE'). \end{aligned} \tag{11.18}$$

These are transformations for $E$ and $p$ between frames. If you want to put the factors of $c$ back in, then the $vE'$ term becomes $vE'/c^2$. These equations are easy to remember, because they look *exactly* like the Lorentz transformations for the coordinates $t$ and $x$ in eq. (10.17). This is no coincidence, as we will see in Chapter 12.

Remark: We can perform a few checks on eqs. (11.18). If $u' = 0$ (so that $p' = 0$ and $E' = m$), then $E = \gamma m$ and $p = \gamma m v$, as they should. Also, if $u' = -v$ (so that $p' = -\gamma m v$ and $E' = \gamma m$), then $E = m$ and $p = 0$, as they should. ♣

Note that since the transformations in eq. (11.18) are linear, they also hold if $E$ and $p$ represent the total energy and momentum of a collection of particles. That is,

$$\begin{aligned} \sum E &= \gamma\left(\sum E' + v\sum p'\right), \\ \sum p &= \gamma\left(\sum p' + v\sum E'\right). \end{aligned} \tag{11.19}$$

Indeed, any (corresponding) linear combinations of the energies and momenta are valid here, in place of the sums. For example, we can use the combinations $(E_1^b +$

$3E_2^a - 7E_5^b$) and $(p_1^b + 3p_2^a - 7p_5^b)$ in eq. (11.18), where the subscripts indicate which particle, and the superscripts indicate before or after a collision. You can verify this by simply taking the appropriate linear combination of eqs. (11.18) for the various particles. This consequence of linearity is a very important and useful result, as will become clear in the remarks below.

You can use eq. (11.18) to show that

$$E^2 - p^2 = E'^2 - p'^2, \tag{11.20}$$

just as we did to obtain the $t^2 - x^2 = t'^2 - x'^2$ result in eq. (10.37). The $E$'s and $p$'s here can represent any (corresponding) linear combinations of the $E$'s and $p$'s of the various particles, due to the linearity of eq. (11.18). For one particle, we already know that eq. (11.20) is true, because both sides are equal to $m^2$, from eq. (11.10). For many particles, the invariant $E_{\text{total}}^2 - p_{\text{total}}^2$ is equal to the square of the total energy in the CM frame (which reduces to $m^2$ for one particle), because $p_{\text{total}} = 0$ in the CM frame, by definition.

Remarks:

1. In the previous section, we said that we needed to show that if $E$ and $p$ are conserved in one reference frame, then they are conserved in any other frame (because a conservation law shouldn't depend on what frame you're in). Eq. (11.18) quickly gives us this result, because the $E$ and $p$ in one frame are linear functions of the $E'$ and $p'$ in another frame. If the total $\Delta E'$ and $\Delta p'$ in $S'$ are zero, then eq. (11.18) says that the total $\Delta E$ and $\Delta p$ in $S$ must also be zero. We have used the fact that $\Delta E$ is a linear combination of the $E$'s, and that $\Delta p$ is a linear combination of the $p$'s, so eq. (11.18) applies to these linear combinations.

2. Eq. (11.18) makes it clear that if you accept the fact that $p = \gamma m v$ is conserved in all frames, then you must also accept the fact that $E = \gamma m$ is conserved in all frames (and vice versa). This is true because the second of eqs. (11.18) says that if $\Delta p$ and $\Delta p'$ are both zero, then $\Delta E'$ must also be zero (again, we have used linearity). $E$ and $p$ have no choice but to go hand in hand. ♣

Eq. (11.18) applies to the $x$-component of the momentum. How do the transverse components, $p_y$ and $p_z$, transform? Just as with the $y$ and $z$ coordinates in the Lorentz transformations, $p_y$ and $p_z$ do not change between frames. The analysis in Chapter 12 makes this obvious, so for now we'll simply state that

$$\begin{aligned} p_y &= p'_y, \\ p_z &= p'_z, \end{aligned} \tag{11.21}$$

if the relative velocity between the frames is in the $x$-direction. If you really want to show explicitly that the transverse components do not change between frames, or if you are worried that a nonzero speed in the $y$ direction will mess up the relationship between $p_x$ and $E$ that we calculated in eq. (11.18), then Exercise 4 is for you. But it's a bit tedious, so feel free to settle for the much cleaner reasoning in Chapter 12.

## 11.3 Collisions and decays

The strategy for studying relativistic collisions is the same as that for studying nonrelativistic ones. You simply have to write down all the conservation of energy and momentum equations, and then solve for whatever variables you want to solve for. The conservation principles are the same as they've always been. It's just that now the energy and momentum take the new forms in eq. (11.1).

In writing down the conservation of energy and momentum equations, it proves extremely useful to put $E$ and $\mathbf{p}$ together into one four-component vector,

$$P \equiv (E, \mathbf{p}) \equiv (E, p_x, p_y, p_z). \tag{11.22}$$

This is called the *energy-momentum 4-vector*, or the *4-momentum*, for short.[5] Our notation in this chapter will be to use an uppercase $P$ to denote a 4-momentum and a lowercase $\mathbf{p}$ or $p$ to denote a spatial momentum. The components of a 4-momentum are generally indexed from 0 to 3, so that $P_0 \equiv E$, and $(P_1, P_2, P_3) \equiv \mathbf{p}$. For one particle, we have

$$P = (\gamma m, \gamma m v_x, \gamma m v_y, \gamma m v_z). \tag{11.23}$$

The 4-momentum for a collection of particles simply consists of the total $E$ and total $\mathbf{p}$ of all the particles.

There are deep reasons for considering this four-component vector (as we will see in Chapter 12), but for now we will view it as simply a matter of convenience. If nothing else, it helps with the bookkeeping. Conservation of energy and momentum in a collision reduce to the concise statement,

$$P_{\text{before}} = P_{\text{after}}, \tag{11.24}$$

where these are the total 4-momenta of all the particles.

If we define the *inner product* between two 4-momenta, $A \equiv (A_0, A_1, A_2, A_3)$ and $B \equiv (B_0, B_1, B_2, B_3)$, to be

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3, \tag{11.25}$$

then the relation $E^2 - p^2 = m^2$ (which is true for one particle) may be concisely written as

$$P^2 \equiv P \cdot P = m^2. \tag{11.26}$$

In other words, the square of a particle's 4-momentum equals the square of its mass. This relation will prove to be very useful in collision problems. Note that it is frame-independent, as we saw in eq. (11.20).

This inner product is different from the one we're used to in three-dimensional space. It has one positive sign and three negative signs, in contrast with the usual three positive signs. But we are free to define it however we wish, and we did indeed pick a good definition, because our inner product is invariant under Lorentz-transformations, just as the usual 3-D inner product is invariant under rotations.[6]

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[5] If we were keeping in the factors of $c$, then the first term would be $E/c$, although some people instead multiply the $\mathbf{p}$ by $c$. Either convention is fine.

[6] For the inner product of a 4-momentum with itself (which could be any linear combination of

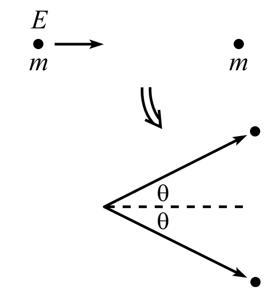

**Figure 11.7**

**Example (Relativistic billiards):** A particle with mass $m$ and energy $E$ approaches an identical particle at rest. They collide (elastically) in such a way that they both scatter at an angle $\theta$ relative to the incident direction (see Fig. 11.7). What is $\theta$ in terms of $E$ and $m$? What is $\theta$ in the relativistic and non-relativistic limits?

**Solution:** The first thing we should always do is write down the 4-momenta. The 4-momenta before the collision are

$$P_1 = (E, p, 0, 0), \qquad P_2 = (m, 0, 0, 0), \tag{11.27}$$

where $p = \sqrt{E^2 - m^2}$. The 4-momenta after the collision are (primes now denote "after")

$$P_1' = (E', p'\cos\theta, p'\sin\theta, 0), \qquad P_2' = (E', p'\cos\theta, -p'\sin\theta, 0), \tag{11.28}$$

where $p' = \sqrt{E'^2 - m^2}$. Conservation of energy gives $E' = (E+m)/2$, and conservation of $p_x$ gives $p'\cos\theta = p/2$. Therefore, the 4-momenta after the collision are

$$P_{1,2}' = \left(\frac{E+m}{2}\,,\, \frac{p}{2}\,,\, \pm\frac{p}{2}\tan\theta,\, 0\right). \tag{11.29}$$

From eq. (11.26), the squares of these 4-momenta must be $m^2$. Therefore,

$$\begin{aligned} m^2 &= \left(\frac{E+m}{2}\right)^2 - \left(\frac{p}{2}\right)^2 (1+\tan^2\theta) \\ \Longrightarrow \quad 4m^2 &= (E+m)^2 - \frac{(E^2-m^2)}{\cos^2\theta} \\ \Longrightarrow \quad \cos^2\theta &= \frac{E^2-m^2}{E^2+2Em-3m^2} = \frac{E+m}{E+3m}\,. \end{aligned} \tag{11.30}$$

The relativistic limit is $E \gg m$, which yields $\cos\theta \approx 1$. Therefore, both particles scatter almost directly forward.

The nonrelativistic limit is $E \approx m$ (it's *not* $E \approx 0$), which yields $\cos\theta \approx 1/\sqrt{2}$. Therefore, $\theta \approx 45^\circ$, and the particles scatter with a $90^\circ$ angle between them. This agrees with the result from the example in Section 4.7.2, a result which pool players are very familiar with.

Decays are basically the same as collisions. All you have to do is conserve energy and momentum, as the following example shows.

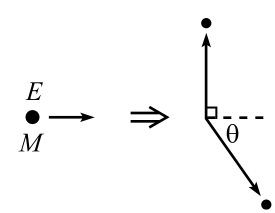

**Figure 11.8**

**Example (Decay at an angle):** A particle with mass $M$ and energy $E$ decays into two identical particles. In the lab frame, they are emitted at angles $90^\circ$ and $\theta$, as shown in Fig. 11.8. What are the energies of the created particles?

We'll give two solutions. The second one shows how 4-momenta can be used in a very clever and time-saving way.

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4-momenta of various particles), this invariance is simply the statement in eq. (11.20). For the inner product of two different 4-momenta, we'll prove the invariance in Section 12.3.

**First Solution:** The 4-momentum before the decay is

$$P = (E, p, 0, 0), \tag{11.31}$$

where $p = \sqrt{E^2 - M^2}$. Let the created particles have mass $m$. The 4-momenta after the collision are

$$P_1 = (E_1, 0, p_1, 0), \qquad P_2 = (E_2, p_2 \cos\theta, -p_2 \sin\theta, 0). \tag{11.32}$$

Conservation of $p_x$ immediately gives $p_2 \cos\theta = p$, which then implies that $p_2 \sin\theta = p \tan\theta$. Conservation of $p_y$ says that the final $p_y$'s are opposites. Therefore, the 4-momenta after the collision are

$$P_1 = (E_1, 0, p \tan\theta, 0), \qquad P_2 = (E_2, p, -p \tan\theta, 0). \tag{11.33}$$

Conservation of energy gives $E = E_1 + E_2$. Writing this in terms of the momenta and masses gives

$$E = \sqrt{p^2 \tan^2\theta + m^2} + \sqrt{p^2(1 + \tan^2\theta) + m^2}. \tag{11.34}$$

Putting the first radical on the left side, squaring, and solving for that radical (which is $E_1$) gives

$$E_1 = \frac{E^2 - p^2}{2E} = \frac{M^2}{2E}\,. \tag{11.35}$$

In a similar manner, we find that $E_2$ equals

$$E_2 = \frac{E^2 + p^2}{2E} = \frac{2E^2 - M^2}{2E}\,. \tag{11.36}$$

These add up to $E$, as they should.

**Second Solution:** With the 4-momenta defined as in eqs. (11.31) and (11.32), conservation of energy and momentum can be combined into the statement, $P = P_1 + P_2$. Therefore,

$$\begin{aligned} P - P_1 &= P_2, \\ \Longrightarrow \quad (P - P_1) \cdot (P - P_1) &= P_2 \cdot P_2, \\ \Longrightarrow \quad P^2 - 2P \cdot P_1 + P_1^2 &= P_2^2, \\ \Longrightarrow \quad M^2 - 2EE_1 + m^2 &= m^2, \\ \Longrightarrow \quad E_1 &= \frac{M^2}{2E}\,. \end{aligned} \tag{11.37}$$

And then $E_2 = E - E_1 = (2E^2 - M^2)/2E$.

This solution should convince you that 4-momenta can save you a lot of work. What happened here was that the expression for $P_2$ was fairly messy, but we arranged things so that it only appeared in the form of $P_2^2$, which is simply $m^2$. 4-momenta provide a remarkably organized method for sweeping unwanted garbage under the rug.

## 11.4 Particle-physics units

A branch of physics that uses relativity as one of its main ingredient is Elementary-Particle Physics, which is the study of the building blocks of matter (electrons, quarks, neutrinos, etc.). It is unfortunately the case that most of the elementary particles we want to study don't exist naturally in the world. We therefore have to create them in particle accelerators by colliding other particles together at very high energies. The high speeds involved require the use of relativistic dynamics. Newtonian physics is essentially useless.

What is a typical size of a rest energy, $mc^2$, of an elementary particle? The rest energy of a proton (which isn't really elementary; it's made up of quarks, but never mind) is

$$E_{\rm p} = m_{\rm p}c^2 = (1.67 \cdot 10^{-27}\,{\rm kg})(3 \cdot 10^{8}\,{\rm m/s})^2 = 1.5 \cdot 10^{-10}\ {\rm joules}. \tag{11.38}$$

This is very small, of course. So a joule is probably not the best unit to work with. We would get very tired of writing the negative exponents over and over.

We could perhaps work with "nanojoules", but particle-physicists like to work instead with the "eV", the *electron-volt*. This is the amount of energy gained by an electron when it passes through a potential of one volt. The electron charge is $e = 1.6022 \cdot 10^{-19}\,$C, and a volt is defined as $1\,$V $= 1\,$J/C. So the conversion from eV to joules is[7]

$$1\ {\rm eV} = (1.6022 \cdot 10^{-19}\,{\rm C})(1\,{\rm J/C}) = 1.6022 \cdot 10^{-19}\,{\rm J}. \tag{11.39}$$

Therefore, in terms of eV, the rest-energy of a proton is $938 \cdot 10^6\,$eV. We now have the opposite problem of having a large exponent hanging around. But this is easily remedied by the prefix "M", which stands for "mega", or "million". So we finally have a proton rest energy of

$$E_{\rm p} = 938\ {\rm MeV}. \tag{11.40}$$

You can work out for yourself that the electron has a rest-energy of $E_{\rm e} = 0.511$ MeV. The rest energies of various particles are listed in the table below. The ones preceded by a "≈" are the averages of differently charged particles, whose energies differ by a few MeV. These (and the many other) elementary particles have specific properties (spin, charge, etc.), but for the present purposes they need only be thought of as point objects having a definite mass.

[7] This is getting a little picky, but "eV" should actually be written as "$e$V", because "eV" stands for two things that are multiplied together (in contrast with, for example, the "kg" symbol for "kilogram"), one of which is the electron charge, which is usually denoted by $e$.

| particle | rest energy (MeV) |
|---|---|
| electron ($e$) | 0.511 |
| muon ($\mu$) | 105.7 |
| tau ($\tau$) | 1784 |
| proton ($p$) | 938.3 |
| neutron ($n$) | 939.6 |
| lambda ($\Lambda$) | 1115.6 |
| sigma ($\Sigma$) | $\approx$ 1193 |
| delta ($\Delta$) | $\approx$ 1232 |
| pion ($\pi$) | $\approx$ 137 |
| kaon ($K$) | $\approx$ 496 |

We now come to a slight abuse of language. When particle-physicists talk about masses, they say things like, "The mass of a proton is 938 MeV." This, of course, makes no sense, because the units are wrong; a mass can't equal an energy. But what they mean is that if you take this energy and divide it by $c^2$, then you get the mass. It would truly be a pain to keep saying, "The mass is such-and-such an energy, divided by $c^2$." For a quick conversion back to kilograms, you can show that

$$1 \text{ MeV}/c^2 = 1.783 \cdot 10^{-30} \text{ kg}. \tag{11.41}$$

## 11.5 Force

### 11.5.1 Force in one dimension

"Force" is a fairly intuitive concept. It is how hard you push or pull on something. We were told long ago that $\mathbf{F}$ equals $m\mathbf{a}$, and this makes sense. If you push an object in a certain direction, then it accelerates in that direction. But, alas, we've now outgrown the $\mathbf{F} = m\mathbf{a}$ definition. It's time to look at things a different way.

The force on an object is hereby *defined* to be the rate of change of momentum (we'll just deal with one-dimensional motion for now),

$$F = \frac{dp}{dt}\,. \tag{11.42}$$

This is actually the definition in nonrelativistic physics too, but in that case, where $p = mv$, we obtain $F = ma$ anyway. So it doesn't matter if we define $F$ to be $dp/dt$ or $ma$. But in the relativistic case, it does matter, because $p = \gamma mv$, and $\gamma$ can change with time. This will complicate things, and it will turn out that $F$ does *not* equal $ma$. Why do we define $F$ to be $dp/dt$ instead of $ma$? One reason is given in the first remark below. Another arises from the general 4-vector formalism in Chapter 12.

To see what form the $F$ in eq. (11.42) takes in terms of the acceleration, $a$, note that

$$\frac{d\gamma}{dt} \equiv \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) = \frac{v\dot{v}}{(1-v^2)^{3/2}} \equiv \gamma^3 va. \tag{11.43}$$

Therefore, assuming that $m$ is constant, we have

$$\begin{aligned} F = \frac{d(\gamma m v)}{dt} &= m(\dot{\gamma} v + \gamma \dot{v}) \\ &= ma\gamma(\gamma^2 v^2 + 1) \\ &= \gamma^3 ma. \end{aligned} \tag{11.44}$$

This doesn't look as nice as $F = ma$, but that's the way it goes.

They *said*, "$F$ is $ma$, bar none."
What they *meant* wasn't quite as much fun.
It's $dp$ by $dt$,
Which just happens to be
Good ol' "$ma$" when $\gamma$ is 1.

Consider now the quantity $dE/dx$, where $E$ is the energy, $E = \gamma m$. We have

$$\begin{aligned} \frac{dE}{dx} = \frac{d(\gamma m)}{dx} &= m\frac{d(1/\sqrt{1-v^2})}{dx} \\ &= \gamma^3 mv\frac{dv}{dx}\,. \end{aligned} \tag{11.45}$$

But $v(dv/dx) = dv/dt \equiv a$. Therefore, $dE/dx = \gamma^3 ma$, and eq. (11.44) gives

$$F = \frac{dE}{dx}\,. \tag{11.46}$$

Note that eqs. (11.42) and (11.46) take exactly the same form as in the nonrelativistic case. The only new thing in the relativistic case is that the expressions for $p$ and $E$ are modified.

REMARKS:

1. Eq. (11.42) is devoid of any physical content, because all it does is define $F$. If $F$ were instead defined through eq. (11.46), then eq. (11.46) would be devoid of any content. The whole point of this section, and the only thing of any substance, is that (with the definitions $p = \gamma mv$ and $E = \gamma m$)

$$\frac{dp}{dt} = \frac{dE}{dx}\,. \tag{11.47}$$

   This is the physically meaningful statement. If we then want to label both sides of the equation with the letter $F$ for "force," so be it. But "force" is simply a name.

2. The result in eq. (11.46) suggests another way to arrive at the $E = \gamma m$ relation. The reasoning is exactly the same as in the nonrelativistic derivation of energy conservation in Section 4.1. Define $F$, as we have done, through eq. (11.42). Then integrate eq. (11.44) from $x_1$ to $x_2$ to obtain

$$\int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} (\gamma^3 ma)\,dx$$

$$\begin{aligned} &= \int_{x_1}^{x_2} \left( \gamma^3 m v \frac{dv}{dx} \right) dx \\ &= \int_{v_1}^{v_2} \gamma^3 m v \, dv \\ &= \gamma m \Big|_{v_1}^{v_2}, \end{aligned} \tag{11.48}$$

where we have used eq. (11.46). If we then define the "potential energy" as

$$V(x) \equiv -\int_{x_0}^{x} F(x)\,dx, \tag{11.49}$$

where $x_0$ is an arbitrary reference point, we obtain

$$V(x_1) + \gamma m\Big|_{v_1} = V(x_2) + \gamma m\Big|_{v_2}. \tag{11.50}$$

We see that the quantity $V + \gamma m$ is independent of $x$. It is therefore worthy of a name, and we use the name "energy" due to the similarity with the Newtonian result.[8]

The work-energy theorem (that is, $\int F\,dx = \Delta E$) holds in relativistic physics, just as it does in the nonrelativistic case. The only difference is that $E$ is $\gamma m$ instead of $mv^2/2$. ♣

### 11.5.2 Force in two dimensions

In two dimensions, the concept of force becomes a little strange. In particular, as we will see, the acceleration of an object need not point in the same direction as the force. We start with the definition,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{11.51}$$

This is a vector equation. Without loss of generality, let us deal with only two spatial dimensions. Consider a particle moving in the $x$-direction, and let us apply a force, $\mathbf{F} = (F_x, F_y)$. The particle's momentum is

$$\mathbf{p} = \frac{m(v_x, v_y)}{\sqrt{1 - v_x^2 - v_y^2}}. \tag{11.52}$$

Taking the derivative of this, and using the fact that $v_y$ is initially zero, we obtain

$$\begin{aligned} \mathbf{F} &= \frac{d\mathbf{p}}{dt}\Big|_{v_y=0} \\ &= m\left( \frac{\dot{v}_x}{\sqrt{1-v^2}} + \frac{v_x(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3}, \frac{\dot{v}_y}{\sqrt{1-v^2}} + \frac{v_y(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3} \right)\Bigg|_{v_y=0} \end{aligned}$$

[8]Actually, this derivation only suggests that $E$ is given by $\gamma m$ up to an additive constant. For all we know, $E$ might take the form, $E = \gamma m - m$, which would make the energy of a motionless particle equal to zero. An argument along the lines of Section 11.1.2 is required to show that the additive constant is zero.

$$
\begin{aligned}
&= m\left(\frac{\dot{v}_x}{\sqrt{1-v^2}}\left(1+\frac{v^2}{1-v^2}\right), \frac{\dot{v}_y}{\sqrt{1-v^2}}\right) \\
&= m\left(\frac{\dot{v}_x}{(\sqrt{1-v^2})^3}, \frac{\dot{v}_y}{\sqrt{1-v^2}}\right) \\
&\equiv m(\gamma^3 a_x, \gamma a_y). \qquad (11.53)
\end{aligned}
$$

We see that this is *not* proportional to $(a_x, a_y)$. The first component agrees with eq. (11.44), but the second component has only one factor of $\gamma$. The difference comes from the fact that $\gamma$ has a first-order change if $v_x$ changes, but not if $v_y$ changes, assuming that $v_y$ is initially zero. The particle therefore responds differently to forces in the $x$- and $y$-directions. It is easier to accelerate something in the transverse direction.

### 11.5.3 Transformation of forces

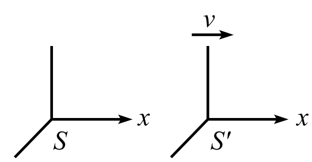

**Figure 11.9**

Let a force act on a particle. How are the components of the force in the particle's frame, $S'$, related to the components of the force in another frame, $S$?[9] Let the relative motion be along the $x$- and $x'$-axes, as in Fig. 11.9. In frame $S$, eq. (11.53) says

$$(F_x, F_y) = m(\gamma^3 a_x, \gamma a_y). \qquad (11.54)$$

And in frame $S'$, the $\gamma$ factor for the particle equals 1, so eq. (11.53) reduces to the usual expression,

$$(F'_x, F'_y) = m(a'_x, a'_y). \qquad (11.55)$$

Let's now try to relate these two forces, by writing the primed accelerations on the right-hand side of eq. (11.55) in terms of the unprimed accelerations.

First, we have $a'_y = \gamma^2 a_y$. This is true because transverse distances are the same in the two frames, but times are shorter in $S'$ by a factor $\gamma$. That is, $dt' = dt/\gamma$. We have indeed put the $\gamma$ in the right place here, because the particle is essentially at rest in $S'$, so the usual time dilation holds. Therefore, $a'_y \equiv d^2y'/dt'^2 = d^2y/(dt/\gamma)^2 \equiv \gamma^2 a_y$.

Second, we have $a'_x = \gamma^3 a_x$. In short, this is true because time dilation brings in two factors of $\gamma$ (as in the $a_y$ case), and length contraction brings in one. In a little more detail: Let the particle move from one point to another in frame $S'$, as it accelerates from rest in $S'$. Mark these two points, which are a distance $a'_x(dt')^2/2$ apart, in $S'$. As $S'$ flies past $S$, the distance between the two marks will be length contracted by a factor $\gamma$, as viewed by $S$. This distance (which is the excess distance the particle has over what it would have had if there were no acceleration) is what $S$ calls $a_x(dt)^2/2$. Therefore,

$$\frac{1}{2}a_x\,dt^2 = \frac{1}{\gamma}\left(\frac{1}{2}a'_x\,dt'^2\right) \qquad \Longrightarrow \qquad a'_x = \gamma a_x\left(\frac{dt}{dt'}\right)^2 = \gamma^3 a_x. \qquad (11.56)$$

[9]To be more precise, $S'$ is the instantaneous inertial frame of the particle. Once the force is applied, the particle's frame will no longer be $S'$. But for a very small elapsed time, the particle will still essentially be in $S'$.

Eq. (11.55) may now be written as

$$(F'_x, F'_y) = m(\gamma^3 a_x, \gamma^2 a_y). \tag{11.57}$$

Finally, comparing eqs. (11.54) and (11.57), we find

$$F_x = F'_x, \qquad \text{and} \qquad F_y = \frac{F'_y}{\gamma}. \tag{11.58}$$

We see that the longitudinal force is the same in the two frames, but the transverse force is larger by a factor of $\gamma$ in the particle's frame.

Remarks:

1. What if someone comes along and relabels the primed and unprimed frames in eq. (11.58), and concludes that the transverse force is *smaller* in the particle's frame? He certainly can't be correct, given that eq. (11.58) is true, but where is the error?

   The error lies in the fact that we (correctly) used $dt' = dt/\gamma$ above, because this is the relevant expression concerning two events along the particle's worldline. We are interested in two such events, because we want to see how the particle moves. The inverted expression, $dt = dt'/\gamma$, deals with two events located at the same position in $S$, and therefore has nothing to do with the situation at hand. Similar reasoning holds for the relation between $dx$ and $dx'$. There is indeed one frame here that is special among all the possible frames, namely the particle's instantaneous inertial frame.

2. If you want to compare forces in two frames, neither of which is the particle's rest frame, then just use eq. (11.58) twice and relate each of the forces to the rest-frame forces. It quickly follows that for another frame $S''$, we have $F''_x = F_x$, and $\gamma'' F''_y = \gamma F_y$, where the $\gamma$'s are measured relative to the rest fame, $S'$. ♣

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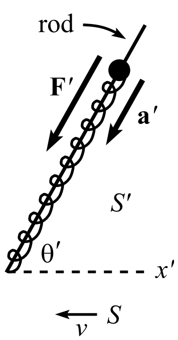


**Figure 11.10**

**Example (Bead on a rod):** A spring with a tension has one end attached to the end of a rod, and the other end attached to a bead which is constrained to move along the rod. The rod makes an angle $\theta'$ with respect to the $x'$-axis, and is fixed at rest in the $S'$ frame (see Fig. 11.10). The bead is released and is pulled along the rod.

When the bead is released, what does the situation look like in the frame, $S$, of someone moving to the left at speed $v$? In answering this, draw the directions of

(a) the rod,

(b) the acceleration of the bead, and

(c) the force on the bead.

In frame $S$, does the wire exert a force of constraint?

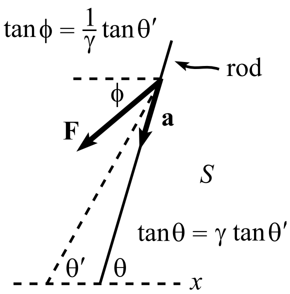


**Figure 11.11**

**Solution:** In frame $S$:

(a) The horizontal span of the rod is decreased by a factor $\gamma$, due to length contraction, and the vertical span is unchanged, so we have $\tan\theta = \gamma \tan\theta'$, as shown in Fig. 11.11.

(b) The acceleration must point along the rod, because the bead always lies on the rod. Quantitatively, the position of the bead in frame $S$ takes the form of $(x, y) = (vt - a_x t^2/2, -a_y t^2/2)$, by the definition of acceleration. The position relative to the starting point on the rod, which has coordinates $(vt, 0)$, is then $(\Delta x, \Delta y) = (-a_x t^2/2, -a_y t^2/2)$. The condition for the bead to stay on the rod is that the ratio of these coordinates be equal the slope of the rod in Frame $S$. Therefore, $a_y/a_x = \tan\theta$, so the acceleration points along the rod.

(c) The $y$-component of the force on the bead is decreased by a factor $\gamma$, by eq. (11.58), so we have $\tan\phi = (1/\gamma)\tan\theta'$, as shown in the figure.

As a double-check that **a** does indeed point along the rod, we can use eq. (11.53) to write $a_y/a_x = \gamma^2 F_y/F_x$. Then eq. (11.58) gives $a_y/a_x = \gamma F'_y/F'_x = \gamma\tan\theta' = \tan\theta$, which is the direction of the wire.

The wire does *not* exert a force of constraint. The bead need not touch the wire in $S'$, so it need not touch it in $S$. Basically, there is no need to have an extra force to combine with **F** to make the result point along **a**, because **F** simply does not have to be collinear with **a**.

## 11.6 Rocket motion

Up to this point, we have dealt with situations where the masses of our particles are constant, or where they change abruptly (as in a decay, where the sum of the masses of the products is less than the mass of the initial particle). But in many problems, the mass of an object changes continuously. A rocket is the classic example of this type of situation. Hence, we will use the term "rocket motion" to describe the general class of problems where the mass changes continuously.

The relativistic rocket itself encompasses all of the important ideas, so let's study that example here. Many more examples are left for the problems. We'll present three solutions to the rocket problem, the last of which is rather slick. In the end, the solutions are all basically the same, but it should be helpful to see the various ways of looking at the problem.

**Example (Relativistic rocket):** Assume that a rocket propels itself by continually converting mass into photons and firing them out the back. Let $m$ be the instantaneous mass of the rocket, and let $v$ be the instantaneous speed with respect to the ground. Show that

$$\frac{dm}{m} + \frac{dv}{1-v^2} = 0. \tag{11.59}$$

If the initial mass is $M$, and the initial $v$ is zero, integrate eq.(11.59) to obtain

$$m = M\sqrt{\frac{1-v}{1+v}}\,. \tag{11.60}$$

**First solution:** The strategy of this solution will be to use conservation of momentum in the ground frame.

Consider the effect of a small mass being converted into photons. The mass of the rocket goes from $m$ to $m + dm$ (where $dm$ is negative). So in the frame of the rocket, photons with total energy $E_r = -dm$ (which is positive) are fired out the back. In the frame of the rocket, these photons have momentum $p_r = dm$ (which is negative). Let the rocket move with speed $v$ with respect to the ground. Then the momentum of the photons in the ground frame, $p_g$, may be found via the Lorentz transformation,

$$p_g = \gamma(p_r + vE_r) = \gamma\big(dm + v(-dm)\big) = \gamma(1-v)\,dm. \tag{11.61}$$

This is still negative, of course.

Remark: A common error is to say that the converted mass $(-dm)$ takes the form of photons of energy $(-dm)$ in the ground frame. This is incorrect, because although the photons have energy $(-dm)$ in the rocket frame, they are redshifted (due to the Doppler effect) in the ground frame. From eq. (10.48), we see that the frequency (and hence the energy) of the photons decreases by a factor of $\sqrt{(1-v)/(1+v)}$ when going from the rocket frame to the ground frame. This factor equals the $\gamma(1-v)$ factor in eq. (11.61). ♣

We may now use conservation of momentum in the ground frame to say that

$$(m\gamma v)_{\text{old}} = \gamma(1-v)\,dm + (m\gamma v)_{\text{new}} \qquad \Longrightarrow \qquad \gamma(1-v)\,dm + d(m\gamma v) = 0. \tag{11.62}$$

The $d(m\gamma v)$ term may be expanded to give

$$\begin{aligned} d(m\gamma v) &= (dm)\gamma v + m(d\gamma)v + m\gamma(dv) \\ &= \gamma v\,dm + m(\gamma^3 v\,dv)v + m\gamma\,dv \\ &= \gamma v\,dm + m\gamma(\gamma^2 v^2 + 1)\,dv \\ &= \gamma v\,dm + m\gamma^3\,dv. \end{aligned} \tag{11.63}$$

Therefore, eq. (11.62) gives

$$\begin{aligned} 0 &= \gamma(1-v)\,dm + \gamma v\,dm + m\gamma^3 dv \\ &= \gamma\,dm + m\gamma^3\,dv. \end{aligned} \tag{11.64}$$

Hence,

$$\frac{dm}{m} + \frac{dv}{1-v^2} = 0, \tag{11.65}$$

in agreement with eq. (11.59). We must now integrate this. With the given initial values, we have

$$\int_M^m \frac{dm}{m} + \int_0^v \frac{dv}{1-v^2} = 0. \tag{11.66}$$

We could simply look up the $dv$ integral in a table, but let's do it from scratch.[10] Writing $1/(1-v^2)$ as the sum of two fractions gives

$$\begin{aligned} \int_0^v \frac{dv}{1-v^2} &= \frac{1}{2}\int_0^v \left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv \\ &= \frac{1}{2}\Big(\ln(1+v) - \ln(1-v)\Big)\Big|_0^v \\ &= \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right). \end{aligned} \tag{11.67}$$

[10] Tables often list the integral of $1/(1-v^2)$ as $\tanh^{-1}(v)$, which you can show is equivalent to the result in eq. (11.67).

Eq. (11.66) therefore gives

$$\begin{aligned}\ln\left(\frac{m}{M}\right) &= -\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\\ \Longrightarrow \qquad m &= M\sqrt{\frac{1-v}{1+v}}\,,\end{aligned}\qquad (11.68)$$

in agreement with eq. (11.60). This result is independent of the rate at which the mass is converted into photons. It is also independent of the frequency of the emitted photons. Only the total mass expelled matters.

Note that eq. (11.68) quickly tells us that the energy of the rocket, as a function of velocity, is

$$E = \gamma m = \gamma M\sqrt{\frac{1-v}{1+v}} = \frac{M}{1+v}\,. \qquad (11.69)$$

This has the interesting property of approaching $M/2$ as $v \to c$. In other words, half of the initial energy remains with the rocket, and half ends up as photons (see Exercise 18).

REMARK: From eq. (11.61), or from the previous remark, we see that the ratio of the energy of the photons in the ground frame to that in the rocket frame is $\sqrt{(1-v)/(1+v)}$. This factor is the same as the factor in eq. (11.68). In other words, the photons' energy decreases in exactly the same manner as the mass of the rocket (assuming that the photons are ejected with the same frequency in the rocket frame throughout the process). Therefore, in the ground frame, the ratio of the photons' energy to the mass of the rocket doesn't change with time. There must be a nice intuitive explanation for this, but it eludes me. ♣

**Second solution:** The strategy of this solution will be to use $F = dp/dt$ in the ground frame.

Let $\tau$ denote the time in the rocket frame. Then in the rocket frame, $dm/d\tau$ is the rate at which the mass of the rocket decreases and is converted into photons ($dm$ is negative). The photons therefore acquire momentum at the rate $dp/d\tau = dm/d\tau$ in the rocket frame. Since force is the rate of change of momentum, we see that a force of $dm/d\tau$ pushes the photons backward, and an equal and opposite force of $F = -dm/d\tau$ pushes the rocket forward in the rocket frame.

Now go to the ground frame. We know from eq. (11.58) that the longitudinal force is the same in both frames, so $F = -dm/d\tau$ is also the force on the rocket in the ground frame. And since $t = \gamma\tau$, where $t$ is the time on the ground (the photon emissions occur at the same place in the rocket frame, so we have indeed put the time-dilation factor of $\gamma$ in the right place), we have

$$F = -\gamma\frac{dm}{dt}\,. \qquad (11.70)$$

REMARK: We can also calculate the force on the rocket by working entirely in the ground frame. Consider a mass $(-dm)$ that is converted into photons. Initially, this mass is traveling along with the rocket, so it has momentum $(-dm)\gamma v$. After it is converted into photons, it has momentum $\gamma(1-v)\,dm$ (from the first solution above). The change in momentum is therefore $\gamma(1-v)\,dm - (-dm)\gamma v = \gamma\,dm$. Since force is the rate of change of momentum, a force of $\gamma\,dm/dt$ pushes the photons backwards, and an equal and opposite force of $F = -\gamma\,dm/dt$ therefore pushes the rocket forwards. ♣

Now things get a little tricky. It is tempting to write down $F = dp/dt = d(m\gamma v)/dt = (dm/dt)\gamma v + m\,d(\gamma v)/dt$. This, however, is not correct, because the $dm/dt$ term is not relevant here. When the force is applied to the rocket at an instant when the rocket has mass $m$, the only thing the force cares about is that the mass of the rocket at the given instant is $m$. It doesn't care that $m$ is changing.[11] Therefore, the correct expression we want is

$$F = m\frac{d(\gamma v)}{dt}\,. \tag{11.71}$$

As in the first solution above, or in eq. (11.44), we have $d(\gamma v)/dt = \gamma^3\,dv/dt$. Using the $F$ from eq. (11.70), we arrive at

$$-\gamma\frac{dm}{dt} = m\gamma^3\frac{dv}{dt}\,, \tag{11.72}$$

which is equivalent to eq. (11.64). The solution proceeds as above.

**Third solution:** The strategy of this solution will be to use conservation of energy and momentum in the ground frame, in a slick way.

Consider a clump of photons fired out the back. The energy and momentum of these photons are equal in magnitude and opposite in sign (with the convention that the photons are fired in the negative direction). By conservation of energy and momentum, the same statement must be true about the changes in energy and momentum of the rocket. That is,

$$d(\gamma m) = -d(\gamma m v) \qquad \Longrightarrow \qquad d(\gamma m + \gamma m v) = 0. \tag{11.73}$$

Therefore, $\gamma m(1+v)$ is a constant. We are given that $m = M$ when $v = 0$. Hence, the constant must be $M$. Therefore,

$$\gamma m(1+v) = M \qquad \Longrightarrow \qquad m = M\sqrt{\frac{1-v}{1+v}}\,. \tag{11.74}$$

Now, *that's* a quick solution, if there ever was one!

## 11.7 Relativistic strings

Consider a "massless" string with a tension that is constant (that is, independent of length).[12] We will call such objects *relativistic strings*, and we will study them for two reasons. First, these strings, or reasonable approximations thereof, actually do occur in nature. For example, the gluon force which holds quarks together is approximately constant over distance. And second, they open the door to a whole new supply of problems we can solve, like the following one.

[11] Said in a different way, the momentum associated with the missing mass still exists. It's just that it's not part of the rocket anymore. This issue is expanded on in Appendix E.

[12] By "massless," we mean that the string has no mass in its unstretched (that is, zero-length) state. Once it is stretched, it will have energy, and hence mass.

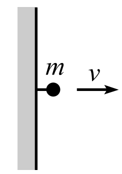


Figure 11.12

**Example (Mass connected to a wall):** A mass $m$ is connected to a wall by a relativistic string with tension $T$. The mass starts next to the wall and has initial speed $v$ away from it (see Fig. 11.12). What is the maximum distance from the wall the mass achieves? How much time does it take to reach this point?

**Solution:** Let $\ell$ be the maximum distance from the wall. The initial energy of the mass is $E = \gamma m$. The final energy at $x = \ell$ is simply $m$, because the mass is instantaneously at rest there. Integrating $F = dE/dx$, and using the fact that the force always equals $-T$, gives

$$F\Delta x = \Delta E \quad \Longrightarrow \quad (-T)\ell = m - \gamma m \quad \Longrightarrow \quad \ell = \frac{m(\gamma - 1)}{T}. \qquad (11.75)$$

Let $t$ be the time it takes to reach this point. The initial momentum of the mass is $p = \gamma m v$. Integrating $F = dp/dt$, and using the fact that the force always equals $-T$, gives

$$F\Delta t = \Delta p \quad \Longrightarrow \quad (-T)t = 0 - \gamma m v \quad \Longrightarrow \quad t = \frac{\gamma m v}{T}. \qquad (11.76)$$

Note that we *cannot* use $F = ma$ to do this problem. $F$ does not equal $ma$. It equals $dp/dt$ (and also $dE/dx$).

Relativistic strings may seem a bit strange, but there is nothing more to solving a one-dimensional problem than the two equations,

$$F = \frac{dp}{dt}, \qquad \text{and} \qquad F = \frac{dE}{dx}. \qquad (11.77)$$

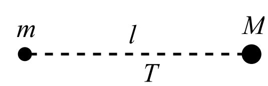


Figure 11.13

**Example (Where the masses meet):** A relativistic string of length $\ell$ and tension $T$ connects a mass $m$ and a mass $M$ (see Fig. 11.13). The masses are released from rest. Where do they meet?

**Solution:** Let the masses meet at a distance $x$ from the initial position of $m$. At this meeting point, $F = dE/dx$ tells us that the energy of $m$ is $m + Tx$, and the energy of $M$ is $M + T(\ell - x)$. Using $p = \sqrt{E^2 - m^2}$ we see that the magnitudes of the momenta at the meeting point are

$$p_m = \sqrt{(m + Tx)^2 - m^2} \qquad \text{and} \qquad p_M = \sqrt{(M + T(\ell - x))^2 - M^2}. \qquad (11.78)$$

But $F = dp/dt$ then tells us that these must be equal, because the same force (in magnitude, but opposite in direction) acts on the two masses for the same time. Equating the above $p$'s gives

$$x = \frac{\ell\big(T(\ell/2) + M\big)}{M + m + T\ell}. \qquad (11.79)$$

This is reassuring, because the answer is simply the location of the initial center of mass, with the string being treated (quite correctly) like a stick of length $\ell$ and mass $T\ell$ (divided by $c^2$).

REMARK: Let's check a few limits. In the limit of large $T$ or $\ell$ (more precisely, in the limit $T\ell \gg Mc^2$ and $T\ell \gg mc^2$), we have $x = \ell/2$. This makes sense, because in this case the masses are negligible and therefore both move at essentially speed $c$, and hence meet in the middle. In the limit of small $T$ or $\ell$ (more precisely, in the limit $T\ell \ll Mc^2$ and $T\ell \ll mc^2$), we have $x = M\ell/(M + m)$, which is simply the Newtonian result for an everyday-strength spring. ♣

## 11.8 Mass

Some treatments of relativity refer to the mass of a motionless particle as the "rest-mass" $m_0$, and the mass of moving particle as the "relativistic mass" $m_{\text{rel}} = \gamma m_0$. This terminology is misleading and should be avoided. There is no such thing as "relativistic mass." There is only one "mass" associated with an object. This mass is what the above treatments would call the "rest mass."[13] And since there is only one type of mass, there is no need to use the qualifier "rest" or the subscript "0." We therefore simply use the notation "$m$." In this section, we will explain why "relativistic mass" is not a good concept to use.[14]

Why might someone want to call $m_{\text{rel}} \equiv \gamma m$ the mass of a moving particle? The basic reason is that the momentum takes the nice Newtonian form of $\mathbf{p} = m_{\text{rel}}\mathbf{v}$. The tacit assumption here is that the goal is to assign a mass to the particle such that all the Newtonian expressions continue to hold, with the only change being a modified mass. That is, we want our particle to act in exactly the same way that a particle of mass $\gamma m$ would, according to our everyday intuition.[15]

If we insist on hanging onto our Newtonian rules, let's see what they imply. If we want our particle to act as a mass $\gamma m$ does, then we must have $\mathbf{F} = (\gamma m)\mathbf{a}$. However, we saw in Section 11.5.2 that although this equation is true for transverse forces, it is *not* true for longitudinal forces. The $\gamma m$ would have to be replaced by $\gamma^3 m$ for a longitudinal force. As far as acceleration goes, a mass reacts differently to forces that point in different directions. We therefore see that it is impossible to assign a unique mass to a moving particle, such that it behaves in a Newtonian way under all circumstances. Not only is the goal of thinking of things in a Newtonian way ill-advised, it is doomed to failure.

[13]For example, the mass of an electron is $9.11 \cdot 10^{-31}$ kg, and the mass of a liter of water is 1 kg, independent of the speed.

[14]Of course, you can *define* the quantity $\gamma m$ with any name you want. You can call it "relativistic mass," or you can call it "pumpkin pie." The point is that the connotations associated with these definitions will mislead you into thinking certain things are true when they are not. The quantity $\gamma m$ does *not* behave as you might want a mass to behave (as we will show). And it also doesn't make for a good dessert.

[15]This goal should send up a red flag. It is similar to trying to think about quantum mechanics in terms of classical mechanics. It simply cannot be done. All analogies will eventually break down and lead to incorrect conclusions. It is quite silly to try to think about a (more) correct theory (relativity or quantum mechanics) in terms of an incorrect theory (classical mechanics), simply because our intuition (which is limited and incorrect) is based on the latter.

"Force is my $a$ times my 'mass',"
Said the driver, when starting to pass.
But from what we've just learned,
He was right when he turned,
But wrong when he stepped on the gas.

The above argument closes the case on this subject, but there are a few other arguments that show why it is not good to think of $\gamma m$ as a mass.

The word "mass" is used to describe what is on the right-hand side of the equation, $E^2 - |\mathbf{p}|^2 = m^2$. The $m^2$ here is an *invariant*, that is, it is something that is independent of the frame of reference. $E$ and the components of $\mathbf{p}$, on the other hand, are components of a 4-vector. They depend on the frame. If "mass" is to be used in this definite way to describe an invariant, then it doesn't make sense to also use it to describe the quantity $\gamma m$, which is frame-dependent. And besides, there is certainly no need to give $\gamma m$ another name. It already goes by the name "$E$," up to factors of $c$.

It is often claimed that $\gamma m$ is the "mass" that appears in the expression for gravitational force. If this were true, then it might be reasonable to use "mass" as a label for the quantity $\gamma m$. But, in fact, it is not true. The gravitational force depends in a somewhat complicated way on the motion of the particle. For example, the force depends on whether the particle is moving longitudinally or transversely to the source. We cannot demonstrate this fact here, but suffice it to say that if one insists on using the naive force law, $F = Gm_1m_2/r^2$, then it is impossible to label the particle with a unique mass.

## 11.9 Exercises

*Section 11.2: Transformations of $E$ and $\vec{p}$*

1. **Energy of two masses** $*$

   Two masses $M$ move at speed $V$, one to the east and one to the west. What is the total energy of the system?

   Now consider the setup as viewed from a frame moving to the west at speed $u$. Find the energy of each mass in this frame. Is the total energy larger or smaller than the total energy in the lab frame?

2. **System of particles** $*$

   Given $p_{\text{total}}$ and $E_{\text{total}}$ for a system of particles, use a Lorentz transformation to find the velocity of the CM. More precisely, find the speed of the frame in which the total momentum is zero.

3. **CM frame** $**$

   A mass $m$ travels at speed $3c/5$, and another mass $m$ sits at rest.

   (a) Find the energy and momentum of the two particles in the lab frame.
   (b) Find the speed of the CM of the system, by using a velocity-addition argument.
   (c) Find the energy and momentum of the two particles in the CM frame.
   (d) Verify that the $E$'s and $p$'s are related by the relevant Lorentz transformations.
   (e) Verify that $E^2_{\text{total}} - p^2_{\text{total}}$ is the same in both frames.

4. **Transformation for 2-D motion** $**$

   A particle has velocity $(u'_x, u'_y)$ in frame $S'$, which travels at speed $v$ in the $x$-direction relative to frame $S$. Use the velocity addition formulas in Section 10.3.3 (eqs. (10.33) and (10.35)) to show that $E$ and $p_x$ transform according to eq. (11.18), and also that $p_y = p'_y$.

   *Hint*: This gets a bit messy, but the main thing you need to show is

   $$\gamma_u = \gamma_{u'}\gamma_v(1 + u'_x v), \quad \text{where} \quad u = \sqrt{u_x^2 + u_y^2} \quad \text{and} \quad u' = \sqrt{u'^2_x + u'^2_y} \tag{11.80}$$

   are the speeds in the two frames.

*Section 11.3: Collisions and decays*

5. **Photon, mass collision** $*$

   A photon with energy $E$ collides with a stationary mass $m$. The combine to form one particle. What is the mass of this particle? What is its speed?

6. **A decay** $*$

   A mass $M$ decays into a mass $m$ and a photon. If the speed of $m$ is $v$, find $m$ and also the energy of the photon (in terms of $M$ and $v$).

7. **Three photons** $*$

A mass $m$ travels with speed $v$. It decays into three photons, one of which travels in the forward direction, and the other two of which move at angles of 120° (in the lab frame) as shown in Fig. 11.14. What are the energies of these three photons?

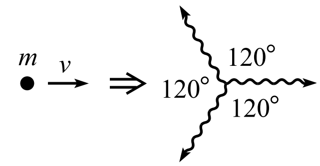


**Figure 11.14**

8. **Perpendicular photon** $*$

A photon with energy $E$ collides with a mass $M$. The mass $M$ scatters off at an angle. If the resulting photon moves perpendicularly to the incident photon's direction, as shown in Fig. 11.15, what is its energy?

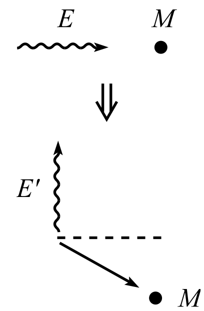


**Figure 11.15**

9. **Another perpendicular photon** $**$

A mass $m$ moving with speed $4c/5$ collides with another mass $m$ at rest. The collision produces a photon with energy $E$ traveling perpendicularly to the original direction, and a mass $M$ traveling in another direction, as shown in Fig. 11.16. In terms of $E$ and $m$, what is $M$? What is the largest value of $E$ (in terms of $m$) for which this setup is possible?

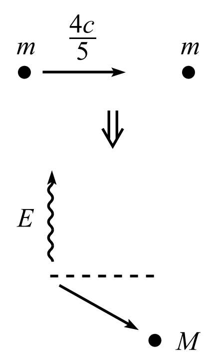


**Figure 11.16**

10. **Colliding diagonally** $*$

A mass $m$ moving northeastward at speed $4c/5$ collides with a photon moving southeastward. The result of the collision is one particle of mass $M$ moving eastward, as shown in Fig. 11.17 Find the energy of the photon, the mass $M$, and the speed of $M$. (Give the first two of these answers in terms of $m$.)

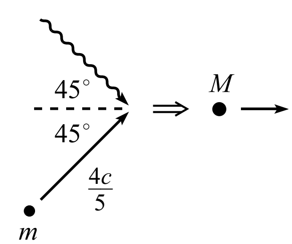


**Figure 11.17**

11. **Decay into photons** $*$

A mass $M$ traveling at $3c/5$ decays into a mass $M/4$ and two photons. One photon moves perpendicularly to the original direction, the other photon moves off at an angle $\theta$, and the mass $M/4$ is at rest, as shown in Fig. 11.18. What is $\theta$?

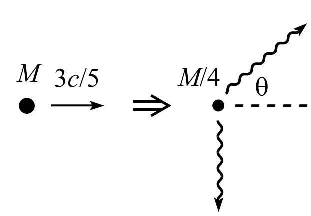


**Figure 11.18**

12. **Three masses colliding** $*$

Three masses $m$, all traveling at speed $v = 4c/5$, collide at the origin and produce a particle of mass $M$. The three original velocities are in the northeast, north, and northwest directions. Find $M$ and its velocity.

13. **Maximum mass** $*$

A photon and a mass $m$ move in opposite directions. They collide head-on and create a new particle. If the total energy of the system is $E$, how should it be divided between the photon and the mass $m$, so that the mass of the resulting particle is as large as possible?

*Section 11.4: Particle-physics units*

14. **Pion-muon race** $*$

A pion and a muon each have energy 10 GeV. They have a 100 m race. By how much distance does the muon win?

*Section 11.5: Force*

15. **Force and a collision** $*$

Two identical masses $m$ are at rest, a distance $x$ apart. A constant force $F$ accelerates one of them towards the other until they collide and stick together. How much time does this take? What is the mass of the resulting particle?

16. **Pushing on a mass** $**$

A mass $m$ starts at rest. You push on it with a constant force $F$.

(a) How much time, $t$, does it take to move the mass a distance $x$? (Both $t$ and $x$ here are measured in the lab frame.)

(b) After a very long time, the speed of $m$ will approach the speed of light. Therefore, after a very long time, $m$ will remain (approximately) a constant distance (as measured in the lab frame) behind a photon that was emitted at $t = 0$ from the starting position of $m$. Show that this distance equals $mc^2/F$.

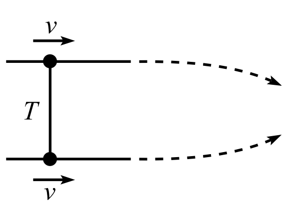

**Figure 11.19**

17. **Momentum paradox** $***$

Two equal masses are connected by a massless string with tension $T$. The masses are constrained to move with speed $v$ along parallel lines, as shown in Fig. 11.19. The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct? If your answer is "no", state what is invalid about whichever of the four sentences is/are invalid.

"The forces on the masses point in the $y$-direction. Therefore, there is no change in momentum in the $x$-direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collided with some relative speed). Therefore, the speed of the resulting blob must be less than $v$ (to keep $p_x$ constant), so the whole apparatus slows down in the $x$-direction."

*Section 11.6: Rocket motion*

18. **Rocket energy** $**$

As mentioned at the end of the first solution to the rocket problem in Section 11.6, the energy of the rocket in the ground frame equals $M/(1 + v)$. Derive this result again, by integrating up the amount of energy that the photons have in the ground frame.

*Section 11.7: Relativistic strings*

19. **Two masses** $*$

    A mass $m$ is placed right in front of an identical one. They are connected by a relativistic string with tension $T$. The front one suddenly acquires a speed of $3c/5$. How far from the starting point will the masses collide with each other?

## 11.10 Problems

*Section 11.1: Energy and momentum*

1. **Deriving $E$ and $p$** $**$

   Accepting the facts that the energy and momentum of a photon are $E = h\nu$ and $p = h\nu/c$ (where $\nu$ is the frequency of the light wave, and $h$ is Planck's constant), derive the relativistic formulas for the energy and momentum of a massive particle, $E = \gamma mc^2$ and $p = \gamma mv$. *Hint:* Consider a mass $m$ that decays into two photons. Look at this decay both in the rest frame of the mass, and in a frame where the mass has speed $v$. You'll need to use the Doppler effect.

*Section 11.3: Collisions and decays*

2. **Colliding photons**

   Two photons each have energy $E$. They collide at an angle $\theta$ and create a particle of mass $M$. What is $M$?

3. **Increase in mass**

   A large mass $M$, moving with speed $V$, collides and sticks to a small mass $m$, initially at rest. What is the mass of the resulting object? Work in the approximation where $M \gg m$.

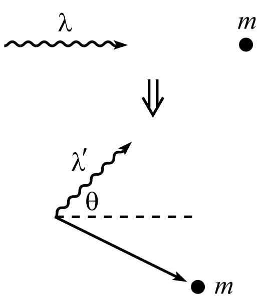

**Figure 11.20**

4. **Compton scattering** $**$

   A photon collides with a stationary electron. If the photon scatters at an angle $\theta$ (see Fig. 11.20), show that the resulting wavelength, $\lambda'$, is given in terms of the original wavelength, $\lambda$, by

   $$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta), \tag{11.81}$$

   where $m$ is the mass of the electron. Note: The energy of a photon is $E = h\nu = hc/\lambda$.

5. **Bouncing backwards** $**$

   A ball of mass $M$ and energy $E$ collides head-on elastically with a stationary ball of mass $m$. Show that the final energy of mass $M$ is

   $$E' = \frac{2mM^2 + E(m^2 + M^2)}{m^2 + M^2 + 2Em}\,. \tag{11.82}$$

   *Hint:* This problem is a little messy, but you can save yourself a lot of trouble by noting that $E' = E$ must be a root of an equation you get for $E'$. (Why?)

6. **Two-body decay** $*$

   A mass $M_A$ decays into masses $M_B$ and $M_C$. What are the energies of $M_B$ and $M_C$? What are their momenta?

7. **Threshold energy** $*$

A particle of mass $m$ and energy $E$ collides with an identical stationary particle. What is the threshold energy for a final state containing $N$ particles of mass $m$? ("Threshold energy" is the minimum energy for which the process occurs.)

*Section 11.5: Force*

8. **Relativistic harmonic oscillator** $**$

A particle of mass $m$ moves along the $x$-axis under a force $F = -m\omega^2 x$. The amplitude is $b$. Show that the period is given by

$$T = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}\, dx, \qquad (11.83)$$

where

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2). \qquad (11.84)$$

9. **System of masses** $**$

Consider a dumbbell made of two equal masses, $m$. The dumbbell spins around, with its center pivoted at the end of a stick (see Fig. 11.21). If the speed of the masses is $v$, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what is going on inside.)

Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the "transverse" position shown in the figure) and showing that $F \equiv dp/dt = Ma$.

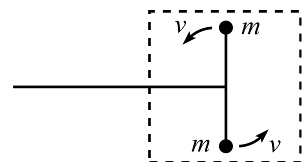

**Figure 11.21**

*Section 11.6: Rocket motion*

10. **Relativistic rocket** $**$

Consider the relativistic rocket from Section 11.6. Let mass be converted to photons at a rate $\sigma$ in the rest frame of the rocket. Find the time, $t$, in the ground frame as a function of $v$.[16] (Alas, it is not possible to invert this, to get $v$ as a function of $t$.)

11. **Relativistic dustpan I** $*$

A dustpan of mass $M$ is given an initial relativistic speed. It gathers up dust with mass density $\lambda$ per unit length on the floor (as measured in the lab frame). At the instant the speed is $v$, find the rate (as measured in the lab frame) at which the mass of the dustpan-plus-dust-inside system is increasing.

[16] This involves a slightly tricky integral. Pick your favorite method – pencil, book, or computer.

12. **Relativistic dustpan II** $**$

Consider the setup in Problem 11. If the initial speed of the dustpan is $V$, what are $v(x)$, $v(t)$, and $x(t)$? All quantities here are measured with respect to the lab frame.

13. **Relativistic dustpan III** $**$

Consider the setup in Problem 11. Calculate, in both the dustpan frame and lab frame, the force on the dustpan-plus-dust-inside system (due to the newly acquired dust particles smashing into it) as a function of $v$, and show that the results are equal.

14. **Relativistic cart I** $****$

A long cart moves at relativistic speed $v$. Sand is dropped into the cart at a rate $dm/dt = \sigma$ in the ground frame. Assume that you stand on the ground next to where the sand falls in, and you push on the cart to keep it moving at constant speed $v$. What is the force between your feet and the ground? Calculate this force in both the ground frame (your frame) and the cart frame, and show that the results are equal.

15. **Relativistic cart II** $****$

A long cart moves at relativistic speed $v$. Sand is dropped into the cart at a rate $dm/dt = \sigma$ in the ground frame. Assume that you grab the front of the cart and pull on it to keep it moving at constant speed $v$ (while running with it). What force does your hand apply to the cart? (Assume that the cart is made of the most rigid material possible.) Calculate this force in both the ground frame and the cart frame (your frame), and show that the results are equal.

*Section 11.6: Relativistic strings*

16. **Different frames** $**$

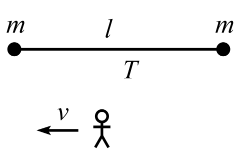

**Figure 11.22**

(a) Two masses $m$ are connected by a string of length $\ell$ and constant tension $T$. The masses are released simultaneously. They collide and stick together. What is the mass, $M$, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left with speed $v$ (see Fig. 11.22). The energy of the resulting blob must be $\gamma Mc^2$, from part (a). Show that you obtain this same result by computing the work done on the two masses.

17. **Splitting mass** $**$

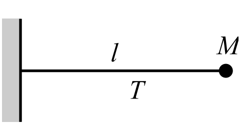

**Figure 11.23**

A massless string with constant tension $T$ has one end attached to a wall and the other end attached to a mass $M$. The initial length of the string is $\ell$ (see Fig. 11.23). The mass is released. Halfway to the wall, the back half of the mass breaks away from the front half (with zero initial relative speed). What is the total time it takes the front half to reach the wall?

18. **Relativistic leaky bucket** ∗∗∗

The mass $M$ in Problem 17 is replaced by a massless bucket containing an initial mass $M$ of sand (see Fig. 11.24). On the way to the wall, the bucket leaks sand at a rate $dm/dx = M/\ell$, where $m$ denotes the mass at later positions. (Note that the rate is constant with respect to distance, not time.)

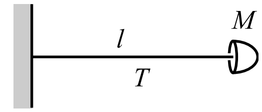


**Figure 11.24**

(a) What is the energy of the bucket, as a function of distance from the wall? What is its maximum value? What is the maximum value of the kinetic energy?

(b) What is the momentum of the bucket, as a function of distance from the wall? Where is it maximum?

19. **Relativistic bucket** ∗∗∗

(a) A massless string with constant tension $T$ has one end attached to a wall and the other end attached to a mass $m$. The initial length of the string is $\ell$ (see Fig. 11.25). The mass is released. How long does it take to reach the wall?

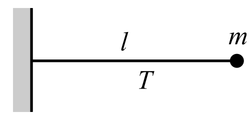


**Figure 11.25**

(b) Let the string now have length $2\ell$, with a mass $m$ on the end. Let another mass $m$ be positioned next to the $\ell$ mark on the string, but not touching the string (see Fig. 11.26). The right mass is released. It heads toward the wall (while the other mass is still motionless), and then sticks to the other mass to make one large blob, which then heads toward the wall.[17] How much time does this whole process take?[18]

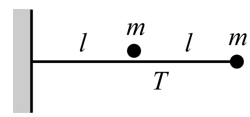


**Figure 11.26**

(c) Let there now be $N$ masses and a string of length $N\ell$ (see Fig. 11.27). How much time does this whole process take?

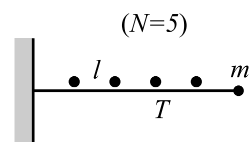


**Figure 11.27**

(d) Consider now a massless bucket at the end of the string (of length $L$) which gathers up a continuous stream of sand (of total mass $M$), as it gets pulled to the wall (see Fig. 11.28). How much time does this whole process take? What is the mass of the contents of the bucket right before it hits the wall?

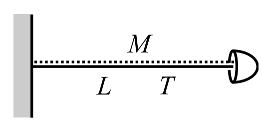


**Figure 11.28**

[17]The left mass could actually be attached to the string, and we would still have the same situation. The mass wouldn't move during the first part of the process, because there would be equal tensions $T$ on both sides of it.

[18]You can do this in various ways, but one method that generalizes nicely for the next part is to show that $\Delta(p^2) = (E_2^2 - E_1^2) + (E_4^2 - E_3^2)$, where the energies of the moving object (that is, the initial $m$ or the resulting blob) are: $E_1$ right at the start, $E_2$ just before the collision, $E_3$ just after the collision, and $E_4$ right before the wall. Note that this method does not require knowledge of the mass of the blob (which is *not* $2m$).

## 11.11 Solutions

1. **Deriving $E$ and $p$**

We'll derive the energy formula, $E = \gamma mc^2$, first. Let the given mass decay into two photons, and let $E_0$ be the energy of the mass in its rest frame. Then each of the resulting photons has energy $E_0/2$ in this frame.

Now look at the decay in a frame where the mass moves at speed $v$. From eq. (10.48), the frequencies of the photons are Doppler-shifted by the factors $\sqrt{(1+v)/(1-v)}$ and $\sqrt{(1-v)/(1+v)}$. Since the photons' energies are given by $E = h\nu$, their energies are shifted by these same factors, relative to the $E_0/2$ value in the original frame. Conservation of energy then says that in the moving frame, the mass (which is moving at speed $v$) has energy

$$E = \frac{E_0}{2}\sqrt{\frac{1+v}{1-v}} + \frac{E_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma E_0. \tag{11.85}$$

We therefore see that a moving mass has an energy that is $\gamma$ times its rest energy.

We will now use the correspondence principle (which says that relativistic formulas must reduce to the familiar nonrelativistic ones, in the nonrelativistic limit) to find $E_0$ in terms of $m$. We just found that the difference between the energies of a moving mass and a stationary mass is $\gamma E_0 - E_0$. This must reduce to the familiar kinetic energy, $mv^2/2$, in the limit $v \ll c$. In other words,

$$\begin{aligned} \frac{mv^2}{2} &\approx \frac{E_0}{\sqrt{1-v^2/c^2}} - E_0 \\ &\approx E_0\left(1+\frac{v^2}{2c^2}\right) - E_0 \\ &= \left(\frac{E_0}{c^2}\right)\frac{v^2}{2}, \end{aligned} \tag{11.86}$$

where we have used the Taylor series, $1/\sqrt{1-\epsilon} \approx 1 + \epsilon/2$. Therefore $E_0 = mc^2$, and hence $E = \gamma mc^2$.

We can derive the momentum formula, $p = \gamma mv$, in a similar way. Let the magnitude of the photons' (equal and opposite) momenta in the particle's rest frame be $p_0/2$.[19] Using the Doppler-shifted frequencies as above, we see that the total momentum of the photons in the frame where the mass moves at speed $v$ is

$$p = \frac{p_0}{2}\sqrt{\frac{1+v}{1-v}} - \frac{p_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma p_0 v. \tag{11.87}$$

Putting the $c$'s back in, we have $p = \gamma p_0 v/c$. By conservation of momentum, this is the momentum of the mass $m$ moving at speed $v$.

We can now use the correspondence principle to find $p_0$ in terms of $m$. If $p = \gamma(p_0/c)v$ is to reduce to the familiar $p = mv$ result in the limit $v \ll c$, then we must have $p_0 = mc$. Therefore, $p = \gamma mv$.

2. **Colliding photons**

The 4-momenta of the photons are (see Fig. 11.29)

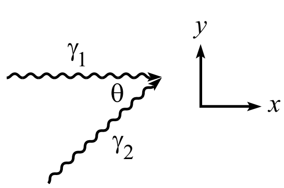


**Figure 11.29**

[19]With the given information that a photon has $E = h\nu$ and $p = h\nu/c$, we can use the preceding $E_0 = mc^2$ result to quickly conclude that $p_0 = mc$. But let's pretend that we haven't found $E_0$ yet. This will give us an excuse to use the correspondence principle again.

$$P_{\gamma_1} = (E, E, 0, 0), \qquad \text{and} \qquad P_{\gamma_2} = (E, E\cos\theta, E\sin\theta, 0). \tag{11.88}$$

Energy and momentum are conserved, so the 4-momentum of the final particle is $P_M = (2E, E + E\cos\theta, E\sin\theta, 0)$. Hence,

$$M^2 = P_M \cdot P_M = (2E)^2 - (E + E\cos\theta)^2 - (E\sin\theta)^2. \tag{11.89}$$

Therefore, the desired mass is

$$M = E\sqrt{2(1-\cos\theta)}\,. \tag{11.90}$$

If $\theta = 180^\circ$ then $M = 2E$, as it should (none of the final energy is kinetic). And if $\theta = 0^\circ$ then $M = 0$, as it should (all of the final energy is kinetic; we simply have a photon with twice the energy).

3. **Increase in mass**

In the lab frame, the energy of the resulting object is $\gamma M + m$, and the momentum is still $\gamma MV$. The mass of the object is therefore

$$M' = \sqrt{(\gamma M + m)^2 - (\gamma MV)^2} = \sqrt{M^2 + 2\gamma Mm + m^2}\,. \tag{11.91}$$

The $m^2$ term is negligible compared to the other two terms, so we may approximate $M'$ as

$$M' \approx M\sqrt{1 + \frac{2\gamma m}{M}} \approx M\left(1 + \frac{\gamma m}{M}\right) = M + \gamma m, \tag{11.92}$$

where we have used the Taylor series, $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$. Therefore, the increase in mass is $\gamma$ times the mass of the stationary object. (This increase must be greater than the nonrelativistic answer of "$m$", because heat is generated during the collision, and this heat shows up as mass in the final object.)

Remark: The $\gamma m$ result is quite clear if we work in the frame where $M$ is initially at rest. In this frame, the mass $m$ comes flying in with energy $\gamma m$, and essentially all of this energy shows up as mass in the final object. That is, essentially none of it shows up as overall kinetic energy of the object.

This is a general result. Stationary large objects pick up negligible kinetic energy when hit by small objects. This is true because the speed of the large object is proportional to $m/M$, by momentum conservation (there's a factor of $\gamma$ if things are relativistic), so the kinetic energy goes like $Mv^2 \propto M(m/M)^2 \approx 0$, if $M \gg m$. In other words, the smallness of $v$ wins out over the largeness of $M$. When a snowball hits a tree, all of the initial energy goes into heat to melt the snowball; (essentially) none of it goes into changing the kinetic energy of the earth. ♣

4. **Compton scattering**

The 4-momenta before the collision are (see Fig. 11.30)

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{11.93}$$

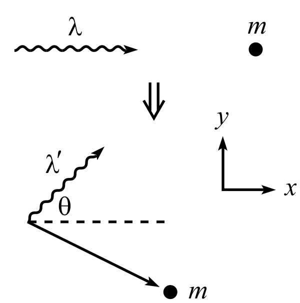

**Figure 11.30**

The 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{11.94}$$

If we wanted to, we could write $P'_m$ in terms of its momentum and scattering angle. But the nice thing about the following method is that we don't need to introduce these quantities which we're not interested in.

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned} (P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\ \Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\ \Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) &= m^2c^4. \end{aligned} \quad (11.95)$$

Multiplying through by $\lambda\lambda'/(hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta). \quad (11.96)$$

The nice thing about this solution is that all the unknown garbage in $P'_m$ disappeared when we squared it.

If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as expected.

If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (that is, $mc^2 \ll hc/\lambda = E_\gamma$, so the photon's energy is much larger than the electron's rest energy), then $\lambda' = 2h/mc$, so

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \quad (11.97)$$

Therefore, the photon bounces back with an essentially fixed $E'_\gamma$, independent of the initial $E_\gamma$ (as long as $E_\gamma$ is large enough). This isn't all that obvious.

5. **Bouncing backwards**

The 4-momenta before the collision are

$$P_M = (E, p, 0, 0), \qquad P_m = (m, 0, 0, 0), \quad (11.98)$$

where $p = \sqrt{E^2 - M^2}$. The 4-momenta after the collision are

$$P'_M = (E', p', 0, 0), \qquad P'_m = (\text{we won't need this}), \quad (11.99)$$

where $p' = \sqrt{E'^2 - M^2}$. If we wanted to, we could write $P'_m$ in terms of its momentum. But we don't need to introduce it. Conservation of energy and momentum give $P_M + P_m = P'_M + P'_m$. Therefore,

$$\begin{aligned} & (P_M + P_m - P'_M)^2 = P'^2_m \\ \Longrightarrow\quad & P^2_M + P^2_m + P'^2_M + 2P_m(P_M - P'_M) - 2P_M P'_M = P'^2_m \\ \Longrightarrow\quad & M^2 + m^2 + M^2 + 2m(E - E') - 2(EE' - pp') = m^2 \\ \Longrightarrow\quad & M^2 - EE' + m(E - E') = pp' \\ \Longrightarrow\quad & \Big((M^2 - EE') + m(E - E')\Big)^2 = \Big(\sqrt{E^2 - M^2}\sqrt{E'^2 - M^2}\,\Big)^2 \\ \Longrightarrow\quad & M^2(E^2 - 2EE' + E'^2) + 2(M^2 - EE')m(E - E') + m^2(E - E')^2 = 0. \end{aligned} \quad (11.100)$$

As claimed, $E' = E$ is a root of this equation. This is true because $E' = E$ and $p' = p$ certainly satisfy conservation of energy and momentum with the initial conditions, by definition. Dividing through by $(E - E')$ gives

$$M^2(E - E') + 2m(M^2 - EE') + m^2(E - E') = 0. \quad (11.101)$$

Solving for $E'$ gives the desired result,

$$E' = \frac{2mM^2 + E(m^2 + M^2)}{m^2 + M^2 + 2Em}\,. \tag{11.102}$$

We can double-check a few limits:

(a) $E \approx M$ (barely moving): then $E' \approx M$, because $M$ is still barely moving.

(b) $m \gg E$ (brick wall): then $E' \approx E$, because the heavy mass $m$ picks up essentially no energy.

(c) $M \gg m$: then $E' \approx E$, because it's essentially like $m$ is not there. Actually, this only holds if $E$ isn't too big; more precisely, we need $Em \ll M^2$.

(d) $M = m$: then $E' = M$, because $M$ stops and $m$ picks up all the energy that $M$ had.

(e) $E \gg m \gg M$: then $E' \approx m/2$. This isn't obvious, but it's similar to an analogous limit in the Compton scattering in Problem 4.

6. **Two-body decay**

$B$ and $C$ have equal and opposite momenta. Therefore,

$$E_B^2 - M_B^2 = p^2 = E_C^2 - M_C^2. \tag{11.103}$$

Also, conservation of energy gives

$$E_B + E_C = M_A. \tag{11.104}$$

Solving the two previous equations for $E_B$ and $E_C$ gives (using the shorthand $a \equiv M_A$, etc.)

$$E_B = \frac{a^2 + b^2 - c^2}{2a}, \qquad \text{and} \qquad E_C = \frac{a^2 + c^2 - b^2}{2a}. \tag{11.105}$$

Eq. (11.103) then gives the momentum of the particles as

$$p = \frac{\sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}}{2a}\,. \tag{11.106}$$

Remark: It turns out that the quantity under the radical may be factored into

$$(a + b + c)(a + b - c)(a - b + c)(a - b - c). \tag{11.107}$$

This makes it clear that if $a = b + c$, then $p = 0$, because there is no leftover energy for the particles to be able to move. ♣

7. **Threshold energy**

The initial 4-momenta are

$$(E, p, 0, 0), \qquad \text{and} \qquad (m, 0, 0, 0), \tag{11.108}$$

where $p = \sqrt{E^2 - m^2}$. Therefore, the final 4-momentum is $(E + m, p, 0, 0)$. The quantity $(E + m)^2 - p^2$ is an invariant, and it equals the square of the energy in the CM frame. At threshold, there is no relative motion among the final $N$ particles (because there is no leftover energy for such motion; see the remark below). So the energy in the CM frame is simply the sum of the rest energies, or $Nm$. We therefore have

$$(E + m)^2 - (E^2 - m^2) = (Nm)^2 \qquad \Longrightarrow \qquad E = \left(\frac{N^2}{2} - 1\right)m. \tag{11.109}$$

Note that $E \propto N^2$, for large $N$.

REMARK: Let's justify rigorously that the final particles should travel as a blob (that is, with no relative motion). Using the invariance of $E^2 - p^2$, and the fact that $p^{\rm CM} = 0$, we have

$$\begin{aligned} \left(E_{\rm f}^{\rm lab}\right)^2 - \left(p_{\rm f}^{\rm lab}\right)^2 &= \left(E_{\rm f}^{\rm CM}\right)^2 - \left(p_{\rm f}^{\rm CM}\right)^2 \\ \Longrightarrow \quad (E+m)^2 - \left(\sqrt{E^2 - m^2}\right)^2 &= \left(E_{\rm f}^{\rm CM}\right)^2 - 0 \\ \Longrightarrow \quad 2Em + 2m^2 &= \left(E_{\rm f}^{\rm CM}\right)^2. \end{aligned} \tag{11.110}$$

Therefore, minimizing $E$ is equivalent to minimizing $E_{\rm f}^{\rm CM}$. But $E_{\rm f}^{\rm CM}$ is clearly minimized when all the final particles are at rest in the CM frame (so there is no kinetic energy added to the rest energy). The minimum $E$ is therefore achieved when there is no relative motion among the final particles in the CM frame, and hence in any other frame. ♣

8. **Relativistic harmonic oscillator**

$F = dp/dt$ gives $-m\omega^2 x = d(m\gamma v)/dt$. Using eq. (11.44), we have

$$-\omega^2 x = \gamma^3 \frac{dv}{dt}\,. \tag{11.111}$$

We must somehow solve this differential equation. A helpful thing to do is to multiply both sides by $v$ to obtain $-\omega^2 x\dot{x} = \gamma^3 v\dot{v}$. The right-hand side of this is simply $d\gamma/dt$, as you can check. Integration then gives $-\omega^2 x^2/2 + C = \gamma$, where $C$ is a constant of integration. We know that $\gamma = 1$ when $x = b$, so we find

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2), \tag{11.112}$$

where we have put the $c$'s back in to make the units right.

The period is given by

$$T = 4\int_0^b \frac{dx}{v}\,. \tag{11.113}$$

But $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and so $v = c\sqrt{\gamma^2 - 1}/\gamma$. Therefore,

$$T = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}\,dx. \tag{11.114}$$

REMARK: In the limit $\omega b \ll c$ (so that $\gamma \approx 1$, from eq. (11.112); that is, the speed is always small), we must recover the Newtonian limit. Indeed, to lowest nontrivial order, $\gamma^2 \approx 1 + (\omega^2/c^2)(b^2 - x^2)$, and so

$$T \approx \frac{4}{c}\int_0^b \frac{dx}{(\omega/c)\sqrt{b^2 - x^2}}\,. \tag{11.115}$$

This is the correct result, because conservation of energy gives $v^2 = \omega^2(b^2 - x^2)$ for a nonrelativistic spring. ♣

9. **System of masses**

Let the speed of the stick go from 0 to $\epsilon$, where $\epsilon \ll v$. Then the final speeds of the two masses are obtained by relativistically adding and subtracting $\epsilon$ from $v$. (Assume that the time involved is small, so that the masses are still essentially moving horizontally.) Repeating the derivation leading to eq. (11.17), we see that the final momenta of the two masses have magnitudes $\gamma_v \gamma_\epsilon (v \pm \epsilon)m$. But since $\epsilon$ is small, we may set $\gamma_\epsilon \approx 1$, to first order.

Therefore, the forward-moving mass now has momentum $\gamma_v(v+\epsilon)m$, and the backward-moving mass now has momentum $-\gamma_v(v-\epsilon)m$. The net increase in momentum is thus (with $\gamma_v \equiv \gamma$) $\Delta p = 2\gamma m\epsilon$. Hence,

$$F \equiv \frac{\Delta p}{\Delta t} = 2\gamma m \frac{\epsilon}{\Delta t} \equiv 2\gamma m a = Ma. \tag{11.116}$$

10. **Relativistic rocket**

The relation between $m$ and $v$ obtained in eq. (11.60) is independent of the rate at which mass is converted to photons. We now assume a certain rate, in order to obtain a relation between $v$ and $t$.

In the frame of the rocket, we have $dm/d\tau = -\sigma$. From the usual time dilation effect, we then have $dm/dt = -\sigma/\gamma$ in the ground frame, because the ground frame sees the rocket's clocks run slow (that is, $dt = \gamma\, d\tau$).

Differentiating eq. (11.60), we have

$$dm = \frac{-M\, dv}{(1+v)\sqrt{1-v^2}}\,. \tag{11.117}$$

Using $dm = -(\sigma/\gamma)dt$, this becomes

$$\int_0^t \frac{\sigma}{M}\, dt = \int_0^v \frac{dv}{(1+v)(1-v^2)}\,. \tag{11.118}$$

We could simply use a computer to do this $dv$ integral, but let's do it from scratch. Using a few partial-fraction tricks, we have

$$\begin{aligned}
\int \frac{dv}{(1+v)(1-v^2)} &= \int \frac{dv}{(1+v)(1-v)(1+v)} \\
&= \frac{1}{2}\int \left(\frac{1}{1+v} + \frac{1}{1-v}\right)\frac{dv}{1+v} \\
&= \frac{1}{2}\int \frac{dv}{(1+v)^2} + \frac{1}{4}\int \left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv \\
&= -\frac{1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,. \qquad (11.119)
\end{aligned}$$

Equation (11.118) therefore gives

$$\frac{\sigma t}{M} = \frac{1}{2} - \frac{1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,. \tag{11.120}$$

Remarks: If $v \ll 1$ (or rather, if $v \ll c$), then we may Taylor-expand the quantities in eq. (11.120) to obtain $\sigma t/M \approx v$. This may be rewritten as $\sigma \approx M(v/t) \equiv Ma$. But $\sigma$ is simply the force acting on the rocket (or rather $\sigma c$, to make the units correct), because this is the change in momentum of the photons. We therefore obtain the expected nonrelativistic $F = ma$ equation.

If $v = 1 - \epsilon$, where $\epsilon$ is very small (that is, if $v$ is very close to $c$), then we can make approximations in eq. (11.120) to obtain $\epsilon \approx 2e/e^{4\sigma t/M}$. We see that the difference between $v$ and 1 decreases exponentially with $t$. ♣

11. **Relativistic dustpan I**

This problem is essentially the same as Problem 3.

Let $M$ be the mass of the dustpan-plus-dust-inside system (which we will label "$S$") when its speed is $v$. After a small time $dt$ in the lab frame, $S$ has moved a distance $v\,dt$, so it has basically collided with an infinitesimal mass $\lambda v\,dt$. Its energy therefore increases to $\gamma M + \lambda v\,dt$. Its momentum is still $\gamma M v$, so its mass is now

$$M' = \sqrt{(\gamma M + \lambda v\,dt)^2 - (\gamma M v)^2} \approx \sqrt{M^2 + 2\gamma M \lambda v\,dt}\,, \tag{11.121}$$

where we have dropped the second-order $dt^2$ terms. Using the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, we may approximate $M'$ as

$$M' \approx M\sqrt{1 + \frac{2\gamma\lambda v\,dt}{M}} \approx M\left(1 + \frac{\gamma\lambda v\,dt}{M}\right) = M + \gamma\lambda v\,dt. \tag{11.122}$$

The rate of increase in $S$'s mass is therefore $\gamma\lambda v$. (This increase must certainly be greater than the nonrelativistic answer of "$\lambda v$", because heat is generated during the collision, and this heat shows up as mass in the final object.)

REMARKS: This result is quite clear if we work in the frame where $S$ is at rest. In this frame, a mass $\lambda v\,dt$ comes flying in with energy $\gamma\lambda v\,dt$, and essentially all of this energy shows up as mass (heat) in the final object. That is, essentially none of it shows up as overall kinetic energy of the object, which is a general result when a small object hits a stationary large object.

Note that the rate at which the mass increases, as measured in $S$'s frame, is $\gamma^2\lambda v$, due to time dilation. (The dust-entering-dustpan events happen at the same location in the dustpan frame, so we have indeed put the extra $\gamma$ factor in the correct place.) Alternatively, you can view things in terms of length contraction. $S$ sees the dust contracted, so its density is increased to $\gamma\lambda$. ♣

12. **Relativistic dustpan II**

The initial momentum is $\gamma_V M V \equiv P$. There are no external forces, so the momentum of the dustpan-plus-dust-inside system (denoted by "$S$") always equals $P$. That is, $\gamma m v = P$, where $m$ and $v$ are the mass and speed of $S$ at later times.

Let's find $v(x)$ first. The energy of $S$, namely $\gamma m$, increases due to the acquisition of new dust. Therefore, $d(\gamma m) = \lambda\,dx$, which we can write as

$$d\left(\frac{P}{v}\right) = \lambda\,dx. \tag{11.123}$$

Integrating this, and using the fact that the initial speed is $V$, gives $P/v - P/V = \lambda x$. Therefore,

$$v(x) = \frac{V}{1 + \frac{V\lambda x}{P}}\,. \tag{11.124}$$

Note that for large $x$, this approaches $P/(\lambda x)$. This makes sense, because the mass of $S$ is essentially equal to $\lambda x$, and it is moving at a slow, nonrelativistic speed.

To find $v(t)$, write the $dx$ in eq. (11.123) as $v\,dt$ to obtain $(-P/v^2)\,dv = \lambda v\,dt$. Hence,

$$-\int_V^v \frac{P\,dv}{v^3} = \int_0^t \lambda\,dt \qquad \Longrightarrow \qquad \frac{P}{v^2} - \frac{P}{V^2} = 2\lambda t$$

$$\Longrightarrow \qquad v(t) = \frac{V}{\sqrt{1 + \frac{2\lambda V^2 t}{P}}}\,. \tag{11.125}$$

At this point, there are various ways to find $x(t)$. The simplest one is to just integrate eq. (11.125). The result is

$$x(t) = \frac{P}{V\lambda}\left(\sqrt{1+\frac{2V^2\lambda t}{P}}-1\right). \tag{11.126}$$

You can show that this reduces to $x = Vt$ for small $t$, as it should. For large $t$, $x$ has the interesting property of being proportional to $\sqrt{t}$.

13. **Relativistic dustpan III**

Let $S$ denote the dustpan-plus-dust-inside system at a given time, and consider a small bit of dust (call this subsystem $s$) that enters the dustpan. In $S$'s frame, the density of the dust is $\gamma\lambda$, due to length contraction. Therefore, in a time $d\tau$ (where $\tau$ is the time in the dustpan frame), a little $s$ system of dust with mass $\gamma\lambda v\, d\tau$ crashes into $S$ and loses its (negative) momentum of $-(\gamma\lambda v\, d\tau)(\gamma v) = -\gamma^2 v^2\lambda\, d\tau$. The force on $s$ is therefore $+\gamma^2v^2\lambda$. The desired force on $S$ is equal and opposite to this, so

$$F = -\gamma^2 v^2\lambda. \tag{11.127}$$

Now consider the lab frame. In a time $dt$ (where $t$ is the time in the lab frame), a little $s$ system of dust with mass $\lambda v\, dt$ gets picked up by the dustpan. What is the change in momentum of $s$? It is tempting to say that it is $(\lambda v\, dt)(\gamma v)$, but this would lead to a force of $-\gamma v^2\lambda$ on the dustpan, which doesn't agree with the result we found above in the dustpan frame. This would be a problem, because longitudinal forces should be the same in different frames.

The key point to realize is that the mass of whatever is moving increases at a rate $\gamma\lambda v$, and not $\lambda v$ (see Problem 11). We therefore see that the change in momentum of the additional moving mass is $(\gamma\lambda v\, dt)(\gamma v) = \gamma^2v^2\lambda\, dt$. The original moving system $S$ therefore loses this much momentum, and so the force on it is $F = -\gamma^2v^2\lambda$, in agreement with the result in the dustpan frame.

14. **Relativistic cart I**

**Ground frame (your frame):** Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate

$$\frac{dP}{dt} = \gamma(\gamma\sigma)v = \gamma^2\sigma v. \tag{11.128}$$

This is the force you exert on the cart, so it is also the force the ground exerts on your feet (because the net force on you is zero).

**Cart frame:** The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate $\sigma/\gamma$. The sand flies in at speed $v$, and then eventually comes at rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$. This is the force your hand applies to the cart.

If this were the only change in momentum in the problem, then we would be have a problem, because the force on your feet would be $\sigma v$ in the cart frame, whereas we found above that it is $\gamma^2\sigma v$ in the ground frame. This would contradict the fact that longitudinal forces are the same in different frames. What is the resolution to this apparent paradox?

The resolution is that while you are pushing on the cart, *your mass is decreasing.* You are moving with speed $v$ in the cart frame, and mass is continually being transferred from you (who are moving) to the cart (which is at rest). This is the missing change in momentum we need. Let's be quantitative about this.

Go back to the ground frame for a moment. We found above that the mass the cart-plus-sand-inside system (call this "$C$") increases at rate $\gamma\sigma$ in the ground frame. Therefore, the energy of $C$ increases at a rate $\gamma(\gamma\sigma)$ in the ground frame. The sand provides $\sigma$ of this energy, so you must provide the remaining $(\gamma^2-1)\sigma$ part. Therefore, since you are losing energy at this rate, you must also be losing mass at this rate in the ground frame (because you are at rest there).

Now go back to the cart frame. Due to time dilation, you lose mass at a rate of only $(\gamma^2-1)\sigma/\gamma$. This mass goes from moving at speed $v$ (that is, along with you), to speed zero (that is, at rest on the cart). Therefore, the rate of decrease in momentum of this mass is $\gamma\big((\gamma^2-1)\sigma/\gamma\big)v = (\gamma^2-1)\sigma v$.

Adding this result to the $\sigma v$ result due to the sand, we see that the total rate of decrease in momentum is $\gamma^2\sigma v$. This is therefore the force that the ground applies to your feet, in agreement with the calculation in the ground frame.

15. **Relativistic cart II**

**Ground frame:** Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate $\gamma(\gamma\sigma)v = \gamma^2\sigma v$.

However, this is *not* the force that your hand exerts on the cart. The reason is that the sand enters the cart at locations that are receding from your hand, so your hand cannot immediately be aware of the additional need for momentum. No matter how rigid the cart is, it can't transmit information faster than $c$. In a sense, there is a sort of Doppler effect going on, and your hand only needs to be responsible for a certain fraction of the momentum increase. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time $t$ apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed $c$), then the relative speed (as measured by someone on the ground) of the signals and your hand is $c-v$. The distance between the two signals is $ct$. Therefore, they arrive at your hand separated by a time of $ct/(c-v)$. In other words, the rate at which you feel sand entering the cart is $(c-v)/c$ times the given $\sigma$ rate. This is the factor by which we must multiply the naive $\gamma^2\sigma v$ result for the force we found above. The force you must apply is therefore

$$F = \left(1 - \frac{v}{c}\right)\gamma^2\sigma v = \frac{\sigma v}{1+v}\,. \tag{11.129}$$

**Cart frame (your frame):** The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame; that is, at a rate $\sigma/\gamma$. The sand flies in at speed $v$, and then eventually comes to rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$.

But again, this is *not* the force that your hand exerts on the cart. As before, the sand enters the cart at a location far from your hand, so your hand cannot immediately be aware of the additional need for momentum. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time $t$ apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed $c$), then the relative speed (as measured by someone on the cart) of the signals and your hand is $c$ (because you are at rest). The distance between the two signals is $ct + vt$, because the sand source is moving away from you at speed $v$. Therefore, the signals arrive at your hand separated by a time of $(c+v)t/c$. In other words, the rate at which you feel sand entering the cart is $c/(c+v)$ times the $\sigma/\gamma$ rate found above. This is the factor by which we must multiply the naive $\sigma v$ result for the force we found above. The force you must apply is therefore

$$F = \left(\frac{1}{1+v/c}\right)\sigma v = \frac{\sigma v}{1+v}\,, \tag{11.130}$$

in agreement with eq. (11.129).

In a nutshell, the two naive results in the two frames, $\gamma^2\sigma v$ and $\sigma v$, differ by two factors of $\gamma$. The ratio of the two "Doppler-effect" factors (which arose from the impossibility of absolute rigidity) precisely remedies this discrepancy.

16. **Different frames**

(a) The energy of the resulting blob is $2m + T\ell$. Since the blob is at rest, we have

$$M = 2m + T\ell. \tag{11.131}$$

(b) Let the new frame be $S$. Let the original frame be $S'$. The critical point to realize is that in frame $S$ the left mass begins to accelerate before the right mass does. This is due to the loss of simultaneity between the frames. Note that the longitudinal force is the same in the two frames, so the masses still feel a tension $T$ in frame $S$.

Consider the two events when the two masses start to move. Let the left mass and right mass start moving at positions $x_l$ and $x_r$ in $S$. The Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$ tells us that $x_r - x_l = \gamma\ell$, because $\Delta x' = \ell$ and $\Delta t' = 0$ for these events.

Let the masses collide at position $x_c$ in $S$. Then the gain in energy of the left mass is $T(x_c - x_l)$, and the gain in energy of the right mass is $(-T)(x_c - x_r)$ (which is negative if $x_c > x_r$). The gain in the sum of the energies is therefore

$$\Delta E = T(x_c - x_l) + (-T)(x_c - x_r) = T(x_r - x_l) = T\gamma\ell. \tag{11.132}$$

The initial sum of energies was $2\gamma m$, so the final energy is

$$E = 2\gamma m + \gamma T\ell = \gamma M, \tag{11.133}$$

as desired.

17. **Splitting mass**

We'll calculate the times for the two parts of the process to occur.

The energy of the mass right before it splits is $E_b = M + T(\ell/2)$, so the momentum is $p_b = \sqrt{E_b^2 - M^2} = \sqrt{MT\ell + T^2\ell^2/4}$. Using $F = dp/dt$, the time for the first part of the process is

$$t_1 = \frac{1}{T}\sqrt{MT\ell + T^2\ell^2/4}\,. \tag{11.134}$$

The momentum of the front half of the mass immediately after it splits is $p_a = p_b/2 = (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The energy at the wall is $E_w = E_b/2 + T(\ell/2) = M/2 + 3T\ell/4$, so the momentum at the wall is $p_w = \sqrt{E_w^2 - (M/2)^2} = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4}$. The change in momentum during the second part of the process is therefore $\Delta p = p_w - p_a = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4} - (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The time for the second part is thus

$$t_2 = \frac{1}{2T}\left(\sqrt{3MT\ell + 9T^2\ell^2/4} - \sqrt{MT\ell + T^2\ell^2/4}\right). \tag{11.135}$$

The total time is $t_1 + t_2$, which simply changes the minus sign in the above expression to a plus sign.

18. **Relativistic leaky bucket**

(a) Let the wall be at $x = 0$, and let the initial position be at $x = \ell$. Consider a small interval during which the bucket moves from $x$ to $x + dx$ (where $dx$ is negative). The bucket's energy changes by $(-T)\,dx$ due to the string, and it also changes by a fraction $dx/x$, due to the leaking. Therefore, $dE = (-T)\,dx + E\,dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}\,. \tag{11.136}$$

In solving this differential equation, it is convenient to introduce the variable $y = E/x$. Then $E' = xy' + y$, where a prime denotes differentiation with respect to $x$. Eq. (11.136) then becomes $xy' = -T$, or

$$dy = \frac{-T dx}{x}\,. \tag{11.137}$$

Integration gives $y = -T\ln x + C$, which we may write as $y = -T\ln(x/\ell) + B$, in order to have a dimensionless argument in the log. Since $E = xy$, we therefore have

$$E = Bx - Tx\ln(x/\ell). \tag{11.138}$$

The reasoning up to this point is valid for both the total energy and the kinetic energy. Let's now look at each of these cases.

- *Total energy:* Eq. (11.138) gives

$$E = M(x/\ell) - Tx\ln(x/\ell), \tag{11.139}$$

where the constant of integration, $B$, has been chosen so that $E = M$ when $x = \ell$. To find the maximum of $E$, it is more convenient to work with the fraction $z \equiv x/\ell$, in terms of which $E = Mz - T\ell z\ln z$. Setting $dE/dz$ equal to zero gives

$$\ln z = \frac{M}{T\ell} - 1 \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{T\ell}{e}\, e^{M/T\ell}. \tag{11.140}$$

The fraction $z$ must satisfy $z \le 1$, so we must have $\ln z \le 0$. Therefore, a solution for $z$ exists only for $M \le T\ell$. If $M \ge T\ell$, then the energy decreases all the way to the wall.

If $M$ is slightly less then $T\ell$, then $z$ is slightly less than 1, so $E$ quickly achieves a maximum of slightly more than $M$, then decreases for the rest of the way to the wall.

If $M \ll T\ell$, then $E$ achieves its maximum at $z \approx 1/e$, where it has the value $T\ell/e$.

- *Kinetic energy:* Eq. (11.138) gives

$$KE = -Tx\ln(x/\ell), \tag{11.141}$$

where the constant of integration, $B$, has been chosen so that $KE = 0$ when $x = \ell$. Equivalently, $E - KE$ must equal the mass $M(x/\ell)$. In terms of the fraction $z \equiv x/\ell$, we have $KE = -T\ell z \ln z$. Setting $d(KE)/dz$ equal to zero gives

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad KE_{\text{max}} = \frac{T\ell}{e}\,, \tag{11.142}$$

which is independent of $M$. Since this result is independent of $M$, it must hold in the nonrelativistic limit. And indeed, the analogous "Leaky-bucket" problem in Chapter 4 (Problem 4.16) gave the same result.

(b) Eq. (11.139) gives, with $z \equiv x/\ell$,

$$\begin{aligned} p = \sqrt{E^2 - (Mz)^2} &= \sqrt{(Mz - T\ell z\ln z)^2 - (Mz)^2} \\ &= \sqrt{-2MT\ell z^2 \ln z + T^2\ell^2 z^2 \ln^2 z}\,. \end{aligned} \tag{11.143}$$

Setting the derivative equal to zero gives $T\ell\ln^2 z + (T\ell - 2M)\ln z - M = 0$. The maximum momentum therefore occurs at

$$\ln z = \frac{2M - T\ell - \sqrt{T^2\ell^2 + 4M^2}}{2T\ell}\,. \tag{11.144}$$

We have ignored the other root, because it gives $\ln z > 0$.

If $M \ll T\ell$, then the maximum $p$ occurs at $z \approx 1/e$. In this case, the bucket immediately becomes relativistic, so we have $E \approx pc$. Therefore, both $E$ and $p$ should achieve their maxima at the same place. This agrees with the result for $E$ above.

If $M \gg T\ell$, then the maximum $p$ occurs at $z \approx 1/\sqrt{e}$. In this case, the bucket is nonrelativistic, so the result should agree with the analogous "Leaky-bucket" problem in Chapter 4 (Problem 4.16), which it does.

19. **Relativistic bucket**

(a) The mass's energy just before it hits the wall is $E = m + T\ell$. Therefore, the momentum just before it hits the wall is $p = \sqrt{E^2 - m^2} = \sqrt{2mT\ell + T^2\ell^2}$. $F = \Delta p/\Delta t$ then gives (using the fact that the tension is constant)

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{2mT\ell + T^2\ell^2}}{T}\,. \tag{11.145}$$

If $m \ll T\ell$, then $\Delta t \approx \ell$ (or $\ell/c$ in normal units), which is correct, because the mass essentially travels at speed $c$.

If $m \gg T\ell$, then $\Delta t \approx \sqrt{2m\ell/T}$. This is the nonrelativistic limit, and it agrees with the result obtained from the familiar $\ell = at^2/2$, where $a = T/m$ is the acceleration.

(b) *Straightforward method:* The energy of the blob right before it hits the wall is $E_w = 2m + 2T\ell$. If we can find the mass, $M$, of the blob, then we can use $p = \sqrt{E^2 - M^2}$ to get the momentum, and then use $\Delta t = \Delta p/F$ to get the time.[20]

[20] Note that although the tension $T$ acts on two different things (the mass $m$ initially, and then the blob), it is valid to use the total $\Delta p$ to obtain the total time $\Delta t$ via $\Delta t = \Delta p/F$, simply because we could break up the $\Delta p$ into its two parts, and then find the two partial times, and then add them back together to get the total $\Delta t$.

The momentum right before the collision is $p_b = \sqrt{2mT\ell + T^2\ell^2}$, and this is also the momentum of the blob right after the collision, $p_a$.
The energy of the blob right after the collision is $E_a = 2m + T\ell$. So the mass of the blob after the collision is $M = \sqrt{E_a^2 - p_a^2} = \sqrt{4m^2 + 2mT\ell}$.
Therefore, the momentum at the wall is $p_w = \sqrt{E_w^2 - M^2} = \sqrt{6mT\ell + 4T^2\ell^2}$, and hence

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{6mT\ell + 4T^2\ell^2}}{T}. \tag{11.146}$$

Note that if $m = 0$ then $\Delta t = 2\ell$, as it should.

*Better method:* In the notation in the footnote in the statement of the problem, the change in $p^2$ from the start to just before the collision is $\Delta(p^2) = E_2^2 - E_1^2$. This is true because

$$E_1^2 - m^2 = p_1^2, \qquad \text{and} \qquad E_2^2 - m^2 = p_2^2, \tag{11.147}$$

and since $m$ is the same throughout the first half of the process, we have $\Delta(E^2) = \Delta(p^2)$.
Likewise, the change in $p^2$ during the second half of the process is $\Delta(p^2) = E_4^2 - E_3^2$, because

$$E_3^2 - M^2 = p_3^2, \qquad \text{and} \qquad E_4^2 - M^2 = p_4^2, \tag{11.148}$$

and since $M$ is the same throughout the second half of the process,[21] we have $\Delta(E^2) = \Delta(p^2)$.
The total change in $p^2$ is the sum of the above two changes, so the final $p^2$ is

$$\begin{aligned} p^2 &= (E_2^2 - E_1^2) + (E_4^2 - E_3^2) \\ &= \Big((m + T\ell)^2 - m^2\Big) + \Big((2m + 2T\ell)^2 - (2m + T\ell)^2\Big) \\ &= 6mT\ell + 4T^2\ell^2, \end{aligned} \tag{11.149}$$

as in eq. (11.146). The first solution above basically performs the same calculation, but in a more obscure manner.

(c) The reasoning in part (b) tells us that the final $p^2$ equals the sum of the $\Delta(E^2)$ terms over the $N$ parts of the process. So we have, using an indexing notation analogous to that in part (b),

$$\begin{aligned} p^2 &= \sum_{k=1}^{N} \Big(E_{2k}^2 - E_{2k-1}^2\Big) \\ &= \sum \Big((km + kT\ell)^2 - \big(km + (k-1)T\ell\big)^2\Big) \\ &= \sum \Big(2kmT\ell + \big(k^2 - (k-1)^2\big)T^2\ell^2\Big) \\ &= N(N+1)mT\ell + N^2T^2\ell^2. \end{aligned} \tag{11.150}$$

Therefore,

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{N(N+1)mT\ell + N^2T^2\ell^2}}{T}. \tag{11.151}$$

This checks with the results from parts (a) and (b).

[21] $M$ happens to be $\sqrt{4m^2 + 2mT\ell}$, but the nice thing about this solution is that we don't need to know this. All we need to know is that it is constant.

(d) We want to take the limit $N \to \infty$, $\ell \to 0$, $m \to 0$, with the restrictions that $N\ell = L$ and $Nm = M$. Written in terms of $M$ and $L$, the result in part (c) is

$$\Delta t = \frac{\sqrt{(1+1/N)MTL + T^2L^2}}{T} \quad \longrightarrow \quad \frac{\sqrt{MTL + T^2L^2}}{T}\,, \tag{11.152}$$

as $N \to \infty$. Note that this time equals the time is takes for one particle of mass $m = M/2$ to reach the wall, from part (a).

The mass, $M_f$, of the final blob at the wall is

$$\begin{aligned} M_f = \sqrt{E_w^2 - p_w^2} &= \sqrt{(M+TL)^2 - (MTL + T^2L^2)} \\ &= \sqrt{M^2 + MTL}\,. \end{aligned} \tag{11.153}$$

If $TL \ll M$, then $M_f \approx M$, which makes sense. If $M \ll TL$, then $M_f \approx \sqrt{MTL}$, so $M_f$ is the geometric mean between the given mass and the energy stored in the string, which isn't entirely obvious.

# Chapter 12

# 4-vectors



We now come to a very powerful concept in relativity, namely that of *4-vectors*. Although it is possible to derive everything in special relativity without the use of 4-vectors (and indeed, this is the route, give or take, that we took in the previous two chapters), they are *extremely* helpful in making calculations and concepts much simpler and more transparent.

I have chosen to postpone the introduction to 4-vectors until now, in order to make it clear that everything in special relativity can be derived without them. In encountering relativity for the first time, it's nice to know that no "advanced" techniques are required. But now that you've seen everything once, let's go back and derive various things in an easier way.

This situation, where 4-vectors are helpful but not necessary, is more pronounced in general relativity, where the concept of *tensors* (the generalization of 4-vectors) is, for all practical purposes, completely necessary for an understanding of the subject. We won't have time to go very deeply into GR in Chapter 13, so you'll have to just accept this fact. But suffice it to say that an eventual understanding of GR requires a firm understanding of special-relativity 4-vectors.

## 12.1 Definition of 4-vectors

**Definition 12.1** *The 4-tuplet, $A = (A_0, A_1, A_2, A_3)$, is a "4-vector" if the $A_i$ transform under a Lorentz transformation in the same way that $(c\,dt, dx, dy, dz)$ do. In other words, they must transform like (assuming the LT is along the $x$-direction; see Fig. 12.1):*

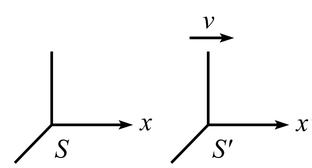

**Figure 12.1**

$$\begin{aligned} A_0 &= \gamma(A'_0 + (v/c)A'_1), \\ A_1 &= \gamma(A'_1 + (v/c)A'_0), \\ A_2 &= A'_2, \\ A_3 &= A'_3. \end{aligned} \qquad (12.1)$$

Remarks:

1. Similar equations must hold, of course, for Lorentz transformations in the $y$- and $z$-directions.

2. Additionally, the last three components must be a vector in 3-space. That is, they must transform like a usual vector under rotations in 3-space.
3. We'll use a capital Roman letter to denote a 4-vector. A bold-face letter will denote, as usual, a vector in 3-space.
4. Lest we get tired of writing the $c$'s over and over, we will henceforth work in units where $c = 1$.
5. The first component of a 4-vector is called the "time" component. The other three are the "space" components.
6. The components in $(dt, dx, dy, dz)$ are sometimes referred to as $(dx_0, dx_1, dx_2, dx_3)$. Also, some treatments use the indices "1" through "4", with "4" being the "time" component. But we'll use "0" through "3".
7. The $A_i$ may be functions of the $dx_i$, the $x_i$ and their derivatives, any invariants (that is, frame-independent quantities) such as the mass $m$, and $v$.
8. 4-vectors are the obvious generalization of vectors in regular space. A vector in 3-dimensions, after all, is something that transforms under a rotation just like $(dx, dy, dz)$ does. We have simply generalized a 3-D rotation to a 4-D Lorentz transformation. ♣

## 12.2 Examples of 4-vectors

So far, we have only one 4-vector at our disposal, namely $(dt, dx, dy, dz)$. What are some others? Well, $(7dt, 7dx, 7dy, 7dz)$ certainly works, as does any other constant multiple of $(dt, dx, dy, dz)$. Indeed, $m(dt, dx, dy, dz)$ is a 4-vector, because $m$ is an invariant (independent of frame).

How about $A = (dt, 2dx, dy, dz)$? No, this isn't a 4-vector, because on one hand it must transform (assuming it's a 4-vector) like

$$\begin{aligned} dt \equiv A_0 &= \gamma(A'_0 + vA'_1) \equiv \gamma\Big(dt' + v(2\,dx')\Big), \\ 2\,dx \equiv A_1 &= \gamma(A'_1 + vA'_0) \equiv \gamma\Big((2\,dx') + v\,dt'\Big), \\ dy \equiv A_2 &= A'_2 \equiv dy', \\ dz \equiv A_3 &= A'_3 \equiv dz', \end{aligned} \tag{12.2}$$

from the definition of a 4-vector. But on the other hand, it transforms like

$$\begin{aligned} dt &= \gamma(dt' + v\,dx'), \\ 2\,dx &= 2\gamma(dx' + v\,dt'), \\ dy &= dy', \\ dz &= dz', \end{aligned} \tag{12.3}$$

because this is how the $dx_i$ transform. The two preceding sets of equations are inconsistent, so $A = (dt, 2dx, dy, dz)$ is not a 4-vector. Note that if we had instead considered the 4-tuplet, $A = (dt, dx, 2dy, dz)$, then the two preceding equations would have been consistent. But if we had then looked at how $A$ transforms under a Lorentz transformation in the $y$-direction, we would have found that it is not a 4-vector.

The moral of this story is that the above definition of a 4-vector is a nontrivial one because there are two possible ways that a 4-tuplet can transform. It can transform according to the 4-vector definition, as in eq. (12.2). Or, it can transform by simply having each of the $A_i$ transform separately (knowing how the $dx_i$ transform), as in eq. (12.3). Only for certain special 4-tuplets do these two methods give the same result. By definition, we label these special 4-tuplets as 4-vectors.

Let us now construct some less trivial examples of 4-vectors. In constructing these, we will make abundant use of the fact that the proper-time interval, $d\tau \equiv \sqrt{dt^2 - d\mathbf{r}^2}$, is an invariant.

- **Velocity 4-vector:** We can divide $(dt, dx, dy, dz)$ by $d\tau$, where $d\tau$ is the proper time between two events (the same two events that yielded the $dt$, etc.). The result is indeed a 4-vector, because $d\tau$ is independent of the frame in which it is measured. Using $d\tau = dt/\gamma$, we obtain

$$V \equiv \frac{1}{d\tau}(dt, dx, dy, dz) = \gamma\left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma, \gamma\mathbf{v}) \tag{12.4}$$

is a 4-vector. This is known as the *velocity 4-vector*. In the rest frame of the object we have $\mathbf{v} = \mathbf{0}$, so $V$ reduces to $V = (1, 0, 0, 0)$. With the $c$'s, we have $V = (\gamma c, \gamma\mathbf{v})$.

- **Energy-momentum 4-vector:** If we multiply the velocity 4-vector by the invariant $m$, we obtain another 4-vector,

$$P \equiv mV = (\gamma m, \gamma m\mathbf{v}) = (E, \mathbf{p}), \tag{12.5}$$

which is known as the *energy-momentum 4-vector* (or the *4-momentum* for short), for obvious reasons. In the rest frame of the object, $P$ reduces to $P = (m, 0, 0, 0)$. With the $c$'s, we have $P = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$. Some treatments multiply through by $c$, so that the 4-momentum is $(E, \mathbf{p}c)$.

- **Acceleration 4-vector:** We can also take the derivative of the velocity 4-vector with respect to $\tau$. The result is indeed a 4-vector, because taking the derivative simply entails taking the difference between two 4-vectors (which results in a 4-vector because eq. (12.1) is linear), and then dividing by the invariant $d\tau$ (which again results in a 4-vector). Using $d\tau = dt/\gamma$, we obtain

$$A \equiv \frac{dV}{d\tau} = \frac{d}{d\tau}(\gamma, \gamma\mathbf{v}) = \gamma\left(\frac{d\gamma}{dt}\,, \frac{d(\gamma\mathbf{v})}{dt}\right). \tag{12.6}$$

Using $d\gamma/dt = v\dot{v}/(1-v^2)^{3/2} = \gamma^3 v\dot{v}$, we have

$$A = (\gamma^4 v\dot{v}\,,\ \gamma^4 v\dot{v}\mathbf{v} + \gamma^2\mathbf{a}), \tag{12.7}$$

where $\mathbf{a} \equiv d\mathbf{v}/dt$. $A$ is known as the *acceleration 4-vector*. In the rest frame of the object (or, rather, in the instantaneous inertial frame), $A$ reduces to $A = (0, \mathbf{a})$.

As we always do, we will pick the relative velocity, $\mathbf{v}$, to point in the $x$-direction. That is, $\mathbf{v} = (v_x, 0, 0)$. This means that $v = v_x$, and also that $\dot{v} = \dot{v}_x \equiv a_x$.[1] Eq. (12.7) then becomes

$$\begin{aligned} A &= (\gamma^4 v_x a_x \,,\, \gamma^4 v_x^2 a_x + \gamma^2 a_x \,,\, \gamma^2 a_y \,,\, \gamma^2 a_z) \\ &= (\gamma^4 v_x a_x \,,\, \gamma^4 a_x \,,\, \gamma^2 a_y \,,\, \gamma^2 a_z). \end{aligned} \tag{12.8}$$

We can keep taking derivatives with respect to $\tau$ to create other 4-vectors, but these have little relevance in the real world.

- **Force 4-vector:** We define the *force 4-vector* as

$$F \equiv \frac{dP}{d\tau} = \gamma\left(\frac{dE}{dt}, \frac{d\mathbf{p}}{dt}\right) = \gamma\left(\frac{dE}{dt}, \mathbf{f}\right), \tag{12.9}$$

where $\mathbf{f} \equiv d(\gamma m \mathbf{v})/dt$ is the usual 3-force. We'll use $\mathbf{f}$ instead of $\mathbf{F}$ in this chapter, to avoid confusion with the 4-force, $F$.

In the case where $m$ is constant,[2] $F$ can be written as $F = d(mV)/d\tau = m\, dV/d\tau = mA$. We therefore still have a nice "$F$ equals $mA$" law of physics, but it's now a 4-vector equation instead of the old 3-vector one. In terms of the acceleration 4-vector, we may use eq. (12.7) to write (if $m$ is constant)

$$F = mA = (\gamma^4 m v \dot{v} \,,\, \gamma^4 m v \dot{v} \mathbf{v} + \gamma^2 m \mathbf{a}). \tag{12.10}$$

In the rest frame of the object (or, rather, the instantaneous inertial frame), $F$ reduces to $F = (0, \mathbf{f})$, because $dE/dt = 0$, as you can verify. Also, $mA$ reduces to $mA = (0, m\mathbf{a})$. Therefore, $F = mA$ reduces to the familiar $\mathbf{f} = m\mathbf{a}$.

## 12.3 Properties of 4-vectors

The appealing thing about 4-vectors is that they have many useful properties. Let's look at some of these.

- **Linear combinations:** If $A$ and $B$ are 4-vectors, then $C \equiv aA + bB$ is also a 4-vector. This is true because the transformations in eq. (12.1) are linear (as we noted above when deriving the acceleration 4-vector). This linearity implies that the transformation of, say, the time component is

$$\begin{aligned} C_0 \equiv (aA + bB)_0 = aA_0 + bB_0 &= a(A'_0 + vA'_1) + b(B'_0 + vB'_1) \\ &= (aA'_0 + bB'_0) + v(aA'_1 + bB'_1) \\ &\equiv C'_0 + vC'_1, \end{aligned} \tag{12.11}$$

which is the proper transformation for the time component of a 4-vector. Likewise for the other components. This property holds, of course, just as it does for linear combinations of vectors in 3-space.

[1] The acceleration vector, $\mathbf{a}$, is free to point in any direction, but you can check that the 0's in $\mathbf{v}$ lead to $\dot{v} = a_x$. See Exercise 1.

[2] The mass $m$ would not be constant if the object were being heated, or if extra mass were being added to it. We won't concern ourselves with such cases here.

- **Inner-product invariance:** Consider two arbitrary 4-vectors, $A$ and $B$. Define their inner product to be

$$A \cdot B \equiv A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 \equiv A_0B_0 - \mathbf{A}\cdot\mathbf{B}. \tag{12.12}$$

Then $A \cdot B$ is invariant. That is, it is independent of the frame in which it is calculated. This can be shown by direct calculation, using the transformations in eq. (12.1):

$$\begin{aligned}
A \cdot B &\equiv A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 \\
&= \Big(\gamma(A_0' + vA_1')\Big)\Big(\gamma(B_0' + vB_1')\Big) - \Big(\gamma(A_1' + vA_0')\Big)\Big(\gamma(B_1' + vB_0')\Big) \\
&\qquad -A_2'B_2' - A_3'B_3' \\
&= \gamma^2\Big(A_0'B_0' + v(A_0'B_1' + A_1'B_0') + v^2A_1'B_1'\Big) \\
&\quad -\gamma^2\Big(A_1'B_1' + v(A_1'B_0' + A_0'B_1') + v^2A_0'B_0'\Big) \\
&\qquad -A_2'B_2' - A_3'B_3' \\
&= A_0'B_0'(\gamma^2 - \gamma^2v^2) - A_1'B_1'(\gamma^2 - \gamma^2v^2) - A_2'B_2' - A_3'B_3' \\
&= A_0'B_0' - A_1'B_1' - A_2'B_2' - A_3'B_3' \\
&\equiv A' \cdot B'.
\end{aligned} \tag{12.13}$$

The importance of this result cannot be overstated. This invariance is analogous to the invariance of the inner product, $\mathbf{A}\cdot\mathbf{B}$, for rotations in 3-space. The above inner product is also invariant under rotations in 3-space, because it involves the combination $\mathbf{A}\cdot\mathbf{B}$.

The minus signs in the inner product may seem a little strange. But the goal was to find a combination of two arbitrary vectors that is invariant under a Lorentz transformation (because such combinations are very useful in seeing what is going on in a problem). The nature of the LT's demands that there be opposite signs in the inner product, so that's the way it is.

- **Norm:** As a corollary to the invariance of the inner product, we can look at the inner product of a 4-vector with itself, which is by definition the square of the norm. We see that

$$A^2 \equiv A \cdot A \equiv A_0A_0 - A_1A_1 - A_2A_2 - A_3A_3 = A_0^2 - |\mathbf{A}|^2 \tag{12.14}$$

is invariant. This is analogous to the invariance of the norm $\sqrt{\mathbf{A}\cdot\mathbf{A}}$ for rotations in 3-space. Special cases of the invariance of the 4-vector norm are the invariance of $c^2t^2 - x^2$ in eq. (10.37), and the invariance of $E^2 - p^2$ in eq. (11.20).

- **A theorem:** Here's a nice little theorem:

*If a certain one of the components of a 4-vector is 0 in every frame, then all four components are 0 in every frame.*

**Proof:** If one of the space components (say, $A_1$) is 0 in every frame, then the other space components must also be 0 in every frame, because otherwise a rotation would make $A_1 \neq 0$. Also, the time component $A_0$ must be 0 in every frame, because otherwise a Lorentz transformation in the $x$-direction would make $A_1 \neq 0$.

If the time component, $A_0$, is 0 in every frame, then the space components must also be 0 in every frame, because otherwise a Lorentz transformation in the appropriate direction would make $A_0 \neq 0$. ■

If someone comes along and says that she has a vector in 3-space that has no $x$-component, no matter how you rotate the axes, then you would certainly say that the vector must obviously be the zero vector. The situation in Lorentzian 4-space is basically the same, because all the coordinates get intertwined with each other in the Lorentz (and rotation) transformations.

## 12.4 Energy, momentum

### 12.4.1 Norm

Many useful things arise from the simple fact that the $P$ in eq. (12.5) is a 4-vector. The invariance of the norm implies that $P \cdot P = E^2 - |\mathbf{p}|^2$ is invariant. If we are dealing with only one particle, we can determine the value of $P^2$ by conveniently working in the rest frame of the particle (so that $\mathbf{v} = \mathbf{0}$). We obtain

$$E^2 - p^2 = m^2, \tag{12.15}$$

or $E^2 - p^2c^2 = m^2c^4$, with the $c$'s. We already knew this, of course, from just writing out $E^2 - p^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 = m^2$.

For a collection of particles, knowledge of the norm is very useful. If a process involves many particles, then we can say that for *any* subset of the particles,

$$\left(\sum E\right)^2 - \left(\sum \mathbf{p}\right)^2 \qquad \text{is invariant,} \tag{12.16}$$

because this is simply the norm of the sum of the energy-momentum 4-vectors of the chosen particles. The sum is again a 4-vector, due to the linearity of eqs. (12.1).

What is the value of the invariant in eq. (12.16)? The most concise description (which is basically a tautology) is that it is the square of the energy in the CM frame, that is, in the frame where $\sum \mathbf{p} = \mathbf{0}$. For one particle, this reduces to $m^2$.

Note that the sums are taken before squaring in eq. (12.16). Squaring before adding would simply give the sum of the squares of the masses.

### 12.4.2 Transformation of $E$,$p$

We already know how the energy and momentum transform (see Section 11.2), but let's derive the transformation again here in a very quick and easy manner. We know that $(E, p_x, p_y, p_z)$ is a 4-vector. So it must transform according to eq. (12.1). Therefore (for an LT in the $x$-direction),

$$\begin{aligned} E &= \gamma(E' + vp'_x), \\ p_x &= \gamma(p'_x + vE'), \\ p_y &= p'_y, \\ p_z &= p'_z, \end{aligned} \qquad (12.17)$$

in agreement with eq. (11.18). That's all there is to it.

REMARK: The fact that $E$ and $\mathbf{p}$ are part of the same 4-vector provides an easy way to see that if one of them is conserved (in every frame) in a collision, then the other is also. Consider an interaction among a set of particles, and look at the 4-vector, $\Delta P \equiv P_{\text{after}} - P_{\text{before}}$. If $E$ is conserved in every frame, then the time component of $\Delta P$ is 0 in every frame. But then the theorem in the previous section says that all four components of $\Delta P$ are 0 in every frame. Therefore, $\mathbf{p}$ is conserved. Likewise for the case where one of the $p_i$ is known to be conserved. ♣

## 12.5 Force and acceleration

Throughout this section, we will deal with objects with constant mass, which we will call "particles". The treatment here can be generalized to cases where the mass changes (for example, the object is being heated, or extra mass is being dumped on it), but we won't concern ourselves with these.

### 12.5.1 Transformation of forces

Let's first look at the force 4-vector in the instantaneous inertial frame of a given particle (frame $S'$). Eq. (12.9) gives

$$F' = \gamma\left(\frac{dE'}{dt}, \mathbf{f}'\right) = (0, \mathbf{f}'). \qquad (12.18)$$

The first component is zero because $dE'/dt = d(m/\sqrt{1 - v'^2}\,)/dt$, and this carries a factor of $v'$, which is zero in this frame. Equivalently, you can just use eq.(12.10), with a speed of zero.

We can now write down two expressions for the 4-force, $F$, in another frame, $S$. First, since $F$ is a 4-vector, it transforms according to eq. (12.1). We therefore have, using eq. (12.18),

$$\begin{aligned} F_0 &= \gamma(F'_0 + vF'_1) = \gamma v f'_x, \\ F_1 &= \gamma(F'_1 + vF'_0) = \gamma f'_x, \\ F_2 &= F'_2 = f'_y, \\ F_3 &= F'_3 = f'_z. \end{aligned} \qquad (12.19)$$

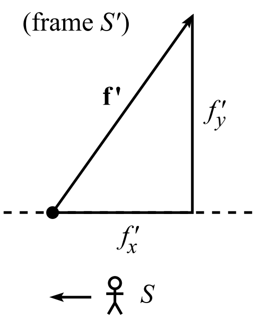

**Figure 12.2**

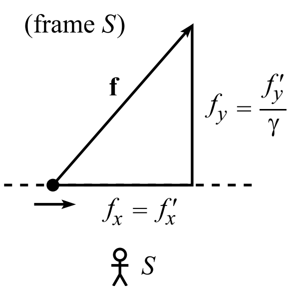

**Figure 12.3**

But second, from the definition in eq. (12.9), we also have

$$\begin{aligned} F_0 &= \gamma\, dE/dt, \\ F_1 &= \gamma f_x, \\ F_2 &= \gamma f_y, \\ F_3 &= \gamma f_z. \end{aligned} \tag{12.20}$$

Combining eqs. (12.19) and (12.20), we obtain

$$\begin{aligned} dE/dt &= v f'_x, \\ f_x &= f'_x, \\ f_y &= f'_y/\gamma, \\ f_z &= f'_z/\gamma. \end{aligned} \tag{12.21}$$

We therefore recover the results of Section 11.5.3. The longitudinal force is the same in both frames, but the transverse forces are larger by a factor of $\gamma$ in the particle's frame. Hence, $f_y/f_x$ decreases by a factor of $\gamma$ when going from the particle's frame to the lab frame (see Fig. 12.2 and Fig. 12.3).

As a bonus, the $F_0$ component in eq. (12.21) tells us (after multiplying through by $dt$) that $dE = f_x\, dx$, which is the work-energy result. In other words, using $f_x \equiv dp_x/dt$, we have just proved again the result, $dE/dx = dp/dt$, from Section 11.5.1.

As noted in Section 11.5.3, we can't switch the $S$ and $S'$ frames and write $f'_y = f_y/\gamma$. When talking about the forces on a particle, there is indeed one preferred frame of reference, namely that of the particle. All frames are not equivalent here. When forming all of our 4-vectors in Section 12.2, we explicitly used the $d\tau$, $dt$, $dx$, etc., from two events, and it was understood that these two events were located at the particle.

### 12.5.2 Transformation of accelerations

The procedure here is similar to the above treatment of the force. Let's first look at the acceleration 4-vector in the instantaneous inertial frame of a given particle (frame $S'$). Eq. (12.7) or eq. (12.8) gives

$$A' = (0, \mathbf{a}'), \tag{12.22}$$

because $v' = 0$ in $S'$.

We can now write down two expressions for the 4-acceleration, $A$, in another frame, $S$. First, since $A$ is a 4-vector, it transforms according to eq. (12.1). So we have, using eq. (12.22),

$$\begin{aligned} A_0 &= \gamma(A'_0 + vA'_1) = \gamma v a'_x, \\ A_1 &= \gamma(A'_1 + vA'_0) = \gamma a'_x, \\ A_2 &= A'_2 = a'_y, \\ A_3 &= A'_3 = a'_z. \end{aligned} \tag{12.23}$$

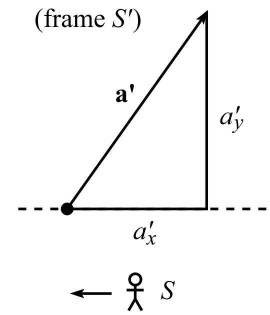


**Figure 12.4**

But second, from the expression in eq. (12.8), we also have

$$\begin{aligned} A_0 &= \gamma^4 v a_x, \\ A_1 &= \gamma^4 a_x, \\ A_2 &= \gamma^2 a_y, \\ A_3 &= \gamma^2 a_z. \end{aligned} \tag{12.24}$$

Combining eqs. (12.23) and (12.24), we obtain

$$\begin{aligned} a_x &= a'_x/\gamma^3, \\ a_x &= a'_x/\gamma^3, \\ a_y &= a'_y/\gamma^2, \\ a_z &= a'_z/\gamma^2. \end{aligned} \tag{12.25}$$

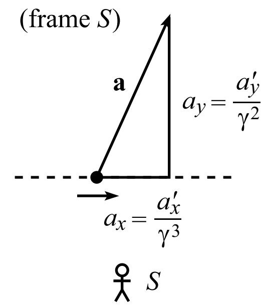


**Figure 12.5**

(The first two equations here are redundant.) We see that $a_y/a_x$ increases by a factor of $\gamma^3/\gamma^2 = \gamma$ when going from the particle's frame to the lab frame (see Fig. 12.4 and Fig. 12.5). This is the opposite of the effect on $f_y/f_x$.[3] This difference makes it clear that an $\mathbf{f} = m\mathbf{a}$ law wouldn't make any sense. If it were true in one frame, then it wouldn't be true in another.

Note also that the increase in $a_y/a_x$ in going to the lab frame is consistent with length contraction, as the Bead-on-a-rod example in Section 11.5.3 showed.

---

**Example (Acceleration for circular motion):** A particle moves with constant speed $v$ along the circle $x^2 + y^2 = r^2$, $z = 0$, in the lab frame. At the instant the particle crosses the negative $y$-axis (see Fig. 12.6), find the 3-acceleration and 4-acceleration in both the lab frame and the instantaneous rest frame of the particle (with axes chosen parallel to the lab's axes).

**Solution:** Let the lab frame be $S$, and let the particle's instantaneous inertial frame be $S'$ when it crosses the negative $y$-axis. Then $S$ and $S'$ are related by a Lorentz transformation in the $x$-direction.

The 3-acceleration in $S$ is simply

$$\mathbf{a} = (0, v^2/r, 0). \tag{12.26}$$

There's nothing fancy going on here; the nonrelativistic proof of $a = v^2/r$ works just fine again in the relativistic case. Eq. (12.7) or (12.8) then gives the 4-acceleration in $S$ as

$$A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.27}$$

To find the acceleration vectors in $S'$, we will use the fact $S'$ and $S$ are related by a Lorentz transformation in the $x$-direction. Therefore, the $A_2$ component of the 4-acceleration is unchanged. So the 4-acceleration in $S'$ is also

$$A' = A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.28}$$

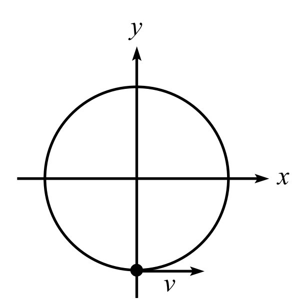


**Figure 12.6**

[3]In a nutshell, this difference is due to the fact that $\gamma$ changes with time. When talking about accelerations, there are $\gamma$'s that we have to differentiate; see eq. (12.6). This isn't the case with forces, because the $\gamma$ is absorbed into the definition of $\mathbf{p} \equiv \gamma m \mathbf{v}$; see eq. (12.9). This is what leads to the different powers of $\gamma$ in eq. (12.24), in contrast with the identical powers in eq. (12.20).

In the particle's frame, $\mathbf{a}'$ is simply the space part of $A$ (using eq. (12.7) or (12.8), with $v = 0$ and $\gamma = 1$). Therefore, the 3-acceleration in $S'$ is

$$\mathbf{a}' = (0, \gamma^2 v^2/r, 0). \tag{12.29}$$

REMARK: We can also arrive at the two factors of $\gamma$ in $\mathbf{a}'$ by using a simple time-dilation argument. We have

$$a'_y = \frac{d^2 y'}{d\tau^2} = \frac{d^2 y'}{d(t/\gamma)^2} = \gamma^2 \frac{d^2 y}{dt^2} = \gamma^2 \frac{v^2}{r}\,, \tag{12.30}$$

where we have used the fact that transverse lengths are the same in the two frames. ♣

## 12.6 The form of physical laws

One of the postulates of special relativity is that all inertial frames are equivalent. Therefore, if a physical law holds in one frame, then it must hold in all frames. Otherwise, it would be possible to differentiate between frames. As noted in the previous section, the statement "$\mathbf{f} = m\mathbf{a}$" cannot be a physical law. The two sides of the equation transform differently when going from one frame to another, so the statement cannot be true in all frames.

If a statement has any chance of being true in all frames, it must involve only 4-vectors. Consider a 4-vector equation (say, "$A = B$") which is true in frame $S$. Then if we apply to this equation a Lorentz transformation (call it $\mathcal{M}$) from $S$ to another frame $S'$, we have

$$\begin{aligned} A &= B, \\ \Longrightarrow \quad \mathcal{M}A &= \mathcal{M}B, \\ \Longrightarrow \quad A' &= B'. \end{aligned} \tag{12.31}$$

The law is therefore also true in frame $S'$.

Of course, there are many 4-vector equations that are simply not true (for example, $F = P$, or $2P = 3P$). Only a small set of such equations (for example, $F = mA$) correspond to the real world.

Physical laws may also take the form of scalar equations, such as $P \cdot P = m^2$. A scalar is by definition a quantity that is frame-independent (as we have shown the inner product to be). So if a scalar statement is true in one inertial frame, then it is true in all inertial frames. Physical laws may also be higher-rank "tensor" equations, such as arise in electromagnetism and general relativity. We won't discuss such things here, but suffice it to say that tensors may be thought of as things built up from 4-vectors. Scalars and 4-vectors are special cases of tensors.

All of this is exactly analogous to the situation in 3-D space. In Newtonian mechanics, $\mathbf{f} = m\mathbf{a}$ is a possible law, because both sides are 3-vectors. But $\mathbf{f} = m(2a_x, a_y, a_z)$ is not a possible law, because the right-hand side is not a 3-vector; it depends on which axis you label as the $x$-axis. Another example is the statement that a given stick has a length of 2 meters. That's fine, but if you say that the stick has an $x$-component of 1.7 meters, then this cannot be true in all frames.

God said to his cosmos directors,
"I've added some stringent selectors.
One is the clause
That your physical laws
Shall be written in terms of 4-vectors."

## 12.7 Exercises

1. **Acceleration at rest**

   Show that the derivative of $v \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$ equals $a_x$, independent of how all the $v_i$'s are changing, provided that $v_y = v_z = 0$ at the moment in question.

2. **Linear acceleration** $*$

   A particle's velocity and acceleration both point in the $x$-direction, with magnitudes $v$ and $\dot{v}$, respectively (as measured in the lab frame). In the spirit of the example in Section 12.5.2, find the 3-acceleration and 4-acceleration in both the lab frame and the instantaneous rest frame of the particle. Verify that 3-accelerations are related according to eq. (12.25).

3. **Same speed** $*$

   Consider the setup in Problem 2. Given $v$, what should $\theta$ be so that the speed of one particle, as viewed by the other, is also $v$? Do your answers make sense for $v \approx 0$ and $v \approx c$?

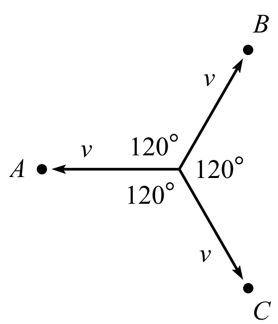


**Figure 12.7**

4. **Three particles** $**$

   Three particles head off with equal speeds $v$, at 120° with respect to each other, as shown in Fig. 12.7. What is the inner product of any two of the 4-velocities in any frame? Use your result to find the angle $\theta$ (see Fig. 12.8) at which two particles travel in the frame of the third.

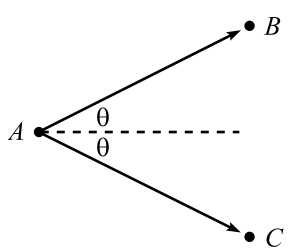


**Figure 12.8**

5. **Doppler effect** $*$

   Consider a photon traveling in the $x$-direction. Ignoring the $y$ and $z$ components, and setting $c = 1$, the 4-momentum is $(p, p)$. In matrix notation, what are the Lorentz transformations for the frames traveling to the left and to the right at speed $v$? What is the new 4-momentum of the photon in these new frames? Accepting the fact the a photon's energy is proportional to its frequency, verify that your results are consistent with the Doppler results in Section 10.6.1.

## 12.8 Problems

1. **Velocity addition**

   In $A$'s frame, $B$ moves to the right with speed $u$, and $C$ moves to the left with speed $v$. What is the speed of $B$ with respect to $C$? In other words, use 4-vectors to derive the velocity-addition formula.

2. **Relative speed** $*$

   In the lab frame, two particles move with speed $v$ along the paths shown in Fig. 12.9. The angle between the trajectories is $2\theta$. What is the speed of one particle, as viewed by the other?

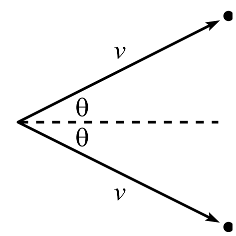


**Figure 12.9**

3. **Another relative speed** $*$

   In the lab frame, two particles, $A$ and $B$, move with speeds $u$ and $v$ along the paths shown in Fig. 12.10. The angle between the trajectories is $\theta$. What is the speed of one particle, as viewed by the other?

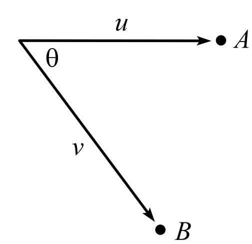


**Figure 12.10**

4. **Acceleration for linear motion** $*$

   A spaceship starts at rest with respect to frame $S$ and accelerates with constant proper acceleration $a$. In Section 10.7, we showed that the speed of the spaceship with respect to $S$ is given by $v(\tau) = \tanh(a\tau)$, where $\tau$ is the spaceship's proper time (and $c = 1$). Let $V$ be the spaceship's 4-velocity, and let $A$ be its 4-acceleration. In terms of the proper time $\tau$,

   (a) Find $V$ and $A$ in frame $S$, by explicitly using $v(\tau) = \tanh(a\tau)$.

   (b) Write down $V$ and $A$ in the spaceship's frame, $S'$.

   (c) Verify that $V$ and $A$ transform like 4-vectors between the two frames.

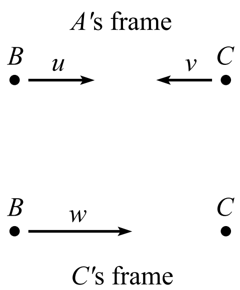


**Figure 12.11**

## 12.9 Solutions

1. **Velocity addition**

   Let the desired speed of $B$ with respect to $C$ be $w$. See Fig. 12.11.

   In $A$'s frame, the 4-velocity of $B$ is $(\gamma_u, \gamma_u u)$, and the 4-velocity of $C$ is $(\gamma_v, -\gamma_v v)$. We have suppressed the $y$ and $z$ components here.

   In $C$'s frame, the 4-velocity of $B$ is $(\gamma_w, \gamma_w w)$, and the 4-velocity of $C$ is $(1, 0)$.

   The invariance of the inner product implies

$$\begin{aligned} (\gamma_u, \gamma_u u) \cdot (\gamma_v, -\gamma_v v) &= (\gamma_w, \gamma_w w) \cdot (1, 0) \\ \Longrightarrow \quad \gamma_u \gamma_v (1 + uv) &= \gamma_w \\ \Longrightarrow \quad \frac{1 + uv}{\sqrt{1 - u^2}\sqrt{1 - v^2}} &= \frac{1}{\sqrt{1 - w^2}}\,. \end{aligned} \tag{12.32}$$

   Squaring and then solving for $w$ gives

$$w = \frac{u + v}{1 + uv}\,. \tag{12.33}$$

2. **Relative speed**

   In the lab frame, the 4-velocities of the particles are (suppressing the $z$ component)

$$(\gamma_v, \gamma_v v \cos\theta, \gamma_v v \sin\theta) \qquad \text{and} \qquad (\gamma_v, \gamma_v v \cos\theta, -\gamma_v v \sin\theta). \tag{12.34}$$

   Let $w$ be the desired speed of one particle as viewed by the other. Then in the frame of one particle, the 4-velocities are (suppressing two spatial components)

$$(\gamma_w, \gamma_w w) \qquad \text{and} \qquad (1, 0), \tag{12.35}$$

   where we have rotated the axes so that the relative motion is along the $x$-axis in this frame. Since the 4-vector inner product is invariant under Lorentz transformations and rotations, we have (using $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

$$\begin{aligned} (\gamma_v, \gamma_v v \cos\theta, \gamma_v v \sin\theta) \cdot (\gamma_v, \gamma_v v \cos\theta, -\gamma_v v \sin\theta) &= (\gamma_w, \gamma_w w) \cdot (1, 0) \\ \Longrightarrow \quad \gamma_v^2 (1 - v^2 \cos 2\theta) &= \gamma_w. \end{aligned} \tag{12.36}$$

   Using the definitions of the $\gamma$'s, squaring, and solving for $w$ gives

$$w = \sqrt{1 - \frac{(1 - v^2)^2}{(1 - v^2\cos 2\theta)^2}} = \frac{\sqrt{2v^2(1 - \cos 2\theta) - v^4 \sin^2 2\theta}}{1 - v^2 \cos 2\theta}\,. \tag{12.37}$$

   If desired, this can be rewritten (using some double-angle formulas) in the form,

$$w = \frac{2v \sin\theta \sqrt{1 - v^2 \cos^2\theta}}{1 - v^2 \cos 2\theta}\,. \tag{12.38}$$

   Remark: If $2\theta = 180^\circ$, then $w = 2v/(1 + v^2)$, in agreement with the standard velocity-addition formula. And if $\theta = 0^\circ$, then $w = 0$, as should be the case. If $\theta$ is very small, then you can show $w \approx 2v \sin\theta/\sqrt{1 - v^2}$, which is simply the relative speed in the lab frame, multiplied by the time dilation factor between the frames. (The particles' clocks run slow, and transverse distances don't change, so the relative speed is larger in a particle's frame.) ♣

3. **Another relative speed**

In the lab frame, the 4-velocities of the particles are (suppressing the $z$ component)

$$V_A = (\gamma_u, \gamma_u u, 0) \qquad \text{and} \qquad V_B = (\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta). \tag{12.39}$$

Let $w$ be the desired speed of one particle as viewed by the other. Then in the frame of one particle, the 4-velocities are (suppressing two spatial components)

$$(\gamma_w, \gamma_w w) \qquad \text{and} \qquad (1, 0), \tag{12.40}$$

where we have rotated the axes so that the relative motion is along the $x$-axis in this frame. Since the 4-vector inner product is invariant under Lorentz transformations and rotations, we have

$$\begin{aligned}(\gamma_u, \gamma_u u, 0)\cdot(\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta) &= (\gamma_w, \gamma_w w)\cdot(1,0)\\ \Longrightarrow \qquad \gamma_u\gamma_v(1 - uv\cos\theta) &= \gamma_w.\end{aligned} \tag{12.41}$$

Using the definitions of the $\gamma$'s, squaring, and solving for $w$ gives

$$w = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv\cos\theta)^2}} = \frac{\sqrt{u^2 + v^2 - 2uv\cos\theta - u^2v^2\sin^2\theta}}{1 - uv\cos\theta}. \tag{12.42}$$

You can check various special cases of this result.

4. **Acceleration for linear motion**

(a) Using $v(\tau) = \tanh(a\tau)$, we have $\gamma = 1/\sqrt{1-v^2} = \cosh(a\tau)$. Therefore,

$$V = (\gamma, \gamma v) = \big(\cosh(a\tau), \sinh(a\tau)\big), \tag{12.43}$$

where we have suppressed the two transverse components of $V$. We then have

$$A = \frac{dV}{d\tau} = a\big(\sinh(a\tau), \cosh(a\tau)\big). \tag{12.44}$$

(b) The spaceship is at rest in its instantaneous inertial frame, so

$$V' = (1, 0). \tag{12.45}$$

In the rest frame, we also have

$$A' = (0, a). \tag{12.46}$$

Equivalently, these are obtained by setting $\tau = 0$ in the results from part (a), because the spaceship hasn't started moving at $\tau = 0$, as is always the case in the instantaneous rest frame.

(c) The Lorentz transformation matrix from $S'$ to $S$ is

$$\mathcal{M} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) \\ \sinh(a\tau) & \cosh(a\tau) \end{pmatrix}. \tag{12.47}$$

We must check that

$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \mathcal{M}\begin{pmatrix} V_0' \\ V_1' \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \mathcal{M}\begin{pmatrix} A_0' \\ A_1' \end{pmatrix}. \tag{12.48}$$

These are easily seen to be true.

# Chapter 13

# General Relativity



This will be somewhat of a strange chapter, because we won't have enough time to get to the heart of General Relativity (GR). But we will still be able to get a flavor of the subject and derive a few interesting GR results.

One crucial idea in GR is the Equivalence Principle. This basically says that gravity is equivalent to acceleration. We will have much to say about this issue in the sections below. Another crucial concept in GR is that of coordinate independence. The laws of physics should not depend on what coordinate system you choose. This seemingly innocuous statement has surprisingly far-reaching consequences. However, a discussion of this topic is one of the many things we won't have time for. We would need a whole class on GR to do it justice. But fortunately, it is possible to get a sense of the nature of GR without having to master such things. This is the route we will take in this chapter.

## 13.1 The Equivalence Principle

Einstein's Equivalence Principle says that it is impossible to locally distinguish between gravity and acceleration. This may be stated more precisely in (at least) three ways.

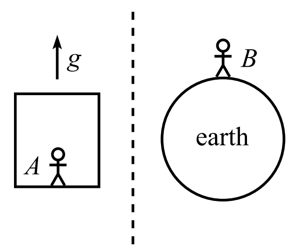


**Figure 13.1**

- Let person $A$ be enclosed in a small box, far from any massive objects, that undergoes uniform acceleration (say, $g$). Let person $B$ stand at rest on the earth (see Fig. 13.1). The Equivalence Principle says that there are no local experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.

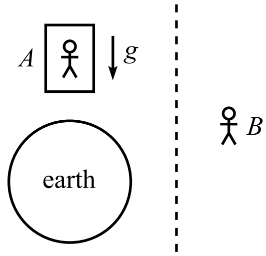


**Figure 13.2**

- Let person $A$ be enclosed in a small box that is in free-fall near a planet. Let person $B$ float freely in space, far away from any massive objects (see Fig. 13.2). The Equivalence Principle says that there are no local experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.

- "Gravitational" mass is equal to (or proportional to) "inertial" mass. Gravitational mass is the $m_g$ that appears in the formula, $F = GMm_g/r^2 \equiv m_g g$.

Inertial mass is the $m_i$ that appears in the formula, $F = m_i a$. There is no *a priori* reason why these two $m$'s should be the same (or proportional). An object that is dropped on the earth will have acceleration $a = (m_g/m_i)g$. For all we know, the ratio $m_g/m_i$ for plutonium is different from that for copper. But experiments with various materials have detected no difference in the ratios. The Equivalence Principle states that the ratios are equal for any type of mass.

This definition of the Equivalence Principle is equivalent to, say, the second one above for the following reason. Two different masses near $B$ will stay right where they are. But two different masses near $A$ will diverge from each other if their accelerations are not equal.

These statements are all quite believable. Consider the first one, for example. When standing on the earth, you have to keep your legs firm to avoid falling down. When standing in the accelerating box, you have to keep your legs firm to maintain the same position relative to the floor (that is, to avoid "falling down"). You certainly can't naively tell the difference between the two scenarios. The Equivalence Principle says that it's not just that you're too inept to figure out a way to differentiate between them, but instead that there is no possible local experiment you can perform to tell the difference, no matter how clever you are.

REMARK: Note the inclusion of the words "small box" and "local" above. On the surface of the earth, the lines of the gravitational force are not parallel; they converge to the center. The gravitational force also varies with height. Therefore, an experiment performed over a non-negligible distance (for example, dropping two balls next to each other, and watching them converge; or dropping two balls on top of each other and watching them diverge) will have different results from the same experiment in the accelerating box. The equivalence principle says that if your laboratory is small enough, or if the gravitational field is sufficiently uniform, then the two scenarios look essentially the same. ♣

## 13.2 Time dilation

The equivalence principle has a striking consequence concerning the behavior of clocks in a gravitational field. It implies that higher clocks run faster than lower clocks. If you put a watch on top of a tower, and then stand on the ground, you will see the watch on the tower tick faster than an identical watch on your wrist. When you take the watch down and compare it to the one on your wrist, it will show more time elapsed.[1] Likewise, someone standing on top of the tower will see a clock on the ground run slow. Let's be quantitative about this. Consider the following two scenarios.

[1]This will be true only if you keep the watch on the tower for a long enough time, because the movement of the watch will cause it to run slow due to the usual special-relativistic time dilation. But the (speeding-up) effect due to the height can be made arbitrarily large compared to the (slowing-down) effect due to the motion, by simply keeping the watch on the tower for an arbitrarily long time.

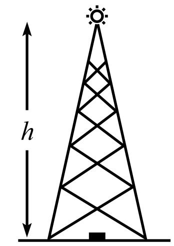

**Figure 13.3**

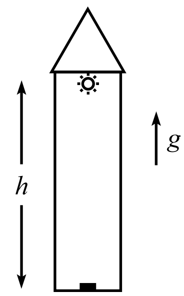

**Figure 13.4**

- A light source on top of a tower of height $h$ emits flashes at time intervals $t_s$. A receiver on the ground receives the flashes at time intervals $t_r$ (see Fig. 13.3). What is $t_r$ in terms of $t_s$?

- A rocket of length $h$ accelerates with acceleration $g$. A light source at the front end emits flashes at time intervals $t_s$. A receiver at the back end receives the flashes at time intervals $t_r$ (see Fig. 13.4). What is $t_r$ in terms of $t_s$?

The equivalence principle tells us that these two scenarios look exactly the same, as far as the sources and receivers are concerned. Hence, the relation between $t_r$ and $t_s$ is the same in each. Therefore, to find out what is going on in the first scenario, we will study the second scenario (because we can figure out how this one behaves).

Consider an instantaneous inertial frame, $S$, of the rocket. In this frame, the rocket is momentarily at rest (at, say, $t = 0$), and then it accelerates out of the frame with acceleration $g$. The following discussion will be made with respect to the frame $S$.

Consider a series of quick light pulses emitted from the source, starting at $t = 0$. The distance the rocket has traveled out of $S$ at time $t$ is $gt^2/2$, so if we assume that $t_s$ is very small, then we may say that many light pulses are emitted before the rocket moves appreciably. Likewise, the speed of the source, namely $gt$, is also very small. We may therefore ignore the motion of the rocket, as far as the light source is concerned.

However, the light takes a finite time to reach the receiver, and by then the receiver will be moving. We therefore *cannot* ignore the motion of the rocket when dealing with the receiver. The time it takes the light to reach the receiver is $h/c$, at which point the receiver has a speed of $v = g(h/c)$.[2] Therefore, by the usual classical Doppler effect, the time between the received pulses is[3]

$$t_r = \frac{t_s}{1 + (v/c)}\,. \tag{13.1}$$

Therefore, the frequencies, $f_r = 1/t_r$ and $f_s = 1/t_s$, are related by

$$f_r = \left(1 + \frac{v}{c}\right) f_s = \left(1 + \frac{gh}{c^2}\right) f_s. \tag{13.2}$$

Returning to the clock-on-tower scenario, we see (using the equivalence principle) that an observer on the ground will see the clock on the tower running fast, by a factor $1 + gh/c^2$. This means that the upper clock really *is* running fast, compared

[2]The receiver moves a tiny bit during this time, so the "$h$" here should really be replaced by a slightly smaller distance. But this yields a negligible second-order effect in the small quantity $gh/c^2$, as you can show. To sum up, the displacement of the source, the speed of the source, and the displacement of the receiver are all negligible. But the speed of the receiver is quite relevant.

[3]Quick proof of the classical Doppler effect: As seen in frame $S$, when the receiver and a particular pulse meet, the next pulse is a distance $ct_s$ behind. The receiver and this next pulse then travel toward each other at relative speed $c + v$ (as measured by someone in $S$). The time difference between receptions is therefore $t_r = ct_s/(c + v)$.

to the lower clock.[4] That is,

$$\Delta t_h = \left(1 + \frac{gh}{c^2}\right) \Delta t_0. \tag{13.3}$$

A twin from Denver will be older than his twin from Boston when they meet up at a family reunion (all other things being equal, of course).

> Greetings! Dear brother from Boulder,
> I hear that you've gotten much older.
> And please tell me why
> My lower left thigh
> Hasn't aged quite as much as my shoulder!

Note that the $gh$ in eq. (13.3) is the gravitational potential energy, divided by $m$.

REMARK: You might object to the above derivation, because $t_r$ is the time measured by someone in the inertial frame, $S$. And since the receiver is eventually moving with respect to $S$, we should multiply the $f_r$ in eq. (13.2) by the usual special-relativistic time-dilation factor, $1/\sqrt{1-(v/c)^2}$ (because the receiver's clocks are running slow relative to $S$, so the frequency measured by the receiver is greater than that measured in $S$). However, this is a second-order effect in the small quantity $v/c = gh/c^2$. We already dropped other effects of the same order, so we have no right to keep this one. Of course, if the leading effect in our final answer was second-order in $v/c$, then we would know that our answer was garbage. But the leading effect happens to be first order, so we can afford to be careless with the second-order effects. ♣

After a finite time has passed, the frame $S$ will no longer be of any use to us. But we can always pick a new instantaneous rest frame of the rocket, so we can repeat the above analysis at any later time. Hence, the result in eq. (13.2) holds at all times.

This GR time-dilation effect was first measured at Harvard by Pound and Rebka in 1960. They sent gamma rays up a 20 m tower and measured the redshift (that is, the decrease in frequency) at the top. This was a notable feat indeed, considering that they were able to measure a frequency shift of $gh/c^2$ (which is only a few parts in $10^{15}$) to within 1% accuracy.

## 13.3 Uniformly accelerated frame

Before reading this section, you should think carefully about the "Break or not break" problem (Problem 25) in Chapter 10. Don't look at the solution too soon, because chances are you will change your answer after a few more minutes of thought. This is a classic problem, so don't waste it by peeking!

[4]Unlike the situation where two people fly past each other (as with the usual twin paradox), we can say here that what an observer *sees* is also what actually *is*. We can say this because everyone here is in the same frame. The "turnaround" effect that was present in the twin paradox is not present now. The two clocks can be slowly moved together without anything exciting or drastic happening to their readings.

Technically, the uniformly accelerated frame we will construct has nothing to do with GR. We will not need to leave the realm of special relativity for the analysis in this section. The reason we choose to study this special-relativistic setup in detail is because it shows many similarities to genuine GR situations, such as black holes.

### 13.3.1 Uniformly accelerated point particle

In order to understand a uniformly accelerated frame, we first need to understand a uniformly accelerated point particle. In Section 10.7, we briefly discussed the motion of a uniformly accelerated particle, that is, one that feels a constant force in its instantaneous rest frame. Let us now take a closer look at such a particle.

Let the particle's instantaneous rest frame be $S'$, and let it start from rest in the inertial frame $S$. Let its mass be $m$. We know from Section 11.5.3 that the longitudinal force is the same in the two frames. Therefore, since it is constant in frame $S'$, it is also constant in frame $S$. Call it $f$. For convenience, let $g \equiv f/m$ (so $g$ is the proper acceleration felt by the particle). Then in frame $S$ we have, using the fact that $f$ is constant,

$$f = \frac{dp}{dt} = \frac{d(m\gamma v)}{dt} \qquad \Longrightarrow \qquad \gamma v = gt \qquad \Longrightarrow \qquad v = \frac{gt}{\sqrt{1+(gt)^2}}\,, \tag{13.4}$$

where we have set $c = 1$. As a double-check, this has the correct behavior for $t \to 0$ and $t \to \infty$. If you want to keep the $c$'s in, then $(gt)^2$ becomes $(gt/c)^2$, to make the units correct.

Having found the speed in frame $S$ at time $t$, the position in frame $S$ at time $t$ is given by

$$x = \int_0^t v\,dt = \int_0^t \frac{gt\,dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right). \tag{13.5}$$

For convenience, let $P$ be the point (see Fig. 13.5)

$$(x_P, t_P) = (-1/g, 0). \tag{13.6}$$

Then eq. (13.5) yields

$$(x - x_P)^2 - t^2 = \frac{1}{g^2}\,. \tag{13.7}$$

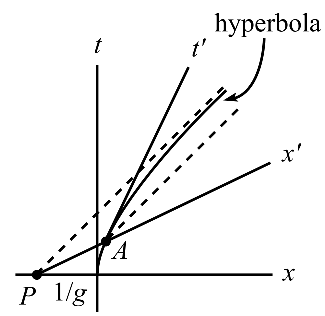

**Figure 13.5**

This is the equation for a hyperbola with its center (defined as the intersection of the asymptotes) at point $P$. For a large acceleration $g$, the point $P$ is very close to the particle's starting point. For a small acceleration, it is far away.

Everything has been fairly normal up to this point, but now the fun begins. Consider a point $A$ on the particle's worldline at time $t$. From eq. (13.5), $A$ has coordinates

$$(x_A, t_A) = \left(\frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right),\, t\right). \tag{13.8}$$

The slope of the line $PA$ is therefore

$$\frac{t_A - t_P}{x_A - x_P} = \frac{gt}{\sqrt{1+(gt)^2}}\,. \tag{13.9}$$

Looking at eq. (13.4), we see that this slope equals the speed of the particle at point $A$. But we know very well that the speed $v$ is the slope of the particle's instantaneous $x'$-axis; see eq. (10.44). Therefore, the line $PA$ and the particle's $x'$-axis are the same line. This holds for any arbitrary time, $t$. So we may say that *at any point along the particle's worldline, the line $PA$ is the instantaneous $x'$-axis of the particle.* Or, said another way, *no matter where the particle is, the event at $P$ is simultaneous with an event located at the particle, as measured in the instantaneous frame of the particle.* In other words, the particle always says that $P$ happens "now."[5]

Here is another strange fact. What is the distance from $P$ to $A$, as measured in an instantaneous rest frame, $S'$, of the particle? The $\gamma$ factor between frames $S$ and $S'$ is, using eq. (13.4), $\gamma = \sqrt{1+(gt)^2}$. The distance between $P$ and $A$ in frame $S$ is $x_A - x_P = \sqrt{1+(gt)^2}\,/g$. So the distance between $P$ and $A$ in frame $S'$ is (using the Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$, with $\Delta t' = 0$)

$$x'_A - x'_P = \frac{1}{\gamma}(x_A - x_P) = \frac{1}{g}\,. \tag{13.10}$$

This is independent of $t$! Therefore, not only do we find that $P$ is always simultaneous with the particle, in the particle's frame; we also find that $P$ *always remains the same distance (namely $1/g$) away from the particle, as measured in the particle's instantaneous rest-frame.* This is rather strange. The particle accelerates away from point $P$, but it does not get further away from it (in its own frame).

Remark: We can give a continuity argument that shows that such a point $P$ must exist. If $P$ is close to you, and if you accelerate away from it, then of course you get farther away from it. Everyday experience is quite valid here. But if $P$ is sufficiently far away from you, and if you accelerate away from it, then the $at^2/2$ distance you travel away from it can easily be compensated by the decrease in distance due to length contraction (brought about by your newly acquired velocity). This effect grows with distance, so we simply need to pick $P$ to be sufficiently far away. What this means is that every time you get out of your chair and walk to the door, there are stars very far away behind you that get closer to you as you walk away from them (as measured in your instantaneous rest frame). By continuity, then, there must exist a point $P$ that remains the same distance from you as you accelerate away from it. ♣

### 13.3.2 Uniformly accelerated frame

Let's now put a collection of uniformly accelerated particles together to make a uniformly accelerated frame. The goal will be to create a frame where the distances between particles (as measured in any particle's instantaneous rest frame) remain constant.

Why is this our goal? We know from the "Break or not break" problem in Chapter 10 that if all the particles accelerate with the same proper acceleration,

[5] The point $P$ is very much like the event horizon of a black hole. Time seems to stand still at $P$. And if we went more deeply into GR, we would find that time seems to stand still at the edge of a black hole, too (as viewed by someone farther away).

$g$, then the distances (as measured in a particle's instantaneous rest frame) grow larger. While this is a perfectly possible frame to construct, it is not desirable here for the following reason. Einstein's Equivalence Principle states that an accelerated frame is equivalent to a frame sitting on, say, the earth. We may therefore study the effects of gravity by studying an accelerated frame. But if we want this frame to look anything like the surface of the earth, we certainly can't have distances that change over time.

We therefore want to construct a *static* frame, that is, one where distances do not change (as measured in the frame). This will allow us to say that if we enclose the frame by windowless walls, then for all a person inside knows, he is standing motionless in a static gravitational field (which has a certain definite form, as we shall see).

Let's figure out how to construct the frame. We'll discuss only the acceleration of two particles here. Others can be added in an obvious manner. In the end, the desired frame as a whole is constructed by accelerating each atom in the floor of the frame with a specific proper acceleration.

From the previous subsection, we already have a particle $A$ which is "centered" around the point $P$.[6] We claim (for reasons that will become clear) that every other particle in the frame should also be "centered" around the same point $P$.

Consider another particle, $B$. Let $a$ and $b$ be the initial distances from $P$ to $A$ and $B$. If both particles are to be centered around $P$, then their proper accelerations must be, from eq. (13.6),

$$g_A = \frac{1}{a}, \qquad \text{and} \qquad g_B = \frac{1}{b}. \tag{13.11}$$

Therefore, in order to have all points in the frame be centered around $P$, we simply have to make their proper accelerations inversely proportional to their initial distances from $P$.

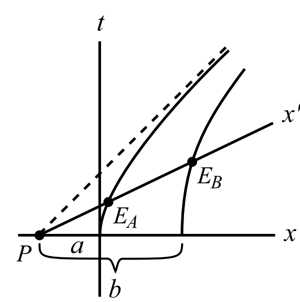

**Figure 13.6**

Why do we want every particle to be centered around $P$? Consider two events, $E_A$ and $E_B$, such that $P$, $E_A$, and $E_B$ are collinear in Fig. 13.6. Due to construction, the line $PE_AE_B$ is the $x'$-axis for both particle $A$ and particle $B$, at the positions shown. From the previous subsection, we know that $A$ is always a distance $a$ from $P$, and $B$ is always a distance $b$ from $P$. Combining these facts with the fact that $A$ and $B$ measure their distances along the $x'$-axis of the same frame (at the events shown in the figure), we see that both $A$ and $B$ measure the distance between them to be $b - a$. This is independent of $t$, so $A$ and $B$ measure a constant distance between them. We have therefore constructed our desired static frame. This frame is often called a "Rindler space."

If a person walks around in the frame, he will think he lives in a static world where the acceleration due to gravity takes the form $g(z) \propto 1/z$, where $z$ is the distance to a certain magical point which is located at the end of the known "universe".

What if a person releases himself from the accelerating frame, so that he forever sails through space at constant speed? He thinks he is falling, and you should

[6]This will be our shorthand notation for "traveling along a hyperbola whose center is the point $P$."

convince yourself that he passes by the "magical point" in a finite proper time. But his friends who are still in the frame will see him take an infinitely long time to get to the "magical point" $P$. This is similar to the situation with a black hole. An outside observer will see it take an infinitely long time for a falling person to reach the "boundary" of a black hole, even though it will take a finite proper time for the person.

Our analysis shows that $A$ and $B$ feel a different proper acceleration, because $a \neq b$. There is no way to construct a static frame where all points feel the same proper acceleration, so it is impossible to mimic a constant gravitational field (over a finite distance) by using an accelerated frame.

## 13.4 Maximal-proper-time principle

The maximal-proper-time principle in General Relativity says: Given two events in spacetime, a particle under the influence of only gravity takes the path in spacetime that maximizes the proper time. For example, if you throw a ball from given coordinates $(\mathbf{x}_1, t_1)$, and it lands at given coordinates $(\mathbf{x}_2, t_2)$, then the claim is that the ball takes the path that maximizes its proper time.[7]

This is clear for a freely-moving ball in outer space, far from any massive objects. The ball travels at constant speed from one point to another, and we know that this constant-speed motion is the motion with the maximal proper time. This is true because a ball ($A$) moving at constant speed would see the clock on any other ball ($B$) slowed down due to the special-relativistic time dilation, if there were a relative speed between them. (It is assumed here that $B$'s non-uniform velocity is caused by a non-gravitational force acting on it.) $B$ would therefore show a shorter elapsed time. This argument does not work the other way around, because $B$ is not in an inertial frame and therefore cannot use the special-relativistic time-dilation result.

### Consistency with Newtonian physics

The maximal-proper-time principle sounds like a plausible idea, but we already know from Chapter 5 that the path an object takes is the one that yields a stationary value of the classical action, $\int(T-V)$. We must therefore demonstrate that the "maximal"-proper-time principle reduces to the stationary-action principle, in the limit of small velocities. If this were not the case, then we'd have to throw out our theory of gravitation.

Consider a ball thrown vertically on the earth. Assume that the initial and final coordinates are fixed to be $(y_1, t_1)$ and $(y_2, t_2)$. Our plan will be to assume that the maximal-proper-time principle holds, and to then show that this leads to the stationary-action principle.

[7] The principle is actually the "*stationary*-proper-time principle." As with the Lagrangian formalism in Chapter 5, any type of stationary point (a maximum, minimum, or saddle point) is allowed. But although we were very careful about stating things properly in Chapter 5, we'll be a little sloppy here and just use the word "maximum," because that's what it will generally turn out to be in the situations we will look at. However, see Problem 8.

Before being quantitative, let's get a qualitative handle on what's going on with the ball. There are two competing effects, as far as maximizing the proper time goes. On one hand, the ball wants to climb very high, because its clock will run faster there (due to the GR time dilation). But on the other hand, if it climbs very high, then it must move very fast to get there (because the total time, $t_2 - t_1$, is fixed), and this will make its clock run slow (due to the SR time dilation). So there is a tradeoff. Let's now look quantitatively at the implications of this tradeoff.

The goal is to maximize

$$\tau = \int_{t_1}^{t_2} d\tau. \tag{13.12}$$

Due to the motion of the ball, we have the usual time dilation, $d\tau = \sqrt{1 - v^2/c^2}\, dt$. But due to the height of the ball, we also have the gravitational time dilation, $d\tau = (1 + gy/c^2)dt$. Combining these effects gives[8]

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} \left(1 + \frac{gy}{c^2}\right) dt. \tag{13.13}$$

Using the Taylor expansion for $\sqrt{1-\epsilon}$, and dropping terms of order $1/c^4$ and smaller, we see that we want to maximize

$$\begin{aligned} \int_{t_1}^{t_2} d\tau \quad &\approx \quad \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2}\right)\left(1 + \frac{gy}{c^2}\right) dt \\ &\approx \quad \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2} + \frac{gy}{c^2}\right) dt. \end{aligned} \tag{13.14}$$

The "1" term gives a constant, so maximizing this integral is the same as minimizing

$$mc^2 \int_{t_1}^{t_2} \left(\frac{v^2}{2c^2} - \frac{gy}{c^2}\right) dt = \int_{t_1}^{t_2} \left(\frac{mv^2}{2} - mgy\right) dt, \tag{13.15}$$

which is the classical action, as desired. For a one-dimensional gravitational problem such as this one, the action will always be a minimum, and the proper time will always be a maximum, as you can show by considering the second-order change in the action (see Exercise 10).

In retrospect, it is not surprising that this all works out. The factor of 1/2 in the kinetic energy here comes about in exactly the same way as in the derivation in eq. (11.9), where we showed that the relativistic form of energy reduces to the familiar Newtonian expression.

## 13.5 Twin paradox revisited

Let's take another look at the standard twin paradox, this time from the perspective of General Relativity. We should emphasize that GR is by no means necessary for an

[8]This result is technically not correct; the two effects are intertwined in a somewhat more complicated way (see Exercise 7). But it is valid up to order $v^2/c^2$, which is all we are concerned with, since we are assuming $v \ll c$.

understanding of the original formulation of the paradox (the first scenario below). We were able to solve it in Section 10.2.2, after all. The present discussion is given simply to show that the answer to an alternative formulation (the second scenario below) is consistent with what we've learned about GR. Consider the two following twin-paradox setups.

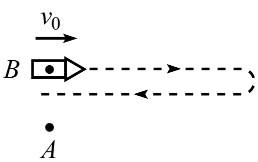


**Figure 13.7**

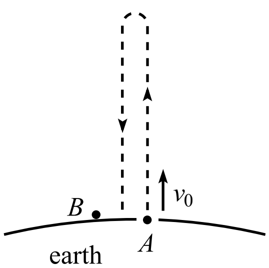


**Figure 13.8**

- Twin $A$ floats freely in outer space. Twin $B$ flies past $A$ in a spaceship, with speed $v_0$ (see Fig. 13.7). At the instant they are next to each other, they both set their clocks to zero. At this same instant, $B$ turns on the reverse thrusters of his spaceship and decelerates with proper deceleration $g$. $B$ eventually reaches a farthest point from $A$ and then accelerates back toward $A$, finally passing him with speed $v_0$ again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?

- Twin $B$ stands on the earth. Twin $A$ is thrown upward with speed $v_0$ (let's say he is fired from a cannon in a hole in the ground). See Fig. 13.8. At the instant they are next to each other, they both set their clocks to zero. $A$ rises up and then falls back down, finally passing $B$ with speed $v_0$ again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?

The first scenario is easily solved using special relativity. Since $A$ is in an inertial frame, he may apply the results of special relativity. In particular, $A$ sees $B$'s clock run slow, due to the usual special-relativistic time dilation. Therefore, $B$ ends up younger at the end. Note that $B$ cannot use the reverse reasoning, because she is not in an inertial frame.

What about the second scenario? The key point to realize is that the Equivalence Principle says that these two scenarios are *exactly the same*, as far as the twins are concerned. Twin $B$ has no way of telling whether she is in a spaceship accelerating at $g$ or on the surface of the earth. And $A$ has no way of telling whether he is floating freely in outer space or in free-fall in a gravitational field.[9] We therefore conclude that $B$ must be younger in the second scenario, too.

At first glance, this seems incorrect, because in the second scenario, $B$ is sitting motionless, while $A$ is the one who is moving. It seems that $B$ should see $A$'s clock running slow, due to the usual special-relativistic time dilation, and hence $A$ should be younger. This reasoning is incorrect because it fails to take into account the gravitational time dilation. The fact of the matter is that $A$ is higher in the gravitational field, and therefore his clock runs faster. This effect does indeed win out over the special-relativistic time dilation, and $A$ ends up older. You can explicitly show this in Problem 11.

Note that the reasoning in this section is another way to conclude that the Equivalence Principle implies that higher clocks must run faster (in one way or

[9]This fact is made possible by the equivalence of inertial and gravitational mass. Were it not for this, different parts of $A$'s body would accelerate at different rates in the gravitational field in the second scenario. This would certainly clue him in to the fact that he was not floating freely in space.

another). The Equivalence Principle implies that $A$ must be older in the second scenario, which means that there must be some height effect that makes $A$'s clock run fast (fast enough to win out over the special-relativistic time dilation). But it takes some more work to show that the factor is actually $1 + gh/c^2$.

Also note that the fact that $A$ is older is consistent with the maximal-proper-time principle. In both scenarios, $A$ is under the influence of only gravity (zero gravity in the first scenario), whereas $B$ feels a normal force from either the spaceship's floor or the ground.

## 13.6 Exercises

*Section 13.1: The Equivalence Principle*

1. **Driving on a hill**

   You drive up and down a hill of height $h$ at constant speed. What should your speed be so that you age the same amount as someone standing at the base of the hill? Assume that the hill is in the shape of an isosceles triangle with altitude $h$.

2. **$Lv/c^2$ and $gh/c^2$** *

   The familiar special-relativistic "head-start" result, $Lv/c^2$, looks rather similar to the $gh/c^2$ term in the GR time-dilation result, eq. (13.3). Imagine standing at the front of a train of length $L$. For small $v$, devise a thought experiment that explains how the $Lv/c^2$ result follows from the $gh/c^2$ result.

3. **Opposite circular motion** ****

   $A$ and $B$ move at speed $v$ ($v \ll c$) in opposite directions around a circle of radius $r$ (so they pass each other after each half-revolution). They both see their clocks ticking at the same rate. Show this in three ways. Work in:

   (a) The lab frame (the inertial frame whose origin is the center of the circle).

   (b) The frame whose origin is $B$ and whose axes remain parallel to an inertial set of axes.

   (c) The rotating frame that is centered at the origin and rotates along with $B$.

   *Hints:* See Problem 4, which is similar, although easier. And take a look at the Einstein limerick in Section 9.2.

*Section 13.2: Uniformly accelerated frame*

4. **Using rapidity** *

   Another way to derive the $v$ in eq. (13.4) is to use the $v = \tanh(g\tau)$ rapidity result (where $\tau$ is the particle's proper time) from Section 10.7. Use time dilation to show that this implies $gt = \sinh(g\tau)$, and hence eq. (13.4).

5. **Various quantities** *

   A particle starts at rest and accelerates with proper acceleration $g$. Let $\tau$ be the time on the particle's clock. Using the $v$ from eq. (13.4), use time dilation to show that the time $t$ in the original inertial frame, the speed of the particle, and the associated $\gamma$ factor are given by (with $c = 1$)

$$gt = \sinh(g\tau), \qquad v = \tanh(g\tau), \qquad \gamma = \cosh(g\tau). \tag{13.16}$$

6. **Redshift** $**$

We found in Section 13.2 that a clock at the back of a rocket will see a clock at the front run fast by a factor $1 + gh/c^2$. However, we ignored higher-order effects in $1/c^2$, so for all we know, the factor is actually, say, $e^{gh/c^2}$, or perhaps $\ln(gh/c^2)$, and we found only the first term in the Taylor series.

(a) For the uniformly accelerated frame in Section 13.3.2, show that the factor is in fact exactly $1 + g_b h/c^2$, where $g_b$ is the acceleration of the back of the rocket. Show this by lining up a series of clocks and looking at the successive factors between them.

(b) By the same reasoning, it follows that the front clock sees the back clock running slow by a factor $1 - g_f h/c^2$, where $g_f$ is the acceleration of the front. Show explicitly that $(1 + g_b/c^2)(1 - g_f h/c^2) = 1$, as must be the case, because a clock can't gain time with respect to itself.

7. **Gravity and speed combined** $**$

Use a Minkowski diagram to do this problem (in the spirit of Problem 10.24, "Acceleration and redshift").

A rocket accelerates with proper acceleration $g$ toward a planet. As measured in the instantaneous inertial frame of the rocket, the planet is a distance $x$ away and moves at speed $v$. Everything is in one dimension here.

As measured in the *accelerating* frame of the rocket, show that the planet's clock runs at a rate (with $c = 1$),

$$dt_p = dt_r(1 + gx)\sqrt{1 - v^2}\,. \tag{13.17}$$

And show that the planet's speed is

$$V = (1 + gx)v. \tag{13.18}$$

Note that if we combine these two results to eliminate $v$, and if we then invoke the equivalence principle, we arrive at the result that a clock moving at height $h$ and speed $V$ in a gravitational field is seen by someone on the ground to run at a rate (putting the $c$'s back in),

$$\sqrt{\left(1 + \frac{gh}{c^2}\right)^2 - \frac{V^2}{c^2}}\,. \tag{13.19}$$

8. **Speed in accelerating frame** $*$

In the setup in Problem 6, use eq. (13.20) to find the speed of the planet, $dx/d\tau$, as a function of $\tau$. What is the maximum value of this speed, in terms of $g$ and the initial distance, $L$?

9. **Accelerating stick's length** $**$

   Consider a uniformly accelerated frame consisting of a stick, the ends of which have worldlines given by the curves in Fig. 13.6 (so the stick has proper length $b-a$). At time $t$ in the lab frame, we know that a point that undergoes acceleration $g$ has position $\sqrt{1+(gt)^2}/g$ relative to the point $P$ in Fig. 13.6.

   An observer in the original inertial frame will see the stick being length-contracted by different factors along its length, because different points move with different speeds (at a given time in the original frame). Show, by doing the appropriate integral, that this observer will conclude that the stick always has proper length $b-a$.

*Section 13.3: Maximal proper-time principle*

10. **Maximum proper time** $*$

    Show that the extremum of the gravitational action in eq. (13.15) is always a minimum. Do this considering a function, $y(t) = y_0(t) + \xi(t)$, where $y_0$ is the path that extremizes the action, and $\xi$ is a small variation.

*Section 13.4: Twin paradox revisited*

11. **Symmetric twin non-paradox** $**$

    Two twins travel in opposite directions at speed $v$ ($v \ll c$) with respect to the earth. They synchronize their clocks when they pass each other. They travel to stars located a distance $\ell$ away, and then decelerate and accelerate back up to speed $v$ in the opposite direction (uniformly, and in a short time compared to the total journey time).

    In the frame of the earth, it is obvious (from symmetry) that both twins age the same amount by the time they pass each other again. Reproduce this result by working in the frame of one of the twins.

## 13.7 Problems

*Section 13.1: The Equivalence Principle*

1. **Airplane's speed**

   A plane flies at constant height $h$. What should its speed be so that an observer on the ground sees the plane's clock tick at the same rate as a ground clock? (Assume $v \ll c$.)

2. **Clock on tower** $**$

   A clock starts on the ground and then moves up a tower at constant speed $v$. It sits on top of the tower for a time $T$ and then descends at constant speed $v$. If the tower has height $h$, how long should the clock sit at the top so that it comes back showing the same time as a clock that remained on the ground? (Assume $v \ll c$.)

3. **Circular motion** $**$

   Person $B$ moves at speed $v$ (with $v \ll c$) in a circle of radius $r$ around person $A$. By what fraction does $B$'s clock run slower than $A$'s? Calculate this in three ways. Work in:

   (a) $A$'s frame.

   (b) The frame whose origin is $B$ and whose axes remain parallel to an inertial set of axes.

   (c) The rotating frame that is centered at $A$ and rotates around $A$ with the same frequency as $B$.

4. **More circular motion** $**$

   $A$ and $B$ move at speed $v$ ($v \ll c$) in a circle of radius $r$, at diametrically opposite points. They both see their clocks ticking at the same rate. Show this in three ways. Work in:

   (a) The lab frame (the inertial frame whose origin is the center of the circle).

   (b) The frame whose origin is $B$ and whose axes remain parallel to an inertial set of axes.

   (c) The rotating frame that is centered at the origin and rotates with the same frequency as $A$ and $B$.

*Section 13.2: Uniformly accelerated frame*

5. **Getting way ahead** $****$

   A rocket with proper length $L$ accelerates from rest, with proper acceleration $g$ (where $gL \ll c^2$). Clocks are located at the front and back of the rocket. If we look at this setup in the frame of the rocket, then the general-relativistic time-dilation effect tells us that the times on the two clocks are related by

$t_f = (1 + gL/c^2)t_b$. Therefore, if we look at things in the ground frame, then the times on the two clocks are related by

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}\,,$$

where the last term is the standard special-relativistic "head-start" result. Derive the above relation by working entirely in the ground frame.[10]

6. **Accelerator's point of view** ***

   A rocket starts at rest relative to a planet, a distance $\ell$ away. It accelerates toward the planet with proper acceleration $g$. Let $\tau$ and $t$ be the readings on the rocket's and planet's clocks, respectively.

   (a) Show that when the astronaut's clock reads $\tau$, he observes the rocket-planet distance, $x$ (as measured in his instantaneous inertial frame), to be given by

   $$1 + gx = \frac{1 + g\ell}{\cosh(g\tau)}\,. \tag{13.20}$$

   (b) Show that when the astronaut's clock reads $\tau$, he observes the time, $t$, on the planet's clock to be given by

   $$gt = (1 + g\ell)\tanh(g\tau). \tag{13.21}$$

   The results from Exercises 5 and 7 will be useful here.

7. **$Lv/c^2$ revisited** **

   You stand at rest relative to a rocket that has synchronized clocks at its ends. It is then arranged for you and the rocket to move with relative speed $v$. A reasonable question to now ask is: As viewed by you, what is the difference in readings on the clocks located at the ends of the rocket?

   It turns out that this question cannot be answered without further information on how you and the rocket got to be moving with relative speed $v$. There are two basic ways this relative speed can come about. The rocket can accelerate while you sit there, or you can accelerate while the rocket sits there. Using the results from Problems 5 and 6, explain what the answers to the above question are in these two cases.

8. **Circling the earth** **

   Clock $A$ sits at rest on the earth, and clock $B$ circles the earth in an orbit that skims along the ground. Both $A$ and $B$ are essentially at the same radius, so

[10] You may find this relation surprising, because it implies that the front clock will eventually be an arbitrarily large time ahead of the back clock, in the ground frame. (The subtractive $Lv/c^2$ term is bounded by $L/c$ and will therefore eventually become negligible compared to the additive, and unbounded, $(gL/c^2)t_b$ term.) But both clocks are doing basically the same thing relative to the ground frame, so how can they eventually differ by so much? Your job is to find out.

the GR time-dilation effect yields no difference in their times. But $B$ is moving relative to $A$, so $A$ will see $B$ running slow, due to the usual SR time-dilation effect. The orbiting clock, $B$, will therefore show a *smaller* elapsed proper time each time it passes $A$. In other words, the clock under the influence of only gravity ($B$) does *not* show the maximal proper time, in conflict with what we have been calling the maximal-proper-time principle. Explain.

*Section 13.4: Twin paradox revisited*

9. **Twin paradox** $*$

   A spaceship travels at speed $v$ ($v \ll c$) to a distant star. Upon reaching the star, it decelerates and then accelerates back up to speed $v$ in the opposite direction (uniformly, and in a short time compared to the total journey time). By what fraction does the traveler age less than her twin on earth? (Ignore the gravity from the earth.) Work in:

   (a) The earth frame.

   (b) The spaceship frame.

10. **Twin paradox again** $**$

    (a) Answer the previous problem, except now let the spaceship turn around by moving in a small semicircle while maintaining speed $v$.

    (b) Answer the previous problem, except now let the spaceship turn around by moving in an arbitrary manner. The only constraints are that the turn-around is done quickly (compared to the total journey time), and that it is contained in a small region of space (compared to the earth-star distance).

11. **Twin paradox times** $***$

    (a) In the first scenario in Section 13.5, calculate the ratio of $B$'s elapsed time to $A$'s, in terms of $v_0$ and $g$. Assume that $v_0 \ll c$, and drop high-order terms.

    (b) Do the same for the second scenario in Section 13.5. Do this from scratch using the time dilations, and then check that your answer agrees (within the accuracy of the calculations) with part (a), as the equivalence principle demands.

## 13.8 Solutions

1. **Airplane's speed**

   An observer on the ground sees the plane's clock run slow by a factor $\sqrt{1-v^2/c^2}$ due to SR time dilation. But he also sees it run fast by a factor $(1+gh/c^2)$ due to GR time dilation. We therefore want the product of these two factors to equal 1. Using the standard Taylor-series approximation for slow speeds in the first factor, we find

$$\left(1-\frac{v^2}{2c^2}\right)\left(1+\frac{gh}{c^2}\right)=1 \qquad \Longrightarrow \qquad 1-\frac{v^2}{2c^2}+\frac{gh}{c^2}-\mathcal{O}\left(\frac{1}{c^4}\right)=1. \tag{13.22}$$

   Neglecting the small $1/c^4$ term, and cancelling the 1's, yields $v=\sqrt{2gh}$.

   Interestingly, $\sqrt{2gh}$ is also the answer to a standard question from Newtonian physics, namely, how fast must you throw a ball straight up if you want it to reach a height $h$?

2. **Clock on tower**

   The SR time-dilation factor is $\sqrt{1-v^2/c^2}\approx 1-v^2/2c^2$. The clock therefore loses a fraction $v^2/2c^2$ of the time elapsed during its motion up and down the tower. The upward journey takes a time $h/v$, and likewise for the downward trip, so the time loss due to the SR effect is

$$\left(\frac{v^2}{2c^2}\right)\left(\frac{2h}{v}\right)=\frac{vh}{c^2}\,. \tag{13.23}$$

   Our goal is to balance this time loss with the time gain due to the GR time-dilation effect. If the clock sits on top of the tower for a time $T$, then the time gain is

$$\left(\frac{gh}{c^2}\right)T. \tag{13.24}$$

   But we must not forget also the increase in time due to the height gained while the clock is in motion. During its motion, the clock's average height is $h/2$. The total time in motion is $2h/v$, so the GR time gain while the clock is moving is

$$\left(\frac{g(h/2)}{c^2}\right)\left(\frac{2h}{v}\right)=\frac{gh^2}{c^2v}\,. \tag{13.25}$$

   Setting the total change in the clock's time equal to zero gives

$$-\frac{vh}{c^2}+\frac{gh}{c^2}T+\frac{gh^2}{c^2v}=0 \quad \Longrightarrow \quad -v+gT+\frac{gh}{v}=0. \tag{13.26}$$

   Therefore,

$$T=\frac{v}{g}-\frac{h}{v}\,. \tag{13.27}$$

   REMARKS: Note that we must have $v>\sqrt{gh}$ in order for a positive solution for $T$ to exist. If $v<\sqrt{gh}$, then the SR effect is too small to cancel out the GR effect, even if the clock spends no time sitting at the top. If $v=\sqrt{gh}$, then $T=0$, and we essentially have the same situation as in Exercise 1. Note also that if $v$ is very large compared to $\sqrt{gh}$ (but still small compared to $c$, so that our $\sqrt{1-v^2/c^2}\approx 1-v^2/2c^2$ approximation is valid), then $T\approx v/g$, which is independent of $h$. ♣

3. **Circular motion**

(a) In $A$'s frame, there is only the SR time-dilation effect. $A$ sees $B$ move at speed $v$, so $B$'s clock runs slow by a factor of $\sqrt{1 - v^2/c^2}$. And since $v \ll c$, we may use the Taylor series to approximate this as $1 - v^2/2c^2$.

(b) In this frame, there are both SR and GR time-dilation effects. $A$ moves at speed $v$ with respect to $B$ in this frame, so there is the SR effect that $A$'s clock runs slow by a factor $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$.

But $B$ undergoes an acceleration of $a = v^2/r$ toward $A$, so there is also the GR effect that $A$'s clock runs fast by a factor $1 + ar/c^2 = 1 + v^2/c^2$.

Multiplying these two effects together, we find (to lowest order) that $A$'s clock runs fast by a factor $1 + v^2/2c^2$. This means (to lowest order) that $B$'s clock runs slow by a factor $1 - v^2/2c^2$, in agreement with the answer to part (a).

(c) In this frame, there is no relative motion between $A$ and $B$, so there is only the GR time-dilation effect. The gravitational field (that is, the centripetal acceleration) at a distance $x$ from the center is $g_x = x\omega^2$. Imagine lining up a series of clocks along a radius, with separation $dx$. Then the GR time-dilation result tells us that each clock loses a fraction $g_x\,dx/c^2 = x\omega^2\,dx/c^2$ of time relative to the clock just inside it. Integrating these fractions from $x = 0$ to $x = r$ shows that $B$'s clock loses a fraction $r^2\omega^2/2c^2 = v^2/2c^2$, compared to $A$'s clock. This agrees with the results in parts (a) and (b).

4. **More circular motion**

(a) In the lab frame, the situation is symmetric with respect to $A$ and $B$. Therefore, if $A$ and $B$ are decelerated in a symmetric manner and brought together, then their clocks must read the same time.

Assume (in the interest of obtaining a contradiction), that $A$ sees $B$'s clock run slow. Then after an arbitrarily long time, $A$ will see $B$'s clock an arbitrarily large time behind his. Now bring $A$ and $B$ to a stop. There is no possible way that the stopping motion can make $B$'s clock gain an arbitrarily large amount of time, as seen by $A$. This is true because everything takes place in a finite region of space, so there is an upper bound on the GR time-dilation effect (because it behaves like $gh/c^2$, and $h$ is bounded). Therefore, $A$ will end up seeing $B$'s clock reading less. This contradicts the result of the previous paragraph.

Remark: Note how this problem differs from the problem where $A$ and $B$ move with equal speeds directly away from each other, and then reverse directions and head back to meet up again.

For this new "linear" problem, the symmetry reasoning in the first paragraph above still holds; they will indeed have the same clock readings when they meet up again. But the reasoning in the second paragraph does not hold (it better not, because each person does *not* see the other person's clock running at the same rate). The error is that in this linear scenario, the experiment is not contained in a small region of space, so the turning-around effects of order $gh/c^2$ become arbitrarily large as the time of travel becomes arbitrarily large, since $h$ grows with time (see Problem 9). ♣

(b) In this frame, there are both SR and GR time-dilation effects. $A$ moves at speed $2v$ with respect to $B$ in this frame (we don't need to use the relativistic velocity-addition formula, because $v \ll c$), so this gives the SR effect that $A$'s clock runs slow by a factor $\sqrt{1 - (2v)^2/c^2} \approx 1 - 2v^2/c^2$.

But $B$ undergoes an acceleration of $a = v^2/r$ toward $A$, so there is also the GR effect that $A$'s clock runs fast by a factor $1 + a(2r)/c^2 = 1 + 2v^2/c^2$ (because they are separated by a distance $2r$).

Multiplying these two effects together, we find (to lowest order) that the two clocks run at the same rate.

(c) In this frame, there is no relative motion between $A$ and $B$. Hence, there is only the GR effect. But $A$ and $B$ are both at the same gravitational potential, because they are at the same radius. Therefore, they both see the clocks running at the same rate.

If you want, you can line up a series of clocks along the diameter between $A$ and $B$, as we did along a radius in part (c) of Problem 3. The clocks will gain time as you march in toward the center, and then lose back the same amount of time as you march back out to the diametrically opposite point.

5. **Getting way ahead**

The explanation of why the two clocks show different times in the ground frame is the following. The rocket becomes increasingly length contracted in the ground frame, which means that the front end isn't traveling as fast as the back end. Therefore, the time-dilation factor for the front clock isn't as large as that for the back clock. So the front clock loses less time relative to the ground, and hence ends up ahead of the back clock. Of course, it's not at all obvious that everything works out quantitatively, and that the front clock eventually ends up an arbitrarily large time ahead of the back clock. In fact, it's quite surprising that this is the case, because the above difference in speeds is rather small. But let's now show that the above explanation does indeed account for the difference in the clock readings.

Let the back of the rocket be located at position $x$. Then the front is located at position $x + L\sqrt{1-v^2}$ (with $c = 1$), due to the length contraction. Taking the time derivatives of the two positions, we see that the speeds of the back and front are (with $v \equiv dx/dt$)[11]

$$v_b = v, \qquad \text{and} \qquad v_f = v(1 - L\gamma\dot{v}). \tag{13.28}$$

For $v_b$, we will simply invoke the result in eq. (13.4),

$$v_b = v = \frac{gt}{\sqrt{1+(gt)^2}}\,, \tag{13.29}$$

where $t$ is the time in the ground frame.

Having found $v$, we must now find the $\gamma$-factors associated with the speeds of the front and back of the rocket. The $\gamma$-factor associated with the speed of the back (namely $v$) is

$$\gamma_b = \frac{1}{\sqrt{1-v^2}} = \sqrt{1+(gt)^2}\,. \tag{13.30}$$

The $\gamma$-factor associated with the speed of the front, $v_f = v(1 - L\gamma\dot{v})$, is a little harder to obtain. We must first calculate $\dot{v}$. From eq. (13.29), we find $\dot{v} = g/(1 + g^2t^2)^{3/2}$, which gives

$$v_f = v(1 - L\gamma\dot{v}) = \frac{gt}{\sqrt{1+(gt)^2}}\left(1 - \frac{gL}{1+g^2t^2}\right). \tag{13.31}$$

[11] Since these speeds are not equal, there is of course an ambiguity concerning which speed we should use in the length-contraction factor, $\sqrt{1-v^2}$. Equivalently, the rocket actually doesn't have one inertial frame that describes all of it. But you can show that any differences arising from this ambiguity are of higher order in $gL/c^2$ than we need to be concerned with.

The $\gamma$-factor (or rather $1/\gamma$, which is what we'll be concerned with) associated with this speed can now be found as follows. In the first line below, we ignore the higher-order $(gL)^2$ term, because it is really $(gL/c^2)^2$, and we are assuming that $gL/c^2$ is small. And in obtaining the third line, we use the Taylor-series approximation, $\sqrt{1-\epsilon} \approx 1-\epsilon/2$.

$$\begin{aligned} \frac{1}{\gamma_f} = \sqrt{1-v_f^2} &\approx \sqrt{1-\frac{g^2t^2}{1+g^2t^2}\left(1-\frac{2gL}{1+g^2t^2}\right)} \\ &= \frac{1}{\sqrt{1+g^2t^2}}\sqrt{1+\frac{2g^3t^2L}{1+g^2t^2}} \\ &\approx \frac{1}{\sqrt{1+g^2t^2}}\left(1+\frac{g^3t^2L}{1+g^2t^2}\right). \end{aligned} \tag{13.32}$$

We can now calculate the time that each clock shows, at time $t$ in the ground frame. The time on the back clock changes according to $dt_b = dt/\gamma_b$, so eq. (13.30) gives

$$t_b = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}}. \tag{13.33}$$

The integral[12] of $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$. Letting $x \equiv gt$, this gives

$$gt_b = \sinh^{-1}(gt). \tag{13.34}$$

The time on the front clock changes according to $dt_f = dt/\gamma_f$, so eq. (13.32) gives

$$t_f = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} + \int_0^t \frac{g^3t^2L\,dt}{(1+g^2t^2)^{3/2}}. \tag{13.35}$$

The integral[13] of $x^2/(1+x^2)^{3/2}$ is $\sinh^{-1} x - x/\sqrt{1+x^2}$. Letting $x \equiv gt$, this gives

$$gt_f = \sinh^{-1}(gt) + (gL)\left(\sinh^{-1}(gt) - \frac{gt}{\sqrt{1+g^2t^2}}\right). \tag{13.36}$$

Using eqs. (13.29), (13.34), and (13.29), we may rewrite this as

$$gt_f = gt_b(1+gL) - gLv. \tag{13.37}$$

Dividing by $g$, and putting the $c$'s back in to make the units correct, we finally have

$$t_f = t_b\left(1+\frac{gL}{c^2}\right) - \frac{Lv}{c^2}, \tag{13.38}$$

as we wanted to show.

REMARK: Looked at from the reverse point of view, this calculation, which uses only special-relativity concepts, demonstrates that someone at the back of a rocket sees a clock at the front running fast by a factor $(1+gL/c^2)$. There are, however, far easier ways of deriving this, as we saw in Section 13.2 and in Problem 10.24 ("Acceleration and redshift"). ♣

[12] To derive this, make the substitution $x \equiv \sinh\theta$.
[13] Again, to derive this, make the substitution $x \equiv \sinh\theta$.

6. **Accelerator's point of view**

(a) **First Solution:** Eq. (13.5) says that the distance traveled by the rocket (as measured in the original inertial frame), as a function of the time in the inertial frame, is

$$d = \frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right). \tag{13.39}$$

An inertial observer on the planet therefore measures the rocket-planet distance to be

$$x = \ell - \frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right). \tag{13.40}$$

The rocket observer will see this length being contracted by a factor $\gamma$. Using the result of Exercise 5, we have $\gamma = \sqrt{1+(gt)^2} = \cosh(g\tau)$. So the rocket-planet distance, as measured in the instantaneous inertial frame of the rocket, is

$$x = \frac{\ell - \frac{1}{g}\big(\cosh(g\tau) - 1\big)}{\cosh(g\tau)} \qquad \Longrightarrow \qquad 1 + gx = \frac{1+g\ell}{\cosh(g\tau)}\,, \tag{13.41}$$

as desired.

**Second Solution:** Eq. (13.18) gives the speed of the planet in the accelerating frame of the rocket. Using the results of Exercise 5 to write $v$ in terms of $\tau$, we have (with $c = 1$)

$$\frac{dx}{d\tau} = -(1+gx)\tanh(g\tau). \tag{13.42}$$

Separating variables and integrating gives

$$\int \frac{dx}{1+gx} = -\int \tanh(g\tau)\,d\tau \qquad \Longrightarrow \qquad \ln(1+gx) = -\ln\big(\cosh(g\tau)\big) + C$$

$$\Longrightarrow \qquad 1+gx = \frac{A}{\cosh(g\tau)}\,. \tag{13.43}$$

Since the initial condition is $x = \ell$ when $\tau = 0$, we must have $A = 1 + g\ell$, which gives eq. (13.20), as desired.

(b) Eq. (13.17) says that the planet's clock runs fast (or slow) according to

$$dt = d\tau\,(1+gx)\sqrt{1-v^2}. \tag{13.44}$$

The results of Exercise 5 yield $\sqrt{1-v^2} = 1/\cosh(g\tau)$. Combining this with the result for $1 + gx$ above, and integrating, gives

$$\int dt = \int \frac{(1+g\ell)\,d\tau}{\cosh^2(g\tau)} \qquad \Longrightarrow \qquad gt = (1+g\ell)\tanh(g\tau), \tag{13.45}$$

as desired.

7. $Lv/c^2$ **revisited**

Consider first the case where the rocket accelerates while you sit there. Problem 5 is exactly relevant here, and it tells us that in your frame the clock readings are related by

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}\,. \tag{13.46}$$

You will eventually see the front clock an arbitrarily large time ahead of the back clock. Note that for small times (before things become relativistic), the standard Newtonian result, $v \approx gt_b$, is valid, so we have

$$t_f \approx \left(t_b + \frac{Lv}{c^2}\right) - \frac{Lv}{c^2} = t_b. \tag{13.47}$$

We see that in the setup where the rocket is the one that accelerates, both clocks show essentially the same time near the start (the leading term in the time difference is of order $(v/c)^2$). This makes sense; both clocks have essentially the same speed at the beginning, so to lowest order their $\gamma$ factors are the same, so the clocks run at the same rate. But eventually the front clock will get ahead of the back clock.

Now consider the case where you accelerate while the rocket sits there. Problem 6 is relevant here, if we let the rocket in that problem now become you, and if we let two planets a distance $L$ apart become the two ends of the rocket. The times you observe on the front and back clocks on the rocket are then, using eq. (13.45) and assuming that you are accelerating toward the rocket,

$$gt_f = (1 + g\ell)\tanh(g\tau), \qquad \text{and} \qquad gt_b = \big(1 + g(\ell + L)\big)\tanh(g\tau). \tag{13.48}$$

But from Exercise 5, we know that your speed relative to the rocket is $v = \tanh g\tau$. Eq. (13.48) therefore gives $t_b = t_f + Lv$, or $t_b = t_f + Lv/c^2$ with the $c^2$. So in this case we arrive at the standard $Lv/c^2$ "head-start" result.

The point here is that in this second case, the clocks are synchronized in the rocket frame, and this is the assumption that went into our derivation of the $Lv/c^2$ result in Chapter 10. In the first case above where the rocket accelerates, the clocks are *not* synchronized in the rocket frame (except right at the start), so it's not surprising that we don't obtain the $Lv/c^2$ result.

8. **Circling the earth**

This is one setup where we really need to use the correct term, *stationary*-proper-time principle. It turns out that $B$'s path yields a saddle point for the proper time. The value at this saddle point is less than $A$'s proper time, but this is irrelevant, because we only care about local extrema, not global ones.

$B$'s path is a saddle point because there exist nearby paths that give both a larger and smaller proper time.[14] The proper time can be made smaller by having $B$ speed up and slow down. This will cause a net increase in the time-dilation effect as viewed by $A$, thereby yielding a smaller proper time.[15] The proper time can be made larger by having $B$ take a nearby path that doesn't quite form a great circle on the earth. (Imagine the curve traced out by a rubber band that has just begun to slip away from a great-circle position.) This path is shorter, so $B$ won't have to travel as fast to get back in a given time, so the time-dilation effect will be smaller as viewed by $A$, thereby yielding a larger proper time.

[14] The differences are in fact second-order ones, because the first-order ones vanish due to the fact that the path satisfies the Euler-Lagrange equations for the Lagrangian in eq. (13.15).

[15] This is true for the same reason that a person who travels at constant speed in a straight line between two points will show a larger proper time than a second person who speeds up and slows down. This follows directly from SR time dilation, as viewed by the first person. If you want, you can imagine unrolling $B$'s circular orbit into a straight line, and then invoke the result just mentioned. As far as SR time-dilation effects from clock $A$'s point of view go, it doesn't matter if the circle is unrolled into a straight line.

9. **Twin paradox**

(a) In the earth frame, the spaceship travels at speed $v$ for essentially the whole time. Therefore, the traveler ages less by a fraction $\sqrt{1-v^2/c^2} \approx 1 - v^2/2c^2$. The fractional loss of time is thus $v^2/2c^2$. The time-dilation effect will be different during the short turning-around period, but this is negligible.

(b) Let the distance to the star be $\ell$, as measured in the frame of the earth (but the difference in lengths in the two frames is negligible in this problem), and let the turnaround take a time $T$. Then the given information says that $T \ll (2\ell)/v$.

During the constant-speed part of the trip, the traveler sees the earth clock running slow by a fraction $\sqrt{1-v^2/c^2} \approx 1 - v^2/2c^2$. The time for this constant-speed part is $2\ell/v$, so the earth clock loses a time of $(v^2/2c^2)(2\ell/v) = v\ell/c^2$.

However, during the turnaround time, the spaceship is accelerating toward the earth, so the traveler sees the earth clock running fast, due to the GR time dilation. The magnitude of the acceleration is $a = 2v/T$, because the spaceship goes from velocity $v$ to $-v$ in time $T$. The earth clock therefore runs fast by a factor $1 + a\ell/c^2 = 1 + 2v\ell/Tc^2$. This happens for a time $T$, so the earth clock gains a time of $(2v\ell/Tc^2)T = 2v\ell/c^2$.

Combining the results of the previous two paragraphs, we see that the earth clock gains a time of $2v\ell/c^2 - v\ell/c^2 = v\ell/c^2$. This is a fraction $(v\ell/c^2)/(2\ell/v) = v^2/2c^2$ of the total time, in agreement with part (a).

10. **Twin paradox again**

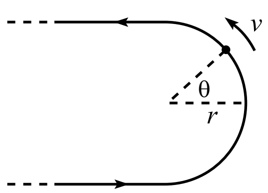


**Figure 13.9**

(a) The only difference between this problem and the previous one is the nature of the turnaround, so all we need to show here is that the traveler still sees the earth clock gain a time of $2v\ell/c^2$ during the turnaround.

Let the radius of the semicircle be $r$. Then the magnitude of the acceleration is $a = v^2/r$. Let $\theta$ be the angle shown in Fig. 13.9. For a given $\theta$, the earth is at a height of essentially $\ell\cos\theta$ in the gravitational field felt by the spaceship. The fractional time that the earth gains while the traveler is at angle $\theta$ is therefore $ah/c^2 = (v^2/r)(\ell\cos\theta)/c^2$. Integrating this over the time of the turnaround, and using $dt = r\,d\theta/v$, we see that the earth gains a time of

$$\Delta t = \int_{-\pi/2}^{\pi/2} \left(\frac{v^2\ell\cos\theta}{rc^2}\right)\left(\frac{r\,d\theta}{v}\right) = \frac{2v\ell}{c^2}\,, \tag{13.49}$$

during the turnaround, as we wanted to show.

(b) Let the acceleration vector at a given instant be $\mathbf{a}$, and let $\boldsymbol{\ell}$ be the vector from the spaceship to the earth. Note that since the turnaround is done in a small region of space, $\boldsymbol{\ell}$ is essentially constant here.

The earth is at a height of essentially $\hat{\mathbf{a}}\cdot\boldsymbol{\ell}$ in the gravitational field felt by the spaceship. (The dot product just gives the cosine term in the above solution to part (a).) The fractional time gain, $ah/c^2$, is therefore equal to $|\mathbf{a}|(\hat{\mathbf{a}}\cdot\boldsymbol{\ell})/c^2 = \mathbf{a}\cdot\boldsymbol{\ell}/c^2$. Integrating this over the time of the turnaround, we see that the earth gains a time of

$$\begin{aligned}\Delta t = \int_{t_i}^{t_f} \frac{\mathbf{a}\cdot\boldsymbol{\ell}}{c^2}\,dt &= \frac{\boldsymbol{\ell}}{c^2}\cdot\int_{t_i}^{t_f}\mathbf{a}\,dt \\ &= \frac{\boldsymbol{\ell}}{c^2}\cdot(\mathbf{v}_f - \mathbf{v}_i)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\boldsymbol{\ell}\cdot(2\mathbf{v}_f)}{c^2} \\
&= \frac{2v\ell}{c^2}\,, \qquad (13.50)
\end{aligned}
$$

during the turnaround, as we wanted to show. The whole point here is that no matter what complicated motion the traveler undergoes during the turnaround, the total effect is to simply change the velocity from **v** outward to **v** inward.

11. **Twin paradox times**

(a) As viewed by $A$, the times of the twins are related by

$$dt_B = \sqrt{1-v^2}\,dt_A. \qquad (13.51)$$

Assuming $v_0 \ll c$, we may say that $v(t_A)$ is essentially equal to $v_0 - gt_A$, so the out and back parts of the trip each take a time of essentially $v_0/g$ in $A$'s frame. The total elapsed time on $B$'s clock is therefore

$$
\begin{aligned}
T_B = \int dt_B &\approx 2\int_0^{v_0/g} \sqrt{1-v^2}\,dt_A \\
&\approx 2\int_0^{v_0/g} \left(1-\frac{v^2}{2}\right) dt_A \\
&\approx 2\int_0^{v_0/g} \left(1-\frac{1}{2}(v_0-gt)^2\right) dt \\
&= 2\left(t+\frac{1}{6g}(v_0-gt)^3\right)\Big|_0^{v_0/g} \\
&= \frac{2v_0}{g} - \frac{v_0^3}{3gc^2}\,, \qquad (13.52)
\end{aligned}
$$

where we have put the $c$'s back in to make the units right. The ratio of $B$'s elapsed time to $A$'s is therefore

$$\frac{T_B}{T_A} \approx \frac{T_B}{2v_0/g} \approx 1 - \frac{v_0^2}{6c^2}\,. \qquad (13.53)$$

(b) As viewed by $B$, the relation between the twins' times is given by eq. (13.13),

$$dt_A = \sqrt{1-\frac{v^2}{c^2}}\left(1+\frac{gy}{c^2}\right) dt_B. \qquad (13.54)$$

Assuming $v_0 \ll c$, we may say that $v(t_B)$ is essentially equal to $v_0 - gt_B$, and $A$'s height is essentially equal to $v_0 t_B - gt_B^2/2$. The up and down parts of the trip each take a time of essentially $v_0/g$ in $B$'s frame. Therefore, the total elapsed time on $A$'s clock is (using the approximation in eq. (13.14), and dropping the $c$'s)

$$
\begin{aligned}
T_A = \int dt_A &\approx 2\int_0^{v_0/g} \left(1-\frac{v^2}{2}+gy\right) dt_B. \\
&\approx 2\int_0^{v_0/g} \left(1-\frac{1}{2}(v_0-gt)^2 + g(v_0 t - gt^2/2)\right) dt.
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left(t + \frac{1}{6g}(v_0 - gt)^3 + g\Big(\frac{v_0 t^2}{2} - \frac{gt^3}{6}\Big)\right)\Bigg|_0^{v_0/g} \\
&= \frac{2v_0}{g} - \frac{v_0^3}{3g} + g\left(\frac{v_0^3}{g^2} - \frac{v_0^3}{3g^2}\right) \\
&= \frac{2v_0}{g} + \frac{v_0^3}{3gc^2}\,,
\end{aligned}
\tag{13.55}
$$

where we have put the $c$'s back in to make the units right. We therefore have

$$
\frac{T_A}{T_B} \approx \frac{T_A}{2v_0/g} \approx 1 + \frac{v_0^2}{6c^2} \qquad \Longrightarrow \qquad \frac{T_B}{T_A} \approx 1 - \frac{v_0^2}{6c^2}\,, \tag{13.56}
$$

up to corrections of order $v_0^4/c^4$. This agrees with the result found in part (a), as the equivalence principle requires.

# Chapter 14

# Appendices



## 14.1 Appendix A: Useful formulas

### 14.1.1 Taylor series

$$f(x_0+\epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{f''(x_0)}{2!}\,\epsilon^2 + \frac{f'''(x_0)}{3!}\,\epsilon^3 + \cdots \tag{14.1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{14.2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{14.3}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \tag{14.4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{14.5}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{14.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{14.7}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{14.8}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{14.9}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{14.10}$$

### 14.1.2 Nice formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{14.11}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{14.12}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}, \qquad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \tag{14.13}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{14.14}$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{14.15}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{14.16}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{14.17}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{14.18}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}) \tag{14.19}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{14.20}$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x \tag{14.21}$$

### 14.1.3 Integrals

$$\int \ln x\, dx = x\ln x - x \tag{14.22}$$

$$\int x\ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} \tag{14.23}$$

$$\int xe^x = e^x(x-1) \tag{14.24}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \qquad \text{or} \qquad -\cot^{-1} x \tag{14.25}$$

$$\int \frac{dx}{x(1+x^2)} = \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right) \tag{14.26}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{or} \qquad \tanh^{-1} x \qquad (x^2<1) \tag{14.27}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad \text{or} \qquad \coth^{-1} x \qquad (x^2>1) \tag{14.28}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \qquad \text{or} \qquad -\cos^{-1} x \tag{14.29}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(x+\sqrt{x^2+1}\,) \qquad \text{or} \qquad \sinh^{-1} x \tag{14.30}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1}\,) \qquad \text{or} \qquad \cosh^{-1} x \tag{14.31}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \qquad \text{or} \qquad -\csc^{-1} x \tag{14.32}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) \qquad \text{or} \qquad -\operatorname{csch}^{-1} x \tag{14.33}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \text{or} \qquad -\operatorname{sech}^{-1} x \tag{14.34}$$

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+\sin x}{\cos x}\right) \tag{14.35}$$

$$\int \frac{dx}{\sin x} = \ln\left(\frac{1-\cos x}{\sin x}\right) \tag{14.36}$$

## 14.2 Appendix B: Units, dimensional analysis

There are two strategies you should invoke without hesitation when solving a problem. One is the consideration of units (that is, dimensions), which is the subject of this appendix. The other is the consideration of limiting cases, which is the subject of the next appendix.

The consideration of units offers two main benefits. First, looking at units before you start a calculation can tell you roughly what the answer has to look like, up to numerical factors. (And in some problems, you can determine the numerical factors by considering a limiting case of a certain parameter. So in some problems, you actually don't have to do *any* calculations!) Second, checking units at the end of a calculation (which is something you should *always* do) tells you if your answer has a chance at being correct. It won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect. If your goal in a problem is to find, say, a length, and if you end up with a mass, then you know it's time to look over your work.

"Your units are wrong!" cried the teacher.
"Your church weighs six joules — what a feature!
And the people inside
Are four hours wide,
And eight gauss away from the preacher!"

In practice, the second of the above two benefits is what you will generally make use of. But let's do a few examples relating to the first benefit, since these can be a little more exciting. To solve the following problems exactly, we would need to invoke results derived in earlier chapters in the text. But let's just see how far we can get by using only dimensional analysis. We'll use the "[ ]" notation for units, and we'll let $M$ stand for mass, $L$ for length, and $T$ for time. For example, we will write a speed as $[v] = L/T$ and the gravitational constant as $[G] = L^3/(MT^2)$ (you can figure this out by noting that $Gm_1m_2/r^2$ has the dimensions of force). Alternatively, you can just use the mks units, kg, m, s, instead of $M$, $L$, $T$, respectively.[1]

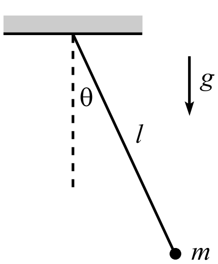

**Figure 14.1**

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**Example 1 (The pendulum):** A mass $m$ hangs from a massless string of length $\ell$ (see Fig. 14.1) and swings back and forth in the plane of the paper. The acceleration due to gravity is $g$. What can we say about the frequency of oscillations?

The only dimensionful quantities given in the problem are $[m] = M$, $[\ell] = L$, and $[g] = L/T^2$. There is one more quantity, the maximum angle $\theta_0$, which is dimensionless (this one is easy to forget). Our goal is to find the frequency, which has units of $1/T$. The only combination of our given dimensionful quantities that has units of $1/T$ is

[1]When you check units at the end of a calculation, you will invariably be working with the kg,m,s notation. So that notation will inevitably get used more. But I'll use the $M$,$L$,$T$ notation in this appendix, because I think it's a little more instructive. At any rate, just remember that the letter m (or $M$) stands for "meter" in one case, and "mass" in the other.

$\sqrt{g/\ell}$.[2] But we can't rule out any $\theta_0$ dependence, so the most general possible form for the frequency (in radians per second) is

$$\omega = f(\theta_0)\sqrt{\frac{g}{\ell}}\,, \tag{14.37}$$

where $f$ is a dimensionless function of the dimensionless variable $\theta_0$.

REMARKS: It just so happens that for small oscillations, $f(\theta_0)$ is essentially equal to 1, and so the frequency is essentially equal to $\sqrt{g/\ell}$. But there is no way to show this by using only dimensional analysis. For larger values of $\theta_0$, the higher-order terms in the expansion of $f$ become important. Exercise 3.8 deals with the leading correction; the answer is $f(\theta_0) = 1 - \theta_0^2/16 + \cdots$.

Note that since there is only one mass scale in the problem, there is no way that the frequency (with units of $1/T$) can depend on $[m] = M$. If it did, there would be nothing to cancel out the units of mass and produce a pure inverse-time. ♣

What can we say about the total energy (relative to the lowest point) of the pendulum? Energy has units of $ML^2/T^2$, and the only combination of the given dimensionful constants of this form is $mg\ell$. Therefore, the energy must be of the form $f(\theta_0)mg\ell$, where $f$ is some function. That's as far as we can go with dimensional analysis. Of course, if we actually invoke a little physics, we know that the total energy equals the potential energy at the highest point, which is $mg\ell(1 - \cos\theta_0)$. Using the Taylor expansion for $\cos\theta$, we see that $f(\theta_0) = \theta_0^2/2 - \theta_0^4/24 + \cdots$. Unlike in the frequency above, the maximum angle, $\theta_0$, plays a critical role in the energy.

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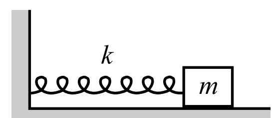


**Figure 14.2**

**Example 2 (The spring):** A spring with spring-constant $k$ has a mass $m$ on its end (see Fig. 14.2). The spring force is $F(x) = -kx$, where $x$ is the displacement from the equilibrium. What can we say about the frequency of oscillations?

The only dimensionful quantities in this problem are $[m] = M$, $[k] = M/T^2$ (obtained by noting that $kx$ has the dimensions of force), and the maximum displacement from the equilibrium, $[x_0] = L$. (There is also the equilibrium length, but the force doesn't depend on this, so there is no way it can come into the answer.) Our goal is to find the frequency, which has units of $1/T$. The only combination of our given dimensionful quantities with these units is

$$\omega = C\sqrt{\frac{k}{m}}\,, \tag{14.38}$$

where $C$ is a dimensionless number. Note that, in contrast with the pendulum above, the frequency cannot have any dependence on the maximum displacement. It just so happens that $C$ is equal to 1, but there is no way to show this by using only dimensional analysis.

What can we say about the total energy of the spring? Energy has units of $ML^2/T^2$, and the only combination of the given dimensionful constants of this form is $Bkx_0^2$,

[2] You can verify this by writing down a general product of the given quantities raised to arbitrary powers (that is, $m^a\ell^b g^c$), and writing out the units of this product in terms of $a$, $b$, and $c$. When you set the units equal to $1/T$, you will obtain a system of equations in $a$,$b$,$c$, and you will find that the solution is $a = 0$, $b = -1/2$, and $c = 1/2$. In more complicated problems, this method may turn out to be necessary. But in most problems, you can quickly see what the correct combination of the given quantities is.

where $B$ is a dimensionless number. It turns out that $B = 1/2$, so the total energy equals $kx_0^2/2$.

REMARK: A real spring doesn't have a perfectly parabolic potential (that is, an exactly linear force), so the force actually looks something like $F(x) = -kx + bx^2 + \cdots$. If we truncate the series at the second term, then we have one more dimensionful quantity to work with, $[b] = M/LT^2$. To form a quantity with the dimensions of frequency, $1/T$, we need the $x_0$ and $b$ to appear in the combination $x_0 b$, because this is the only way to get rid of the $L$. You can then see (by using the strategy of writing out a general product of the variables, discussed in the above footnote) that the frequency must be of the form $f(x_0 b/k)\sqrt{k/m}$. We therefore can have $x_0$ dependence in this case. Note that this answer must reduce to $C\sqrt{k/m}$, for $b = 0$. Hence, $f$ must be of the form $f(y) = C + c_1 y + c_2 y^2 + \cdots$. ♣

---

**Example 3 (Low-orbit satellite):** A satellite of mass $m$ travels in a circular orbit just above the earth's surface. What can we say about its speed?

The only dimensionful quantities in the problem are $[m] = M$, $[g] = L/T^2$, and the radius of the earth $[R] = L$.[3] Our goal is to find the speed, which has units of $L/T$. The only combination of our dimensionful quantities with these units is

$$v = C\sqrt{gR}. \tag{14.39}$$

It turns out that $C = 1$.

---

### 14.2.1 Exercises

1. **Pendulum on Pluto**

   If a pendulum has a period of 3 s on the earth, what would its period be if it were placed on the moon? Use $g_M/g_E \approx 1/6$.

2. **Earth and moon radii**

   The value of $g$ on the surface of a planet is given by

   $$g = \frac{GM}{R^2}\,, \tag{14.40}$$

   where $M$ and $R$ are the mass and radius of the planet, respectively, and $G$ is Newton's gravitational constant. If the densities of the moon and the earth are related by $\rho_M/\rho_E = 3/5$, and if $g_M/g_E = 1/6$, what is $R_M/R_E$?

[3] You might argue that the mass of the earth, $M_e$, and Newton's gravitational constant, $G$, should be included here. But for a particle located at the surface of the earth, these quantities appear only in the gravitational force through the combination $(GM_e/R^2)m \equiv mg$. So we can absorb the effects of $M_e$ and $G$ into $g$.

3. **Escape velocity**

   The *escape velocity*[4] on the surface of a planet is given by

$$v = \sqrt{\frac{2GM}{R}}\,, \tag{14.41}$$

   where $M$ and $R$ are the mass and radius of the planet, respectively, and $G$ is Newton's gravitational constant.

   (a) Write $v$ in terms of the average mass density $\rho$, instead of $M$.

   (b) Assuming that the average density of the earth is four times that of Jupiter, and that the radius of Jupiter is 11 times that of the earth, what is $v_J/v_E$?

4. **Waves on a string**

   How does the speed of waves on a string depends its mass $M$, length $L$, and tension (that is, force) $T$?

5. **Vibrating water drop**

   Consider a vibrating water drop, whose frequency ($\nu$) depends on its radius ($R$), mass density ($\rho$), and surface tension ($S$).[5] How does $\nu$ depend on $R$, $\rho$, and $S$?

### 14.2.2 Problems

1. **Escape velocity**

   Show that the escape velocity from the earth is given by eq. (14.41), up to numerical factors.

2. **Mass in tube**

   A tube of mass $M$ and length $\ell$ is free to swing by a pivot at one end. A mass $m$ is positioned inside the tube at this end. The tube is held horizontal and then released (see Fig. 14.3). Let $\eta$ be the fraction of the tube the mass has traversed by the time the tube becomes vertical. Does $\eta$ depend on $\ell$?

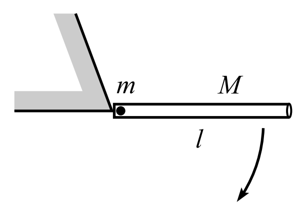


**Figure 14.3**

3. **Waves in a fluid**

   How does the speed of waves in a fluid depend on its density, $\rho$, and "Bulk Modulus," $B$ (which has units of pressure, which is force per area)?

4. **Vibrating star**

   Consider a vibrating star, whose frequency ($\nu$) depends on its radius ($R$), mass density ($\rho$), and Newton's gravitational constant ($G$). How does $\nu$ depend on $R$, $\rho$, and $G$?

[4]This is the velocity needed to refute the "What goes up must come down" maxim (neglecting air resistance and such).

[5]The units of surface tension are (Force)/(Length).

5. **Damping** $**$

   A particle with mass $m$ and initial speed $V$ is subject to a velocity-dependent damping force of the form $bv^n$.

   (a) For $n = 0, 1, 2, \ldots$, determine how the stopping time depends on $m$, $V$, and $b$.

   (b) For $n = 0, 1, 2, \ldots$, determine how the stopping distance depends on $m$, $V$, and $b$.

   Be careful! See if your answers make sense. Dimensional analysis only gives the answer up to a numerical factor. This is a tricky problem, so don't let it discourage you from using dimensional analysis. Most applications of dimensional analysis are quite straightforward.

### 14.2.3 Solutions

1. **Escape velocity**

   The reasoning is the same as in the satellite example above. Therefore, the answer is $v = C\sqrt{gR} = C\sqrt{GM_e/R}$, where we have used $g = GM_e/R^2$. It turns out that $C = \sqrt{2}$.

   This quick solution is actually not quite rigorous, considering the footnote in the above satellite example. Since the particle is not always at the same radius, the force changes, so it isn't obvious that we can absorb the $M_e$ and $G$ dependence into one quantity, $g$, as we did with the orbiting satellite. Let us therefore be more rigorous with the following reasoning.

   The dimensionful quantities in the problem are $[m] = M$, the radius of the earth $[R] = L$, Newton's gravitational constant $[G] = L^3/MT^2$, and the mass of the earth $[M_e] = M$.

   If we use no information other than these given quantities, then there is no way to arrive at the speed of $C\sqrt{GM_e/R}$, because for all we know, there could be a factor of $(m/M_e)^7$ in the answer. This number is dimensionless, so it wouldn't mess up the units.

   If we want to make any progress in this problem, we have to use the fact that the force takes the form of $GM_e m/r^2$. This then implies that the acceleration is independent of $m$. And since the path of the particle is determined by its acceleration, we see that our answer cannot depend on $m$. We are therefore left with the quantities $G$, $R$, and $M$, and you can show that the only combination of these quantities that gives a speed is $v = C\sqrt{GM_e/R}$.

2. **Mass in tube**

   The dimensionful quantities are $[g] = L/T^2$, $[\ell] = L$, $[m] = M$, and $[M] = M$. We want to produce a dimensionless number $\eta$. Since $g$ is the only constant involving time, $\eta$ cannot depend on $g$. This then implies that $\eta$ cannot depend on $\ell$, the only length remaining. Therefore, $\eta$ depends only on the ratio $m/M$. So the answer to the stated problem is, "No."

   It turns out that you have to solve the problem numerically to find $\eta$ (see Problem 7.4). Some results are: If $m \ll M$, then $\eta \approx 0.349$. If $m = M$, then $\eta \approx 0.378$. And if $m = 2M$, then $\eta \approx 0.410$.

3. **Waves in a fluid**

We want to make a speed, $[v] = L/T$, out of the quantities $[\rho] = M/L^3$, and $[B] = [F/A] = (ML/T^2)/(L^2) = M/(LT^2)$. We can play around with these quantities to find the combination that has the correct units, but let's do it the no-fail way. If $v = \rho^a B^b$, then we have

$$\frac{L}{T} = \left(\frac{M}{L^3}\right)^a \left(\frac{M}{LT^2}\right)^b. \tag{14.42}$$

Matching up the powers of the three kinds of units on each side of this equation gives

$$M: \ 0 = a + b, \qquad L: \ 1 = -3a - b, \qquad T: \ -1 = -2b. \tag{14.43}$$

The solution to this system of equations is $a = -1/2$ and $b = 1/2$. Therefore, our answer is $v \propto \sqrt{B/\rho}$. Fortunately, there was a solution to this system of three equations in two unknowns.

4. **Vibrating star**

We want to make a frequency, $[\nu] = 1/T$, out of the quantities $[R] = L$, $[\rho] = M/L^3$, and $[G] = L^3/(MT^2)$. These units for $G$ follow from the gravitational force law, $F = Gm_1m_2/r^2$. We can play around with these quantities to find the combination that has the correct units, but let's do it the no-fail way. If $\nu = R^a\rho^bG^c$, then we have

$$\frac{1}{T} = L^a \left(\frac{M}{L^3}\right)^b \left(\frac{L^3}{MT^2}\right)^c. \tag{14.44}$$

Matching up the powers of the three kinds of units on each side of this equation gives

$$M: \ 0 = b - c, \qquad L: \ 0 = a - 3b + 3c, \qquad T: \ -1 = -2c. \tag{14.45}$$

The solution to this system of three equations is $a = 0$, and $b = c = 1/2$. Therefore, our answer is $\nu \propto \sqrt{\rho G}$.

Remark: Note the difference in the given quantities in this problem ($R$, $\rho$, and $G$) and the ones in Exercise 5 ($R$, $\rho$, and $S$). In this problem with the star, the mass is large enough so that we can ignore the surface tension, $S$. And in Exercise 5 with the drop, the mass is small enough so that we can ignore the gravitational force, and hence $G$. ♣

5. **Damping**

(a) The constant $b$ has units $[b] = [\text{Force}][v^{-n}] = (ML/T^2)(T^n/L^n)$. The other constants are $[m] = M$ and $[V] = L/T$. There is also $n$, which is dimensionless. You can show that the only combination of these constants that has units of $T$ is

$$t = f(n)\frac{m}{bV^{n-1}}, \tag{14.46}$$

where $f(n)$ is a dimensionless function of $n$.

For $n = 0$, we have $t = f(0)mV/b$. This increases with $m$ and $V$, and decreases with $b$, as it should.

For $n = 1$ we have $t = f(1)m/b$. So we *seem* to have $t \sim m/b$. This, however, cannot be correct, because $t$ should definitely grow with $V$. A large initial speed $V_1$ requires some non-zero time to slow down to a smaller speed $V_2$, after which point we simply have the same scenario with initial speed $V_2$. Where did we go wrong? After all, dimensional analysis tells us that the answer *does* have to look like $t = f(1)m/b$, where $f(1)$ is a numerical factor.

The resolution to this puzzle is that $f(1)$ is infinite. If we worked out the problem using $F = ma$, we would encounter an integral that diverges. So for any $V$, we would find an infinite $t$.[6]

Similarly, for $n \geq 2$, there is at least one power of $V$ in the denominator of $t$. This certainly cannot be correct, because $t$ should not decrease with $V$. So $f(n)$ must likewise be infinite for all of these cases.

The moral of this exercise is that you have to be careful when using dimensional analysis. The numerical factor in front of your answer nearly always turns out to be of order 1, but in some strange cases it turns out to be 0 or $\infty$.

Remark: For $n \geq 1$, the expression in eq. (14.46) still has relevance. For example, for $n = 2$, the $m/(Vb)$ expression is relevant if you want to know how long it takes to go from $V$ to some final speed $V_f$. The answer involves $m/(V_f b)$, which diverges as $V_f \to 0$. ♣

(b) You can show that the only combination of these constants that has units of $L$ is

$$\ell = g(n)\frac{m}{bV^{n-2}}\,, \tag{14.47}$$

where $g(n)$ is a dimensionless function of $n$.

For $n = 0$, we have $\ell = g(0)mV^2/b$. This increases with $V$, as it should.

For $n = 1$, we have $\ell = g(1)mV/b$. This increases with $V$, as it should.

For $n = 2$ we have $\ell = g(2)m/b$. So we *seem* to have $\ell \sim m/b$. But as in part (a), this cannot be correct, because $\ell$ should definitely depend on $V$. A large initial speed $V_1$ requires some non-zero distance to slow down to a smaller speed $V_2$, after which point we simply have the same scenario with initial speed $V_2$. So, from the reasoning in part (a), the total distance is infinite for $n \geq 2$, because the function $g$ is infinite.

Remark: Note that for $n \neq 1$, $t$ and $\ell$ are either both finite or both infinite. For $n = 1$, however, the total time is infinite, whereas the total distance is finite. This situation actually holds for $1 \leq n < 2$, if we want to consider fractional $n$. ♣

[6] The total time $t$ is actually undefined, because the particle never comes to rest. But $t$ does grow with $V$, in the sense that if $t$ is defined to be the time to attain some given small speed, then $t$ grows with $V$.

## 14.3 Appendix C: Approximations, limiting cases

Along with checking units, checking limiting cases (or special cases) is something you should always do at the end of a calculation. As in the case with checking dimensions, this won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect. It is generally true that your intuition about limiting cases is much better than your intuition about generic values of the parameters. You should use this fact to your advantage.

A main ingredient in checking limiting cases is the Taylor series approximations. The series for many functions are given in Appendix A.

The examples presented below have been taken from various problems throughout the book. For the most part, we'll just repeat here what we've already said in the remarks given earlier in the text.

---

**Example 1 (Dropped ball):** A beach-ball is dropped from rest at height $h$. Assume that the drag force from the air is $F_d = -m\alpha v$. We found in Section 2.3 that the ball's speed and position are given by

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right), \qquad \text{and} \qquad y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\Big(1 - e^{-\alpha t}\Big)\right). \tag{14.48}$$

Let's look at some limiting cases. If $t$ is very small (more precisely, if $\alpha t \ll 1$; see the remark following this example), then we can use the Taylor series, $e^{-x} \approx 1-x+x^2/2$, to make approximations to leading order in $\alpha t$. The $v(t)$ in eq. (14.48) becomes

$$\begin{aligned} v(t) &= -\frac{g}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right) \\ &\approx -gt, \end{aligned} \tag{14.49}$$

plus terms of higher order in $\alpha t$. This answer is expected, because the drag force is negligible at the start, so we essentially have a freely falling body. Eq. (14.48) also gives

$$\begin{aligned} y(t) &= h - \frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right)\right] \\ &\approx h - \frac{gt^2}{2}, \end{aligned} \tag{14.50}$$

plus terms of higher order in $\alpha t$. Again, this answer is expected, because we essentially have a freely falling body.

We may also look at large $t$ (or rather, large $\alpha t$). In this case, $e^{-\alpha t}$ is essentially zero, so the $v(t)$ in eq. (14.48) becomes

$$v(t) \approx -\frac{g}{\alpha}. \tag{14.51}$$

This is the "terminal velocity". Its value makes sense, because it is the velocity for which the total force, $-mg - m\alpha v$, vanishes. Eq. (14.48) also gives

$$y(t) \approx h - \frac{gt}{\alpha} + \frac{g}{\alpha^2}. \tag{14.52}$$

Apparently, for large $t$, $g/\alpha^2$ is the distance our ball lags behind another ball which started out already at the terminal speed, $g/\alpha$.

---

Whenever you derive approximate answers as we did above, you gain something and you lose something. You lose some truth, of course, because your new answer is technically not correct. But you gain some aesthetics. Your new answer is invariably much cleaner (sometimes involving only one term), and this makes it a lot easier to see what's going on.

REMARK: In the above example, it makes no sense to look at the limit where $t$ is small or large, because $t$ has dimensions. Is a year a large or small time? How about a hundredth of a second? There is no way to answer this without knowing what problem you're dealing with. A year is short on the time scale of galactic evolution, but a hundredth of a second is long on the time scale of a nuclear process.

It only makes sense to look at the limit of a small (or large) *dimensionless* quantity. In the above example, this quantity was $\alpha t$. The given constant $\alpha$ had units of $T^{-1}$, and so $1/\alpha$ set a typical time scale for the system. It therefore made sense to look at the limit where $t \ll 1/\alpha$ (that is, $\alpha t \ll 1$), or likewise $t \gg 1/\alpha$ (that is, $\alpha t \gg 1$). In the limit of a small dimensionless quantity, a Taylor series can be used to expand an answer in powers of the small quantity, as we did above.

We sometimes get sloppy and say things like, "In the limit of small $t$." But you know that we really mean, "In the limit of some small dimensionless quantity that has a $t$ in the numerator," or, "In the limit where $t$ is much smaller that a certain quantity that has the dimensions of time." ♣

The results of checking limits generally fall into two categories. Most of the time you know what the result should be, so this provides a double-check on your answer. But sometimes an interesting limit pops up that you might not expect. Such is the case in the following examples.

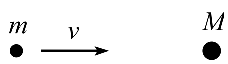


**Figure 14.4**

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**Example 2 (Two masses in 1-D):** A mass $m$ with speed $v$ approaches a stationary mass $M$ (see Fig. 14.4). The masses bounce off each other elastically. Assume all motion takes place in one dimension. We found in Section 4.6.1 that the final speeds of the particles are

$$v_m = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}. \tag{14.53}$$

There are three special cases that beg to be checked:

- If $m = M$, then eq. (14.53) tells us that $m$ stops, and $M$ picks up a speed of $v$. This is fairly believable. And it becomes quite obvious once you realize that these final speeds clearly satisfy conservation of energy and momentum with the initial conditions.
- If $M \gg m$, then $m$ bounces backward with speed $\approx v$, and $M$ hardly moves. This is clear, because $M$ is basically a brick wall.

- If $m \gg M$, then $m$ keeps plowing along at speed $\approx v$, and $M$ picks up a speed of $\approx 2v$. This $2v$ is an unexpected and interesting result (it becomes clearer if you consider what is happening in the reference frame of the heavy mass $m$), and it leads to some neat effects, as in Problem 4.22.

---

**Example 3 (Circular pendulum):** A mass hangs from a massless string of length $\ell$. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle $\theta$ with the vertical (see Fig. 14.5). We found in Section 2.5 that the angular frequency, $\omega$, of this motion is

$$\omega = \sqrt{\frac{g}{\ell \cos\theta}}\,. \tag{14.54}$$

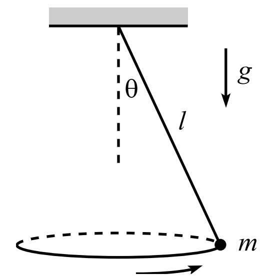

**Figure 14.5**

As far as $\theta$ is concerned, there are two limits we should definitely check:

- If $\theta \to 90°$, then $\omega \to \infty$. This makes sense; the mass has to spin very quickly to avoid flopping down.
- If $\theta \to 0$, then $\omega \to \sqrt{g/\ell}$. This is the same as the frequency of a plane pendulum of length $\ell$ (for small oscillations). You can convince yourself why this is true. Hint: look at the projection of the force on a given horizontal line.

---

In the above examples, we checked limiting and special cases of answers that were correct (I hope). This whole process is more useful (and a bit more fun) when you check the limits of an answer that is *incorrect*. In this case, you gain the unequivocal information that your answer is wrong. But rather than leading you into despair, this information is actually something you should be quite happy about, considering that the alternative is to carry on in a state of blissful ignorance. Personally, if there's any way I'd want to discover that my answer is garbage, this is it. At any rate, checking limiting and special cases can often save you a lot of trouble in the long run...

The lemmings get set for their race.
With one step and two steps they pace.
They take three and four,
And then head on for more,
Without checking the limiting case.

### 14.3.1 Exercise

1. **Atwood's machine** $*$

   Consider the "Atwood's" machine shown in Fig. 14.6, consisting of three masses and three frictionless pulleys. It can be shown that the acceleration of $m_1$ is given by (just accept this):

$$a_1 = g\frac{3m_2m_3 - m_1(4m_3 + m_2)}{m_2m_3 + m_1(4m_3 + m_2)}\,, \tag{14.55}$$

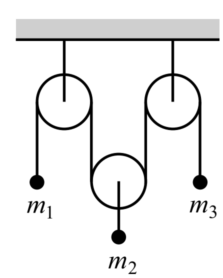

**Figure 14.6**

   with upward taken to be positive. Find $a_1$ in the following special cases:

(a) $m_2 = 2m_1 = 2m_3$.

(b) $m_1$ much larger than both $m_2$ and $m_3$.

(c) $m_1$ much smaller than both $m_2$ and $m_3$.

(d) $m_2 \gg m_1 = m_3$

(e) $m_1 = m_2 = m_3$.

## 14.4 Appendix D: Solving differential equations numerically

Sooner or later you will encounter a differential equation that you cannot solve exactly. Having resigned yourself to not getting the exact answer, you should ponder how to obtain a decent approximation to it. In this marvellously advanced technological era (which your children will dismiss with nothing more than a bewildered chuckle), it's easy to write a short program that will give you a very good numerical answer to your problem. Given enough computer time, you can obtain any desired accuracy (assuming the system isn't chaotic, but no need to worry about that for the systems we'll be dealing with). We'll demonstrate the procedure by using an easy problem, one that we actually do know the answer to.

Consider the equation,

$$\ddot{x} = -\omega^2 x. \tag{14.56}$$

This is of course the equation for a mass on a spring (with $\omega = \sqrt{k/m}$), and we know that the solution can be written, among other ways, in the form,

$$x(t) = A\cos(\omega t + \phi). \tag{14.57}$$

But let's pretend we don't know this. If someone comes along and gives us values for $x(0)$ and $\dot{x}(0)$, it seems that somehow we should be able to find $x(t)$ and $\dot{x}(t)$ for any later $t$, just by using eq. (14.56). Here's how we do it.

The plan is to discretize time into intervals of some small unit (call it $\epsilon$), and then to determine what happens at each successive point in time. If we know $x(t)$ and $\dot{x}(t)$, then we can easily find (approximately) the value of $x$ at a slightly later time, by using the definition of $\dot{x}$. Similarly, if we know $\dot{x}(t)$ and $\ddot{x}(t)$, then we can easily find (approximately) the value of $\dot{x}$ at a slightly later time, by using the definition of $\ddot{x}$. Using the definitions of the derivatives, the relations are simply

$$\begin{aligned} x(t+\epsilon) &\approx x(t) + \epsilon\dot{x}(t), \\ \dot{x}(t+\epsilon) &\approx \dot{x}(t) + \epsilon\ddot{x}(t). \end{aligned} \tag{14.58}$$

These two equations, combined with (14.56), which gives us $\ddot{x}$ in terms of $x$, allow us to march along in time, obtaining successive values for $x$, $\dot{x}$, and $\ddot{x}$.[7]

Here's what a typical program might look like. (This is Maple, but even if you aren't familiar with this, the general idea should be clear.) Let's say that the particle starts from rest at position $x = 2$, and let's pick $\omega^2 = 5$. We'll use the notation where `x1` stands for $\dot{x}$, and `x2` stands for $\ddot{x}$. And `e` stands for $\epsilon$. Let's calculate $x$ at $t = 3$.

[7] Of course, another expression for $\ddot{x}$ is the definitional one, analogous to eqs. (14.58), involving the third derivative. But this would then require knowledge of the third derivative, and so on with higher derivatives, and we would end up with an infinite chain of relations. An equation of motion such as eq. (14.56) (which in general could be an $F = ma$, $\tau = I\alpha$, or Euler-Lagrange equation) relates $\ddot{x}$ back to $x$ (and possibly $\dot{x}$), thereby creating an intertwined relation among $x$, $\dot{x}$, and $\ddot{x}$, and eliminating the need for an infinite and useless chain.

```
x:=2:      # initial position
x1:=0:      # initial speed
e:=.01:      # a small time interval
for i to 300 do     # do 300 steps (ie, up to 3 seconds)
x2:=-5*x:      # the given equation
x:=x+e*x1:      # how x changes, by definition of x1
x1:=x1+e*x2:      # how x1 changes, by definition of x2
end do:      # the Maple command to stop the do loop
x;     # print the value of x
```

This procedure of course won't give the exact value for $x$, because $x$ and $\dot{x}$ don't really change according to eqs. (14.58). These equations are just first-order approximations to the full Taylor series with higher-order terms. Said differently, there is no way the above procedure can be exactly correct, because there are ambiguities in how the program can be written. Should line 5 come before or after line 7? That is, in determining $\dot{x}$ at time $t+\epsilon$, should you use the $\ddot{x}$ at time $t$ or $t+\epsilon$? And should line 7 come before or after line 6? The point is that for very small $\epsilon$, the order doesn't matter much. And in the limit $\epsilon \to 0$, the order doesn't matter at all.

If we want to obtain a better approximation, we can just shorten $\epsilon$ down to .001 and increase the number of steps to 3000. If the result looks basically the same as with $\epsilon = .01$, then we know we pretty much have the right answer.

In the present example, $\epsilon = .01$ yields $x \approx 1.965$ after 3 seconds. If we set $\epsilon = .001$, then we obtain $x \approx 1.836$. And if we set $\epsilon = .0001$, then we get $x \approx 1.823$. The correct answer must therefore be somewhere around $x = 1.82$. And indeed, if we solve the problem exactly, we would obtain $x(t) = 2\cos(\sqrt{5}t)$. Plugging in $t = 3$ gives $x \approx 1.822$.

This is a wonderful procedure, but it shouldn't be abused. It's nice to know that we can always obtain a decent numerical approximation if all else fails. But we should set our initial goal on obtaining the correct algebraic expression, because this allows us to see the overall behavior of the system. And besides, nothing beats the truth. People tend to rely a bit too much on computers and calculators nowadays, without pausing to think about what is actually going on in a problem.

The skill to do math on a page
Has declined to the point of outrage.
Equations quadratica
Are solved on Math'matica,
And on birthdays we don't know our age.

## 14.5 Appendix E: $F = ma$ vs. $F = dp/dt$

In nonrelativistic mechanics,[8] the equations $F = ma$ and $F = dp/dt$ say exactly the same thing, provided that $m$ is constant. But if $m$ in not constant, then $dp/dt = d(mv)/dt = ma + (dm/dt)v$, which does not equal $ma$. In this case, should we use $F = ma$ or $F = dp/dt$? Which law correctly describes the physics? The answer to this depends on what you label as the "system" to which you associate the quantities $m$, $p$, and $a$. You can generally do a problem using either $F = ma$ or $F = dp/dt$, but you must be very careful about how you label things and how you treat them. The subtleties are best understood through two examples.

---

**Example 1 (Sand dropping into cart):** Consider a cart into which sand is dropped (vertically) at a rate $dm/dt = \sigma$. With what force must you push on the cart to keep it moving (horizontally) at a constant speed $v$?

**First solution:** Let $m(t)$ be the mass of the cart-plus-sand-inside system (label this system as "$C$"). If we use $F = ma$ (where $a$ is the acceleration of the cart, which is zero), then we obtain $F = 0$, which is incorrect. The correct expression to use is $F = dp/dt$. This gives

$$F = \frac{dp}{dt} = ma + \frac{dm}{dt}v = 0 + \sigma v. \tag{14.59}$$

This makes sense, because your force is what increases the momentum of $C$, and this momentum increases simply because the mass of $C$ increases.

**Second solution:** It is possible to solve this problem by using $F = ma$, provided that you let your system be a small piece of mass that is being added to the cart. Your force is what accelerates this mass from rest to speed $v$. Consider a mass $\Delta m$ that falls into the cart during a time $\Delta t$. Imagine that it falls into the cart in one lump at the start of the $\Delta t$, and then accelerates (via friction) up to speed $v$ after time $\Delta t$ (and then this process repeats during each successive $\Delta t$ interval). Then $F = ma = \Delta m(v/\Delta t)$. Writing this as $(\Delta m/\Delta t)v$ gives the $\sigma v$ result in the first solution.

---

**Example 2 (Sand leaking from cart):** Consider a cart that leaks sand out of the bottom at a rate $dm/dt = \sigma$. If you apply a force $F$ to the cart, what is its acceleration?

**Solution:** Let $m(t)$ be the mass of the cart-plus-sand-inside system (label this system as "$C$"). In this example, we want to use $F = ma$. So the acceleration is simply

$$a = \frac{F}{m}\,. \tag{14.60}$$

Note that since $m$ decreases with time, $a$ increases with time.

[8] We won't bother with relativity in this Appendix, because nonrelativistic mechanics contains all the critical aspects we want to address.

We used $F = ma$ because at any instant, the mass $m$ is what is being accelerated by the force $F$. If you want, you can imagine the process occurring in discrete steps: The force pushes on the mass for a short period of time, then a little piece instantaneously leaks out; then the force pushes again on the new mass, then another little piece leaks out; and so on. In this discretized scenario, it is clear that $F = ma$ is the appropriate formula, because it holds for each step in the process. The only ambiguity is whether to use $m$ or $m + dm$ at a certain time, but this yields a negligible error.

REMARKS: It is still true that $F = dp/dt$ in this problem, provided that you let $F$ be *total* force, and let $p$ be the *total* momentum. In this problem, $F$ is the only force. However, the total momentum consists of both the sand in the cart and the sand that has leaked out and is falling through the air.[9] A common mistake is to use $F = dp/dt$, with $p$ being only the system $C$'s momentum.

There is an simple example that demonstrates why $F = dp/dt$ doesn't work when $p$ refers only to $C$. Imagine that $F = 0$, and let the cart move with speed $v$. Cut the cart in half, and label the back part as the "leaked sand", and the front part as the "cart". If you want the cart's $p$ to have $dp/dt = F = 0$, then the cart's speed must double if its mass gets cut in half. But this is nonsense. Both halves of the cart simply continue to move at the same rate. ♣

---

To sum up, $F = dp/dt$ is always valid, provided that you use the *total* force and *total* momentum of a given system of particles. This approach, however, can get messy in certain situations. So in some cases it is easier to use an $F = ma$ argument, but you must be careful to correctly identify the system that is being accelerated by the force. The asymmetry in the above two examples is that in the first example, the force does indeed accelerate the incoming sand; whereas in the second example, the force does *not* accelerate (or decelerate) the outgoing sand. $F$ has nothing to do with the leaked sand.

[9] If there were air resistance, we would have to worry about its effect on the falling sand if we wanted to use $F = dp/dt$ to solve the problem, where $p$ is the total momentum. This is clearly not the best way to do the problem. If complicated things happen with the sand in the air, it would be foolish to consider this part of the sand if we don't have to.

## 14.6 Appendix F: Existence of principal axes

In this Appendix, we will prove Theorem 8.4. That is, we will show that an orthonormal set of principal axes does indeed exist for any object, and for any choice of origin. It is not crucial that you study this proof. If you want to simply accept the fact that principal axes exist, that's perfectly fine. But the method we'll use in this proof is one you'll see again and again in your physics career, in particular when you study quantum mechanics (see the remark following the proof).

**Theorem 14.1** *Given a real, symmetric* $3 \times 3$ *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* $I_k$*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k \hat{\boldsymbol{\omega}}_k. \tag{14.61}$$

**Proof:** This theorem holds more generally with 3 replaced by $N$ (all the steps below easily generalize), but we'll work with $N = 3$, to be concrete.

Consider a general matrix, $\mathbf{I}$ (we don't need to assume yet that it's real or symmetric). Assume that $\mathbf{Iu} = I\mathbf{u}$ for some vector $\mathbf{u}$ and some number $I$.[10] This may be rewritten as

$$\begin{pmatrix} (I_{xx} - I) & I_{xy} & I_{xz} \\ I_{yx} & (I_{yy} - I) & I_{yz} \\ I_{zx} & I_{zy} & (I_{zz} - I) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14.62}$$

In order for there to be a nontrivial solution for the vector $\mathbf{u}$ (that is, one where $\mathbf{u} \neq (0, 0, 0)$), the determinant of this matrix must be zero.[11] Taking the determinant, we see that we get an equation for $I$ of the form

$$aI^3 + bI^2 + cI + d = 0. \tag{14.63}$$

The constants $a$, $b$, $c$, and $d$ are functions of the matrix entries $I_{ij}$, but we won't need their precise form to prove this existence theorem. The only thing we need this equation for is to say that there do exist three (generally complex) solutions for $I$, because the equation is of third degree.

We will now show that the solutions for $I$ are real. This will imply that there exist three real vectors $\mathbf{u}$ satisfying $\mathbf{Iu} = I\mathbf{u}$, because we can plug the real $I$'s back into eq. (14.62) and solve for the real components $u_x$, $u_y$, and $u_z$, up to an overall constant. We will then show that these vectors are orthogonal.

- *Proof that the I's are real:* This follows from the real and symmetric conditions on $\mathbf{I}$. Start with the equation $\mathbf{Iu} = I\mathbf{u}$, and take the dot product with $\mathbf{u}^*$ to obtain

$$\begin{aligned} \mathbf{u}^* \cdot \mathbf{Iu} &= \mathbf{u}^* \cdot I\mathbf{u} \\ &= I\mathbf{u}^* \cdot \mathbf{u}. \end{aligned} \tag{14.64}$$

[10] Such a vector $\mathbf{u}$ is called an *eigenvector* of $\mathbf{I}$, and $I$ is the associated *eigenvalue*. But don't let these names scare you. They're just definitions.

[11] If the determinant were not zero, then we could explicitly construct the inverse of the matrix, which involves cofactors divided by the determinant. Multiplying both sides by this inverse would show that $\mathbf{u} = \mathbf{0}$.

The vector $\mathbf{u}^*$ is the vector obtained by simply complex conjugating each component of $\mathbf{u}$ (we don't know yet that $\mathbf{u}$ can be chosen to be real). On the right side, $I$ is a scalar, so we can take it out from between the $\mathbf{u}^*$ and $\mathbf{u}$.

The fact that $\mathbf{I}$ is real implies that if we complex conjugate the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$, we obtain $\mathbf{I}\mathbf{u}^* = I^*\mathbf{u}^*$ (we know that $\mathbf{I}$ is real, but we don't know yet that $I$ is real). If we then take the dot product of this equation with $\mathbf{u}$, we obtain

$$\mathbf{u} \cdot \mathbf{I}\mathbf{u}^* = I^*\mathbf{u} \cdot \mathbf{u}^*. \tag{14.65}$$

We now claim that if $\mathbf{I}$ is symmetric, then $\mathbf{a} \cdot \mathbf{I}\mathbf{b} = \mathbf{b} \cdot \mathbf{I}\mathbf{a}$, for any vectors $\mathbf{a}$ and $\mathbf{b}$. (We'll leave this for you to show by simply multiplying each side out.) In particular, $\mathbf{u}^* \cdot \mathbf{I}\mathbf{u} = \mathbf{u} \cdot \mathbf{I}\mathbf{u}^*$, so eqs. (14.64) and (14.65) give

$$(I - I^*)\mathbf{u} \cdot \mathbf{u}^* = 0. \tag{14.66}$$

Since $\mathbf{u} \cdot \mathbf{u}^* = |u_1|^2 + |u_2|^2 + |u_3|^2 \neq 0$, we must have $I = I^*$. Therefore, $I$ is real.

- *Proof that the* $\mathbf{u}$ *are orthogonal:* This follows from the symmetric condition on $\mathbf{I}$. Let $\mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_2$. Take the dot product of the former equation with $\mathbf{u}_2$ to obtain

$$\mathbf{u}_2 \cdot \mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_2 \cdot \mathbf{u}_1, \tag{14.67}$$

and take the dot product of the latter equation with $\mathbf{u}_1$ to obtain

$$\mathbf{u}_1 \cdot \mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_1 \cdot \mathbf{u}_2. \tag{14.68}$$

As above, the symmetric condition on $\mathbf{I}$ implies that the left-hand sides of eqs. (14.67) and (14.68) are equal. Therefore,

$$(I_1 - I_2)\mathbf{u}_1 \cdot \mathbf{u}_2 = 0. \tag{14.69}$$

There are two possibilities here: (1) If $I_1 \neq I_2$, then we are done, because $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, which says that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. (2) If $I_1 = I_2 \equiv I$, then we have $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$, for any $a$ and $b$. So any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ has the same property that $\mathbf{u}_1$ and $\mathbf{u}_2$ have (namely, that applying $\mathbf{I}$ is the same as simply multiplying by $I$). We therefore have a whole plane of such vectors, so we can pick any two orthogonal vectors in this plane to be called $\mathbf{u}_1$ and $\mathbf{u}_2$. ■

This theorem proves the existence of principal axes, because the inertia tensor in eq. (8.8) is indeed a real and symmetric matrix.

REMARK: (Warning: This remark has nothing to do with classical mechanics. It is simply an ill-disguised excuse to write down another limerick.) In Quantum Mechanics, it turns out that any observable quantity, such as position, energy, momentum, angular momentum, etc., can be represented by a *Hermitian* matrix, with the observed value being an eigenvalue of the matrix. A Hermitian matrix is a (generally complex) matrix with the property that the transpose of the matrix equals the complex conjugate of the matrix. For example, a $2 \times 2$ Hermitian matrix must be of the form,

$$\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}, \tag{14.70}$$

for real numbers $a$, $b$, $c$, and $d$. Now, if observed values are to be given by the eigenvalues of such a matrix, then the eigenvalues had *better* be real, because no one (in this world, at least) is about to go for a jog of $4+3i$ miles, or pay an electric bill for $17-43i$ kilowatt-hours. And indeed, you can show via a slightly modified version of the above "Proof that the $I$ are real" procedure that the eigenvalues of any Hermitian matrix are real. (And likewise, the eigenvectors are orthogonal.) This is, to say the least, very fortunate.

God's first tries were hardly ideal,
For complex worlds have no appeal.
So in the present edition,
He made things Hermitian,
And *this* world, it seems, is quite real. ♣

## 14.7 Appendix G: Diagonalizing matrices

This appendix is relevant to Section 8.3, which covers principal axes. The process of diagonalizing matrices (that is, finding the *eigenvectors* and *eigenvalues*) has applications in countless types of problems in a wide variety of subjects. We will describe the process here as it applies to principle axes and moments of inertia.

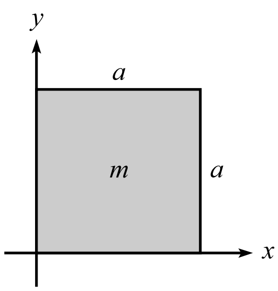


**Figure 14.7**

Let's find the three principal axes and moments of inertia for a square with side length $a$, mass $m$, and one corner at the origin. The square lies in the $x$-$y$ plane, with sides along the $x$- and $y$-axes (see Fig. 14.7).

We'll choose the given $x$- $y$- and $z$-axes as our initial basis axes. From eq. (8.8), the matrix $\mathbf{I}$ (with respect to this initial basis) is easily shown to be

$$\mathbf{I} = \rho \begin{pmatrix} \int y^2 & -\int xy & 0 \\ -\int xy & \int x^2 & 0 \\ 0 & 0 & \int (x^2+y^2) \end{pmatrix} = ma^2 \begin{pmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}, \qquad (14.71)$$

where $\rho$ is the mass per unit area, so that $a^2\rho = m$. We have used the fact that $z = 0$, and we have not bothered to write the $dx\,dy$ in the integrals.

Our goal is to find the basis in which $\mathbf{I}$ is diagonal. That is, we want to find three solutions[12] for $\mathbf{u}$ (and $I$) in the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$. Letting $I \equiv \lambda ma^2$ (to make things look a little cleaner), and using the above explicit form of $\mathbf{I}$, the equation $(\mathbf{I} - I)\mathbf{u} = 0$ becomes

$$ma^2 \begin{pmatrix} 1/3-\lambda & -1/4 & 0 \\ -1/4 & 1/3-\lambda & 0 \\ 0 & 0 & 2/3-\lambda \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (14.72)$$

In order for there to be a nonzero solution for the components $u_x, u_y, u_z$, the determinant of this matrix must be zero. The resulting cubic equation for $\lambda$ is easy to solve, because the determinant is simply $[(1/3-\lambda)^2 - (1/4)^2](2/3-\lambda) = 0$. The solutions are $\lambda = 1/3 \pm 1/4$, and $\lambda = 2/3$. So our three moments of inertia, $I \equiv \lambda ma^2$, are

$$I_1 = \frac{7}{12}ma^2, \qquad I_2 = \frac{1}{12}ma^2, \qquad I_3 = \frac{2}{3}ma^2. \qquad (14.73)$$

These are the *eigenvalues* of $\mathbf{I}$.

What are the vectors, $\mathbf{u}_1$, $\mathbf{u_2}$, and $\mathbf{u}_3$, associated with each of these $I$'s? Plugging $\lambda = 7/12$ into (14.72) gives the three equations (one for each component), $-u_x - u_y = 0$, $-u_x - u_y = 0$, and $u_z = 0$. These are redundant equations (that was the whole point of setting the determinant equal to zero). So $u_x = -u_y$, and $u_z = 0$. The vector may therefore be written as $\mathbf{u}_1 = (c, -c, 0)$, where $c$ is any constant.[13] If we want a normalized vector, then $c = 1/\sqrt{2}$. In a similar manner, plugging $\lambda = 1/12$ into eq. (14.72) gives $\mathbf{u}_2 = (c, c, 0)$. And finally, plugging $\lambda = 1/3$ into eq.

[12] One obvious solution is $\mathbf{u} = \hat{\mathbf{z}}$, because $\mathbf{I}\hat{\mathbf{z}} = (2/3)ma^2\hat{\mathbf{z}}$. From the orthogonality result of Theorem 8.4, we know that the other two vectors must lie in the $x$-$y$ plane. So we could quickly reduce this problem to a two-dimensional one, but let's forge ahead with the general method.

[13] We can only solve for $\mathbf{u}$ up to an overall constant, because if $\mathbf{I}\mathbf{u} = I\mathbf{u}$ is true for a certain $\mathbf{u}$, then it is also true that $\mathbf{I}(c\mathbf{u}) = I(c\mathbf{u})$, where $c$ is any constant.

(14.72) gives $\mathbf{u}_3 = (0, 0, c)$, as claimed in the above footnote. Our three orthonormal principal axes corresponding to the moments in eq. (14.73) are therefore

$$\hat{\boldsymbol{\omega}}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_3 = (0, 0, 1). \tag{14.74}$$

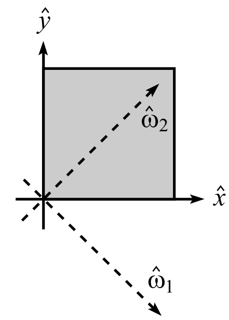

**Figure 14.8**

These are the *eigenvectors* of $\mathbf{I}$. These axes are shown in Fig. 14.8. In the new basis of the principal axes, the matrix $\mathbf{I}$ takes the form

$$\mathbf{I} = ma^2 \begin{pmatrix} 7/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}. \tag{14.75}$$

The basic idea is that from now on we should use the principal axes as our basis vectors. We can forget we ever had anything to do with the original $x$-$y$-$z$-axes.

REMARKS: (1) $I_1 + I_2 = I_3$, as the perpendicular axis theorem demands. (2) $I_2$ is the moment around one diagonal through the center of the square, which of course equals the moment around the other diagonal through the center. But the latter is related to $I_1$ by the parallel axis theorem; and indeed, $I_1 = I_2 + m(a/\sqrt{2})^2$. (3) Convince yourself why $I_2$ should equal the moment around the center of a stick of mass $m$ and length $a$. (Any axis through the center of a square, in the plane of the square, has the same moment.) (4) An application of the parallel and perpendicular axis theorems gives (considering an axis through the center and parallel to the $I_3$ axis) $I_3 = 2I_2 + m(a/\sqrt{2})^2$. ♣

## 14.8 Appendix H: Qualitative relativity questions

1. Is there such a thing as a perfectly rigid body?

   **Answer:** No. Since information can move no faster than the speed of light, it takes time for the atoms in the body to communicate with each other. If you push on one end of a rod, then the other end will not move right away.

2. Moving clocks run slow. Does this result have anything to do with the time it takes light to travel from the clock to your eye?

   **Answer:** No. When we talk about how fast a clock is running in a given frame, we are referring to what the clock actually reads in that frame. It will of course take time for the light from the clock to reach an observer's eye, but it is understood that the observer subtracts off this transit time in order to calculate the time at which the clock actually shows a particular reading.

   Likewise, other relativistic effects, such as length contraction and loss of simultaneity, have nothing to do with the time it takes light to reach your eye. They deal only with what really *is*, in your frame.

3. Does time dilation depend on whether a clock is moving across your vision or directly away from you?

   **Answer:** No. A moving clock runs slow, no matter which way it is moving.

4. Does the special-relativistic time dilation depend on the acceleration of the moving clock?

   **Answer:** No. The time-dilation factor is $\gamma = 1/\sqrt{1-v^2/c^2}$, which does not depend on $a$. The only relevant quantity is the $v$ at a given instant. It doesn't matter if $v$ is changing.

   Of course, if *you* are accelerating, then you can't naively apply the results of special relativity. (To do things correctly, it is perhaps easiest to think in terms of general relativity. But GR is actually not required; see Chapter 13 for a discussion of these issues.) But as long as you represent an inertial frame, then the clock you are viewing can undergo whatever motion it wants, and you will observe it running slow by the simple factor, $\gamma$.

5. Someone says, "A stick that is length-contracted isn't *really* shorter, it just *looks* shorter." Do you agree?

   **Answer:** Hopefully not. The stick really *is* shorter in your frame. Length contraction has nothing to do with how things look. It has to do with where the ends of the stick are at simultaneous times in your frame. (That is, after all, how you measure the length of something.) At a given instant in time (in your frame), the distance between the ends of the stick is indeed less than the proper length of the stick.

6. Consider a stick that moves in the direction in which it points. Does its length contraction depend on whether this direction is across your vision or directly away from you?

**Answer:** No. The stick is length-contracted in both cases. Of course, if you look at the stick in the latter case, then all you will see is the end, which will just be a dot. But the stick is indeed shorter in your reference frame.

7. A mirror moves toward you at speed $v$. You shine a light towards it and the light beam bounces back at you. What is the speed of the reflected beam?

   **Answer:** The speed is $c$, as always. You will observe the light having a higher frequency, due to the Doppler effect. But the speed is still $c$.

8. In relativity, the order of two events in one frame may be reversed in another frame. Does this imply that there exists a frame in which I get off a bus before I get on it?

   **Answer:** No. The order of two events can be reversed in another frame only if the events are spacelike separated. That is, if $\Delta x > c\Delta t$ (in other words, the events are too far apart for even light to get from one to the other). The two relevant events here (getting on the bus, and getting off the bus) are not spacelike separated, because the bus travels at a speed less than $c$, of course. They are timelike separated. Therefore, in all frames it is the case that I get off the bus after I get on it.

   There would be causality problems if there existed a frame in which I got off the bus before I got on it. If I break my ankle getting off a bus, then I wouldn't be able to make the fast dash that I made to catch the bus is the first place, in which case I wouldn't have the opportunity to break my ankle getting off the bus, in which case I could have made the fast dash to catch the bus and get on, and, well, you get the idea.

9. You are in a spaceship sailing along in outer space. Is there any way you can measure your speed without looking outside?

   **Answer:** There are two points to be made here. First, the question is meaningless, because absolute speed does not exist. The spaceship does not have a speed; it only has a speed relative to something else.

   Second, even if the question asked for the speed with respect to, say, a piece of stellar dust, the answer would be "no." Uniform speed is not measurable from within the spaceship. Acceleration, on the other hand, is measurable (assuming there is no gravity around to confuse it with).

10. If you move at the speed of light, what shape does the universe take in your frame?

    **Answer:** The question is meaningless, because it is impossible for you to move at the speed of light. A meaningful question to ask is: What shape does the universe take if you move at a speed very close to $c$? The answer is that in your frame everything would be squashed along the direction of your motion. Any given region of the universe would be squashed down to a pancake.

11. Two objects fly toward you, one from the east with speed $u$, and the other from the west with speed $v$. Is it correct that their relative speed, as measured by you, is $u + v$? Or should you use the velocity-addition formula, $V = (u+v)/(1+uv/c^2)$? Is it possible for their relative speed, as measured by you, to exceed $c$?

    **Answer:** Yes, no, yes, to the three questions. It is legal to simply add the two speeds to obtain $u + v$. There is no need to use the velocity-addition formula, because both speeds here are measured with respect to the *same thing*, namely you. It is perfectly legal for the result to be greater than $c$ (but it must be less than $2c$).

    You need to use the velocity-addition formula when, for example, you are given the speed of a ball with respect to a train, and also the speed of the train with respect to the ground, and your goal is to find the speed of the ball with respect to the ground. The point is that now the two given speeds are measured with respect to *different* things, namely the train and the ground.

12. Two clocks at the ends of a train are synchronized with respect to the train. If the train moves past you, which clock shows the higher time?

    **Answer:** The rear clock shows the higher time. It shows $Lv/c^2$ more than the front clock, where $L$ is the proper length of the train.

13. A train moves at speed $4c/5$. A clock is thrown from the back of the train to the front. As measured in the ground frame, the time of flight is 1 second. Is the following reasoning correct? "The $\gamma$-factor between the train and the ground is $\gamma = 1/\sqrt{1-(4/5)^2} = 5/3$. And since moving clocks run slow, the time elapsed on the clock during the flight is 3/5 of a second."

    **Answer:** No. It is incorrect, because the time-dilation result holds only for two events that happen at the *same place* in the relevant reference frame (the train, here). The clock moves with respect to the train, so the above reasoning is not correct.

    Another way of seeing why it must be incorrect is the following. A certainly valid way to calculate the clock's elapsed time is to find the speed of the clock with respect to the ground (more information would have to be given to determine this), and to then apply time dilation with the associated $\gamma$-factor to arrive at the answer of $1/\gamma$. Since the clock's $v$ is definitely not $4c/5$, the correct answer is definitely not 3/5 s.

14. Person $A$ chases person $B$. As measured in the ground frame, they have speeds $4c/5$ and $3c/5$, respectively. If they start a distance $L$ apart (as measured in the ground frame), how much time will it take (as measured in the ground frame) for $A$ to catch $B$?

    **Answer:** As measured in the ground frame, the relative speed is $4c/5 - 3c/5 = c/5$. Person $A$ must close the initial gap of $L$, so the time it takes is $L/(c/5) = 5L/c$. There is no need to use any fancy velocity-addition or length-contraction formulas, because all quantities in this problem are measured with respect to the *same* frame. So it quickly reduces to a simple "(rate)(time) = (distance)" problem.

15. Is the "the speed of light is the same in all inertial frames" postulate really necessary? That is, is it not already implied by the "the laws of physics are the same in all inertial frames"?

    **Answer:** Yes, it is necessary. It turns out that nearly all the results in relativity can be deduced by using only the "the laws of physics are the same in all inertial frames" postulate. What you can find (with some work) is that there is some limiting speed (which may or may not be infinite). But you still have to postulate that light is the thing that moves with this speed. See Section 10.8.

16. Imagine closing a very large pair of scissors. It is quite possible for the point of intersection of the blades to move faster than the speed of light. Does this violate anything in relativity?

    **Answer:** No. If the angle between the blades is small enough, then the tips of the blades (and all the other atoms in the scissors) can move at a speed well below $c$, while the intersection point moves faster than $c$. But this does not violate anything in relativity. The intersection point is not an actual object, so there is nothing wrong with it moving faster than $c$.

    We should check that this setup cannot be used to send a signal down the scissors at a speed faster than $c$. Since there is no such thing as a rigid body, it is impossible to get the far end of the scissors to move right away, when you apply a force with your hand. The scissors would have to already be moving, in which case the motion is independent of any decision you make at the handle to change the motion of the blades.

17. Two twins travel away from each other at relativistic speed. The time-dilation result from relativity says that each twin sees the other's clock running slow, so each says the other has aged less. How would you reply to someone who asks, "But which twin really *is* younger?"

    **Answer:** It makes no sense to ask which twin really is younger, because the two twins aren't in the same reference frame; they are using different coordinates to measure time. It's as silly as having two people run away from each other into the distance (so that each person sees the other become very small), and then asking: Who is really smaller?

18. The momentum of an object with mass $m$ and speed $v$ is $p = \gamma mv$. "A photon has zero mass, so it should have zero momentum." Correct or incorrect?

    **Answer:** Incorrect. True, $m$ is zero, but the $\gamma$ factor is infinite because $v = c$. Infinity times zero is undefined. A photon does indeed have momentum, and it equals $E/c$ (which equals $h\nu/c$, where $\nu$ is the frequency of the light).

19. It is not necessary to postulate the impossibility of accelerating an object to speed $c$. It follows as a consequence of the relativistic form of energy. Explain.

**Answer:** $E = \gamma mc^2$, so if $v = c$ then $\gamma = \infty$, and the object must have an infinite amount of energy (unless $m = 0$, as for a photon). All the energy in the universe, let alone all the king's horses and all the king's men, can't accelerate something to speed $c$.

## 14.9 Appendix I: Lorentz transformations

In this Appendix, we will give an alternate derivation of the Lorentz transformations, eqs. (10.13). The goal here is to derive them from scratch, using only the two postulates of relativity. We will *not* use any of the results derived in Section 10.2. Our strategy will be to use the relativity postulate ("all inertial frames are equivalent") to figure out as much as we can, and to then invoke the speed-of-light postulate at the end. The main reason for doing things in this order is that it will allow us to derive an very interesting result in Section 10.8.

As in Section 10.3, consider a coordinate system, $S'$, moving relative to another system, $S$ (see Fig. 14.9). Let the constant relative speed of the frames be $v$. Let the corresponding axes of $S$ and $S'$ point in the same direction, and let the origin of $S'$ move along the $x$-axis of $S$, in the positive direction.

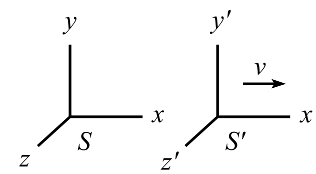


**Figure 14.9**

As in Section 10.3, we want to find the constants, $A$, $B$, $C$, and $D$, in the relations,

$$\begin{aligned}\Delta x &= A\,\Delta x' + B\,\Delta t',\\ \Delta t &= C\,\Delta t' + D\,\Delta x'.\end{aligned} \tag{14.76}$$

The four constants here will end up depending on $v$ (which is constant, given the two inertial frames).

There are four unknowns in eqs. (14.76), so we need four facts. The facts we have at our disposal (using only the two postulates of relativity) are the following.

1. The physical setup: $S'$ travels at speed $v$ with respect to $S$.
2. The principle of relativity: $S$ should see things in $S'$ in exactly the same way as $S'$ sees things in $S$ (except for perhaps a minus sign in some relative positions, but this is just convention).
3. The speed-of-light postulate: A light pulse with speed $c$ in $S'$ also has speed $c$ in $S$.

The second statement here contains two independent bits of information. (It contains at least two, because we will indeed be able to solve for our four unknowns. And it contains no more than two, because then our four unknowns would be overconstrained.) The two bits that are used depend on personal preference. Three that are commonly used are: (a) the relative speed looks the same from either frame, (b) time dilation (if any) looks the same from either frame, and (c) length contraction (if any) looks the same from either frame. It is also common to recast the second statement in the form: The Lorentz transformations are equal to their inverse transformations (up to a possible minus sign). We'll choose to work with (a) and (b). Our four independent facts are then:

1. $S'$ travels at speed $v$ with respect to $S$.
2. $S$ travels at velocity $-v$ with respect to $S'$. The minus sign here is due to the convention that we picked the positive $x$-axes of the two frames to point in the same direction.

3. Time dilation (if any) looks the same from either frame.
4. A light pulse with speed $c$ in $S'$ also has speed $c$ in $S$.

Let's see what these imply, in the above order.[14]

- (1) says that a given point in $S'$ moves at speed $v$ with respect to $S$. Letting $x' = 0$ (which is understood to be $\Delta x' = 0$, but we'll drop the $\Delta$'s from here on) in eqs. (14.76) gives $x/t = B/C$. This must equal $v$. Therefore, $B = vC$, and the transformations become

$$\begin{aligned} x &= Ax' + vCt', \\ t &= Ct' + Dx'. \end{aligned} \tag{14.77}$$

- (2) says that a given point in $S$ moves at velocity $-v$ with respect to $S'$. Letting $x = 0$ in the first of eqs. (14.77) gives $x'/t' = -vC/A$. This must equal $-v$. Therefore, $C = A$, and the transformations become

$$\begin{aligned} x &= Ax' + vAt', \\ t &= At' + Dx'. \end{aligned} \tag{14.78}$$

- (3) can be used in the following way. How fast does a person in $S$ see a clock in $S'$ tick? (The clock is assumed to be at rest with respect to $S'$.) Let our two events be two successive ticks of the clock. Then $x' = 0$, and the second of eqs. (14.78) gives

$$t = At'. \tag{14.79}$$

In other words, one second on $S'$'s clock takes a time of $A$ seconds in $S$'s frame.

Consider the analogous situation from $S'$'s point of view. How fast does a person in $S'$ see a clock in $S$ tick? (The clock is now assumed to be at rest with respect to $S$, in order to create the analogous setup. This is important.) If we invert eqs. (14.78) to solve for $x'$ and $t'$ in terms of $x$ and $t$, we find

$$\begin{aligned} x' &= \frac{x - vt}{A - vD}, \\ t' &= \frac{At - Dx}{A(A - vD)}. \end{aligned} \tag{14.80}$$

Two successive ticks of the clock in $S$ satisfy $x = 0$, so the second of eqs. (14.80) gives

$$t' = \frac{t}{A - vD}. \tag{14.81}$$

In other words, one second on $S$'s clock takes a time of $1/(A - vD)$ seconds in $S'$'s frame.

[14]In what follows, we could obtain the final result a little quicker if we invoked the speed-of-light fact prior the time-dilation one. But we'll do things in the above order so that we can easily carry over the results of this Appendix to the discussion in Section 10.8.

Both eqs. (14.79) and (14.81) apply to the same situation (someone looking at a clock flying by). Therefore, the factors on the right-hand sides must be equal, that is,

$$A = \frac{1}{A - vD} \qquad \Longrightarrow \qquad D = \frac{1}{v}\left(A - \frac{1}{A}\right). \tag{14.82}$$

Our transformations in eqs. (14.78) therefore take the form

$$\begin{aligned} x &= A(x' + vt'), \\ t &= A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)x'\right). \end{aligned} \tag{14.83}$$

- (4) may now be used to say that if $x' = ct'$, then $x = ct$. In other words, if $x' = ct'$, then

$$c = \frac{x}{t} = \frac{A((ct') + vt')}{A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)(ct')\right)} = \frac{c + v}{1 + \frac{c}{v}\Big(1 - \frac{1}{A^2}\Big)}. \tag{14.84}$$

Solving for $A$ gives

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{14.85}$$

We have chosen the positive square root so that the positive $x$ and $x'$ axes point in the same direction.

The constant $A$ is commonly denoted by $\gamma$, so we may finally write our Lorentz transformations, eqs. (14.83), in the form,

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{14.86}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{14.87}$$

in agreement with eq. (10.13).

## 14.10 Appendix J: Resolutions to the twin paradox

We have discussed the twin paradox in Chapters 10 and 13, in both the text and in various problems. To summarize, the twin paradox deals with twin $A$ who stays on the earth,[15] and twin $B$ who travels quickly to a distant star and back. When they meet up again, they discover that $B$ is younger. This is true because $A$ can use the standard special-relativistic time-dilation result to say that $B$'s clock runs slow by a factor $\gamma$.

The "paradox" arises from the fact that the situation *seems* symmetrical. That is, it seems as though each twin should be able to consider herself to be at rest, so that she sees the other twin's clock running slow. So why does $B$ turn out to be younger?

The resolution of the paradox is that the setup is in fact *not* symmetrical, because $B$ will have to turn around and will thus undergo acceleration. She will therefore not always be in an inertial frame. Therefore, she cannot apply the simple special-relativistic time-dilation result.

While the above reasoning is sufficient to get rid of the paradox, it is not quite complete, because (a) it does not explain how the result from $B$'s point of view quantitatively agrees with the result from $A$'s point of view, and (b) the paradox can actually be formulated without any mention of acceleration, in which case slightly different reasoning applies.

Below is a list of all the complete resolutions I can think of. The descriptions are terse, but I refer you to the specific problem or section in the text where things are discussed in more detail. Of course, many of these resolutions are simply slight variations of each other, so it isn't quite clear whether some of them should count as separate resolutions. But here's my list:

1. **Head-start effect**: Let the distant star be labeled as $C$. Then on the outward part of the journey, $B$ sees $C$'s clock ahead of $A$'s by $Lv/c^2$, because $C$ is the rear clock in the universe as the universe flies by. But after $B$ turns around, $A$ becomes the rear clock and is therefore now ahead of $C$. This means that $A$'s clock must jump forward very quickly, from $B$'s point of view. (See Problem 10.2 and Section 10.2.1.)

2. **Looking out the portholes**: Imagine many clocks lined up between the earth and the star, all synchronized in the earth-star frame. And imagine looking out the portholes of the spaceship and making a movie of the clocks as you fly past them. Although you see each individual clock running slow, you will see the "effective" clock in the movie (which is really many successive clocks) running fast. This effect is simply a series small applications of the head-start effect mentioned above.

3. **Minkowski diagram**: Draw a Minkowski diagram with the axes in $A$'s frame perpendicular. Then the lines of simultaneity (that is, the successive

[15] We should actually have $A$ floating in space, to avoid any GR time-dilation effects from the earth's gravity. But if $B$ travels quickly enough, the SR effects will dominate the GR ones.

$x$-axes) in $B$'s frame will be titled in different directions for the outward and inward parts of the journey. The change in the tilt at the turnaround causes a large amount of time to advance on $A$'s clock, as measured in $B$'s frame. (See Section 10.5 and Figure 10.50.)

4. **General Relativistic turnaround effect**: The acceleration that $B$ feels when she turns around may equivalently be thought of as a gravitational field. Twin $A$ on the earth is then high up in the gravitational field, so $A$ sees her clock run very fast during the turnaround. This causes $A$'s clock to show more time in the end. (See Problem 13.9.)

5. **Doppler effect**: By equating the total number of signals one twin sends out with the total number of signals the other twin receives, we can relate the total times on their clocks. (See Exercise 10.32.)

## 14.11 Appendix K: Physical constants and data

### Earth

| | |
|---|---|
| Mass | $M_E = 5.98 \cdot 10^{24}$ kg |
| Mean radius | $R_E = 6.37 \cdot 10^6$ m |
| Mean density | 5.52 g/cm$^3$ |
| Surface acceleration | $g = 9.81$ m/s$^2$ |
| Mean distance from sun | $1.5 \cdot 10^{11}$ m |
| Orbital speed | 29.8 km/s |
| Period of rotation | 23 h 56 min 4 s $= 8.6164 \cdot 10^4$ s |
| Period of orbit | 365 days 6 h $= 3.16 \cdot 10^7$ s |

### Moon

| | |
|---|---|
| Mass | $M_L = 7.35 \cdot 10^{22}$ kg |
| Radius | $R_L = 1.74 \cdot 10^6$ m |
| Mean density | 3.34 g/cm$^3$ |
| Surface acceleration | 1.62 m/s$^2 \approx g/6$ |
| Mean distance from earth | $3.84 \cdot 10^8$ m |
| Orbital speed | 1.0 km/s |
| Period of rotation | 27.3 days $= 2.36 \cdot 10^6$ s |
| Period of orbit | 27.3 days $= 2.36 \cdot 10^6$ s |

### Sun

| | |
|---|---|
| Mass | $M_S = 1.99 \cdot 10^{30}$ kg |
| Radius | $R_S = 6.96 \cdot 10^8$ m |
| Surface acceleration | 274 m/s$^2 \approx 28g$ |

### Fundamental constants

| | |
|---|---|
| Speed of light | $c = 2.998 \cdot 10^8$ m/s |
| Gravitational constant | $G = 6.673 \cdot 10^{-11}$ N m$^2$/kg$^2$ |
| Planck's constant | $h = 6.63 \cdot 10^{-34}$ J s |
| Electron charge | $e = 1.602 \cdot 10^{-19}$ C |
| Electron mass | $m_e = 9.11 \cdot 10^{-31}$ kg $= 0.511$ MeV$/c^2$ |
| Proton mass | $m_p = 1.673 \cdot 10^{-27}$ kg $= 938.3$ MeV$/c^2$ |
| Neutron mass | $m_n = 1.675 \cdot 10^{-27}$ kg $= 939.6$ MeV$/c^2$ |