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Real homotopy invariance of configuration spaces

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Abstract

To a manifold M and an integer $k \geq 0$ one associates the configuration space $F_k(M)$ of k ordered distinct points in M, together with its unordered quotient $B_k(M) = F_k(M)/\Sigma_k$. These are among the most elementary spaces one can build from M. A question raised early in the subject asks how much of $F_k(M)$ is determined by the homotopy type of M alone; for a long time it was expected that homotopy equivalent closed manifolds would have homotopy equivalent configuration spaces. This thesis is organised around that question.

The expectation is false. Chapter 1 gives a self-contained account of the counterexample of Longoni and Salvatore [LS05]: the lens spaces $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent, yet their ordered two-point configuration spaces are not and the difference detected by a triple Massey product on the universal cover present for one space, killed by formality for the other. The argument depends essentially on the fundamental group, and this is the gap through which the positive results enter: once π_1 is removed, the question reopens.

Chapter 2 assembles the rational and real homotopy theory the thesis relies on commutative differential graded algebras, Sullivan minimal models, formality, and Poincaré-duality CDGAs and Lambrechts–Stanley models [LS08, LS04] that feed the main invariance theorem. In chapter 3 we then prove Idrissi's theorem [Idr19, Idr22]: for a smooth, simply connected, closed manifold of dimension at least four, the real homotopy type of $F_k(M)$ depends only on that of M. The proof runs through a labelled graph complex interpolating between the Lambrechts–Stanley model and the piecewise semi-algebraic forms on the Fulton–MacPherson compactification, and turns on the vanishing of a partition function of vacuum graphs; we close the chapter by putting the model to work on an explicit computation of $H^*(F_2(\mathbb{HP}^2); \mathbb{Q})$.

Idrissi's theorem settles the real question and leaves the integral one open. Chapter 4 surveys this exact question: the loop-space and stable invariance theorems, the rational results of Lambrechts–Stanley and Cordova Bulens, and the constraints any simply connected counterexample must meet [Lev95, AK04, CB15, RS18] and adds an explicit computation by a second route: Knudsen's factorisation-homology model [Knu17] for the rational homology of unordered configuration spaces. We compute $H_*(B_2(\mathbb{HP}^2); \mathbb{Q})$ and $H_*(B_3(\mathbb{HP}^2); \mathbb{Q})$, cross-check the first against the Lambrechts–Stanley computation of the previous chapter, and note that the whole Eells–Kuiper family of exotic $\mathbb{HP}^2$'s shares these Betti numbers, so that the first Pontryagin class is invisible to them.

Chapter 5 is the little new contribution, and we shifts perspective in three ways at once: from ordered to unordered configuration spaces, from homotopy type to rational Betti numbers, and from invariance to homological stability as k grows. Using the Chevalley–Eilenberg model of Félix–Thomas and Knudsen [FT00, Knu17], a model

which is different from the Lambrechts–Stanley CDGA of Chapter 3, we determine the complete rational Betti numbers of $B_k(\mathbb{HP}^2)$ for every k, and likewise those of the Cayley plane $B_k(\mathbb{OP}^2)$ and of every *rational homology projective plane*: a closed oriented manifold whose Poincaré polynomial has the projective-like shape $1 + t^{2m} + t^{4m}$. The low-k values already suggest that these Poincaré polynomials stabilise completely, and a theorem of Berceanu and Yameen [BY18] confirms it: from $k = 4$ onwards the polynomial is constant, equal to $1+t^{2m}+t^{4m}+t^{8m-1}+t^{10m-1}+t^{12m-1}$. Berceanu and Yameen show that this strong stability holds, among closed even-dimensional manifolds, exactly for rational homology spheres and rational homology projective planes, so it is precisely the projective-like Poincaré polynomial that places $\mathbb{HP}^2$ and $\mathbb{OP}^2$ on the stable side. Our own additions are little but, to our knowledge, absent from the literature: explicit polynomial cocycle representatives for the six stable classes of $B_k(\mathbb{HP}^2)$ and $B_k(\mathbb{OP}^2)$; a modular block decomposition of the Chevalley–Eilenberg complex, special to the higher projective planes ($m \geq 2$) and failing for $\mathbb{CP}^2$, which collapses the cohomology in each degree to a single rank computation; a fully explicit computation at $k = 6$; and the observation that the exotic Eells–Kuiper families [EK62] agree at the level of all these Betti numbers, so that the Eells–Kuiper μ-invariant, like the Pontryagin class before it, is washed out by the configuration space. It is possible, even if not present in this thesis, to obtain (in the same way as our explicit computation) an easy proof on one of the implications of the strong stability theorem of Berceanu and Yameen that the Poincaré polynomial of projective-like manifolds determine the (ultimately) stable Poincaré polynomial of its configuration spaces.

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1

Configuration Spaces Are Not Homotopy Invariant

This chapter presents a self-contained account of the Longoni–Salvatore proof [LS05] that the two-point ordered configuration spaces of the lens spaces $L(7,1)$ and $L(7,2)$ are not homotopy equivalent, disproving the long-standing conjecture on the homotopy invariance of configuration spaces. The argument compares the universal covers of the two deleted squares: the first is formal, while the second carries a non-trivial triple Massey product. Additional details and refinements are taken from [ELS16, Bas15, Idr22, Mil11].

1.1 Introduction and Statement of the Main Result

Given a topological space X and a positive integer n, the *ordered configuration space* of n points in X is

$$F_n(X) \;=\; \bigl\{(x_1,\dots,x_n)\in X^n \;\big|\; x_i\neq x_j \text{ for all } i\neq j\bigr\} \;\subset\; X^n.$$

A long-standing conjecture asserted that if two closed manifolds M and N are homotopy equivalent, then their configuration spaces $F_n(M)$ and $F_n(N)$ should also be homotopy equivalent.

Levitt [Lev95] proved that the *based loop space* $\Omega F_k(M)$ is a homotopy invariant of M; that is, a homotopy equivalence $M\simeq N$ induces a homotopy equivalence $\Omega F_k(M)\simeq \Omega F_2(N)$ (just as topological space, not as loop space).

1.1.1 Counterexample

In their 2005 paper, Longoni and Salvatore [LS05] disproved the conjecture by exhibiting an explicit counterexample.

Theorem 1.1.1 (Longoni–Salvatore, 2005)**.** *The lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent closed 3-manifolds, but their configuration spaces of two ordered points are **not** homotopy equivalent:*

$$F_2\bigl(L(7,1)\bigr) \;\not\simeq\; F_2\bigl(L(7,2)\bigr).$$

The proof sketch is the following:

We will show that $L(7,1)$ and $L(7,2)$ are homotopy equivalent (but not homeomorphic), then describe the universal covering space $\widetilde{X}_0$ of the deleted square $X_0 = F_2(L(7,q))$ for $q = 1, 2$.
After that, we will compute the cohomology ring $H^*(\widetilde{X}_0;\mathbb{Z})$ together with its $\pi_1(X_0)$-module structure.
We then compute the triple Massey products in $H^*(\widetilde{X}_0;\mathbb{Z}/2)$ and show that for $q = 1$, all Massey products vanish (the universal cover is formal), while for $q = 2$, certain Massey products are nontrivial.

1.2 Lens Spaces

1.2.1 Definition

Definition 1.2.1 (Lens space)**.** Let p and q be coprime positive integers with $p > q \geq 1$. Consider the unit 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

The cyclic group $\mathbb{Z}/p$ acts freely on S^3 via the rotation

$$\rho(z_1, z_2) = (\zeta z_1,\, \zeta^q z_2), \qquad \zeta = e^{2\pi i/p}.$$

The *lens space* $L(p,q)$ is the orbit space $S^3/(\mathbb{Z}/p)$.

Since the action is free and properly discontinuous, $L(p,q)$ is a smooth closed oriented 3-manifold. The quotient map $\pi\colon S^3 \to L(p,q)$ is the universal covering projection, and $\pi_1(L(p,q)) \cong \mathbb{Z}/p$.

1.2.2 CW-structure and homology

The standard CW-complex structure on $L(p,q)$ consists of exactly one cell e^k in each dimension $k = 0, 1, 2, 3$. The resulting cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

giving the homology groups:

$$H_0(L(p,q)) = \mathbb{Z}, \qquad H_1(L(p,q)) = \mathbb{Z}/p,$$
$$H_2(L(p,q)) = 0, \qquad H_3(L(p,q)) = \mathbb{Z}.$$

1.2.3 Classification of lens spaces

The following classical result classifies the homotopy of 3-dimensional lens spaces.

Theorem 1.2.2 (Classification of lens spaces)**.** *Let $L(p,q_1)$ and $L(p,q_2)$ be two 3-dimensional lens spaces.*

(i) $L(p,q_1) \simeq L(p,q_2)$ if and only if $q_1 q_2 \equiv \pm n^2 \pmod p$ for some integer n.

(ii) $L(p,q_1) \cong L(p,q_2)$ if and only if $q_1 \equiv \pm q_2^{\pm 1} \pmod p$.

The homotopy classification is due to J. H. C. Whitehead and the homeomorphism classification is due to Reidemeister [Rei35] (PL case) and Brody [Bro60] (topological case), using Reidemeister torsion.

Corollary 1.2.3. *The lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent but **not** homeomorphic.*

Remark 1.2.4. Lens spaces are parallelizable (as quotients of S^3, which is a Lie group and hence parallelizable, by a free orientation-preserving action). This fact will be used when discussing the normal bundle of the diagonal in $L(p,q) \times L(p,q)$.

1.3 Configuration Spaces and Deleted Squares

Definition 1.3.1. Let $L = L(p,q)$. The *configuration space of two ordered points* (also called the *deleted square*) is

$$X_0 = F_2(L) = (L \times L) \setminus \Delta,$$

where $\Delta = \{(x,x) \mid x \in L\} \subset L \times L$ is the diagonal.

We write $X = L \times L$ for the full square.

1.3.1 Homology of the square $X = L \times L$

Using the product CW-structure on $X = L \times L$ with cells $e^i \times e^j$, the homology is computed via the Künneth theorem:

$$\begin{aligned} H_0(X) &= \mathbb{Z}, & H_3(X) &= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/p, \\ H_1(X) &= \mathbb{Z}/p \oplus \mathbb{Z}/p, & H_4(X) &= \mathbb{Z}/p \oplus \mathbb{Z}/p, \\ H_2(X) &= \mathbb{Z}/p, & H_5(X) &= 0, \\ & & H_6(X) &= \mathbb{Z}. \end{aligned}$$

Remark 1.3.2 (Explicit generators of $H_3(X)$). The two free summands $\mathbb{Z} \oplus \mathbb{Z}$ in $H_3(X)$ are generated by the fundamental classes of the two copies of L embedded as $L \times \{*\}$ and $\{*\} \times L$, corresponding to the cells $e^3 \times e^0$ and $e^0 \times e^3$ respectively. The torsion summand $\mathbb{Z}/p$ is generated by the homology class of the cycle $e^2 \times e^1 + e^1 \times e^2$. To see this is a cycle, note that $\partial(e^2 \times e^1 + e^1 \times e^2) = p \cdot e^1 \times e^1 - e^1 \times p \cdot e^1 = 0$. Its p-th multiple is a boundary: $\partial(e^2 \times e^2) = p(e^1 \times e^2 + e^2 \times e^1)$.

1.3.2 Fundamental group and the fibration structure

The deleted square X_0 is path-connected and its fundamental group satisfies

$$\pi_1(X_0) \cong \pi_1(X) \cong \mathbb{Z}/p \oplus \mathbb{Z}/p,$$

since removing the diagonal (which has codimension 3 in the 6-manifold X) does not affect π_1 by a standard geometric argument.

Proposition 1.3.3 (Fiber bundle structure)**.** *The projection onto the first factor $p_1\colon X_0 \to L$, defined by $p_1(x, y) = x$, is a locally trivial fiber bundle with fiber*

$$L \setminus \{*\} \;\simeq\; L_{\mathrm{punct}},$$

the punctured lens space. Since L is parallelizable, this bundle admits a section.

Proof. The fiber over $x \in L$ is $\{x\} \times (L \setminus \{x\})$. Since L is a manifold, local triviality follows from the existence of local coordinates. The section is obtained by pushing off the diagonal using a nowhere-vanishing vector field on L (which exists because L is parallelizable): given $x \in L$, let $s(x) = (x, \exp_x(\varepsilon v(x)))$ where v is a nonvanishing vector field and $\varepsilon > 0$ is sufficiently small. ☐

1.3.3 Homology of the deleted square X_0

Proposition 1.3.4. *The homology groups of the deleted square $X_0 = F_2(L(p, q))$ are:*

$$\begin{aligned} H_0(X_0) &= \mathbb{Z}, & H_3(X_0) &= \mathbb{Z} \oplus \mathbb{Z}/p,\\ H_1(X_0) &= \mathbb{Z}/p \oplus \mathbb{Z}/p, & H_4(X_0) &= \mathbb{Z}/p,\\ H_2(X_0) &= \mathbb{Z}/p, & H_5(X_0) &= 0. \end{aligned}$$

Proof. We use Poincaré–Lefschetz duality. Since L is parallelizable, the normal bundle of the diagonal $\Delta \hookrightarrow X$ is trivial (it is isomorphic to the tangent bundle of L, which is trivial). Let $N(\Delta)$ be a closed tubular neighborhood of Δ in X. Since the normal bundle is trivial, $\partial N(\Delta) \cong L \times S^2$. Then $X_0 = X \setminus \Delta$ deformation retracts onto $X \setminus \operatorname{Int} N(\Delta)$.

We use the long exact sequence of the pair (X, Δ). The inclusion $i\colon \Delta \hookrightarrow X$ satisfies: the maps $i_*\colon H_k(\Delta) \to H_k(X)$ are injective (they split via the projections $X \to L$). Hence the long exact sequence breaks into short exact sequences

$$0 \to H_k(\Delta) \xrightarrow{i_*} H_k(X) \to H_k(X, \Delta) \to 0.$$

Computing $H_k(X, \Delta)$ is immediate except for $k = 3$. For $k = 3$: the map $i_*\colon H_3(\Delta) \to H_3(X)$ sends the fundamental class $[\Delta]$ to $(1, 1, a) \in \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/p = H_3(X)$ (via the two projections $X \to L$), where a is some element of $\mathbb{Z}/p$. Therefore, $H_3(X, \Delta) = \mathbb{Z} \oplus \mathbb{Z}/p$.

By excision and the Thom isomorphism,

$$H_k(X, X_0) \cong H_k(N(\Delta), \partial N(\Delta)) \cong H_{k-3}(\Delta).$$

Using the Poincaré duality isomorphism $H^{6-k}(X \setminus \Delta) \cong H_k(X, \Delta)$, and the universal coefficient theorem, we obtain the stated homology groups. ☐

Lemma 1.3.5. *The inclusion induced homomorphism $i_*\colon H_3(X_0) \to H_3(X)$ is an isomorphism $\mathbb{Z}/p \to \mathbb{Z}/p$ on the torsion subgroups.*

Proof. Consider the long exact sequence of the pair (X, X_0):

$$H_4(X_0) \xrightarrow{i_*} H_4(X) \xrightarrow{j} H_4(X, X_0) \xrightarrow{\delta} H_3(X_0) \xrightarrow{i_*} H_3(X).$$

We have computed $H_4(X_0) = \mathbb{Z}/p$, $H_4(X) = \mathbb{Z}/p \oplus \mathbb{Z}/p$, $H_4(X, X_0) \cong H_1(\Delta) = \mathbb{Z}/p$, $H_3(X_0) = \mathbb{Z} \oplus \mathbb{Z}/p$, $H_3(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/p$. The connecting homomorphism $\delta\colon \mathbb{Z}/p \to \mathbb{Z}\oplus\mathbb{Z}/p$ must have the form $\delta(1) = (0, k)$ for some $k \in \mathbb{Z}/p$ (since there are no nontrivial homomorphisms $\mathbb{Z}/p \to \mathbb{Z}$). Then $\ker\delta = \operatorname{im} j = \mathbb{Z}/m$ where $m = \gcd(p, k)$. By the first isomorphism theorem applied to j, we need $(\mathbb{Z}/p\oplus\mathbb{Z}/p)/\operatorname{im} i_* \cong \mathbb{Z}/m$, which forces $m = p$, hence $k \equiv 0 \pmod p$ and $\delta = 0$. Therefore i_* is injective on the torsion. ☐

Lemma 1.3.6. *One can choose generators of $H_3(X_0) = \mathbb{Z} \oplus \mathbb{Z}/p$ so that the inclusion-induced map $i_*\colon H_3(X_0) \to H_3(X)$ satisfies*

$$i_*(1,0) = (1,1,0), \qquad i_*(0,1) = (0,0,1),$$

with respect to the canonical basis $(e^3 \times e^0,\ e^0 \times e^3,\ e^2 \times e^1 + e^1 \times e^2)$ of $H_3(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/p$.

Proof. By Lemma 1.3.5, i_* is an isomorphism on the torsion. From the proof above, $\delta = 0$, so the long exact sequence yields

$$0 \to H_3(X_0) \xrightarrow{i_*} H_3(X) \to H_3(X, X_0) \to H_2(X_0) \xrightarrow{i_*} H_2(X) \to 0.$$

Since $H_2(X_0) \cong H_2(X) \cong \mathbb{Z}/p$, the last map i_* is an isomorphism. Modding out the torsion we get a short exact sequence

$$0 \to H_3(X_0)/\mathrm{Tor} \xrightarrow{i_*} H_3(X)/\mathrm{Tor} \to \mathbb{Z} \to 0,$$

i.e., $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$.

To identify the generator: the section $s\colon L \to X_0$ from Proposition 1.3.3 satisfies $i \circ s \simeq \Delta_L$, the diagonal inclusion $x \mapsto (x,x)$. Hence $i_*(s_*[L]) = [\Delta] = (1,1,0) \in H_3(X)/\mathrm{Tor}$. Take $s_*[L]$ as the generator $(1,0)$ of the free part. Combined with the torsion isomorphism, the claim follows. $\square$

1.4 The Universal Cover of the Deleted Square

The universal covering space of $X = L \times L$ is $S^3 \times S^3$. The deck transformation group $\pi_1(X) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts on $S^3 \times S^3$ by

$$\tau_{m,n}\big((z_1,z_2),(z_3,z_4)\big) = \big((\zeta^m z_1, \zeta^{qm} z_2),\ (\zeta^n z_3, \zeta^{qn} z_4)\big), \qquad \zeta = e^{2\pi i/p}.$$

The diagonal $\Delta \subset L \times L$ lifts to the orbit of the diagonal $\Delta_0 \subset S^3 \times S^3$ under this action. The orbit consists of p disjoint copies of S^3, labeled by $k \in \mathbb{Z}/p$:

$$\Delta_k = \big\{\big((z_1,z_2),(z_3,z_4)\big) \in S^3 \times S^3 \ \big|\ (z_1,z_2) = (\zeta^k z_3, \zeta^{qk} z_4)\big\}.$$

Definition 1.4.1 (Orbit configuration space)**.** The *universal covering space* of the deleted square X_0 is the *orbit configuration space*

$$\widetilde{X}_0 = (S^3 \times S^3) \setminus \bigcup_{k \in \mathbb{Z}/p} \Delta_k = \big\{\big((z_1,z_2),(z_3,z_4)\big) \in S^3 \times S^3 \ \big|\ (z_1,z_2) \neq (\zeta^k z_3, \zeta^{qk} z_4)\ \forall\, k \in \mathbb{Z}/p\big\}.$$

1.4.1 Cohomology of the universal cover

Lemma 1.4.2. *The only nontrivial reduced cohomology groups of $\widetilde{X}_0$ are:*

$$H^2(\widetilde{X}_0) = \mathbb{Z}^{p-1}, \qquad H^3(\widetilde{X}_0) = \mathbb{Z}, \qquad H^5(\widetilde{X}_0) = \mathbb{Z}^{p-1}.$$

Moreover, the cup product with a generator of $H^3(\widetilde{X}_0)$ provides an isomorphism

$$H^2(\widetilde{X}_0) \xrightarrow{\sim} H^5(\widetilde{X}_0).$$

Proof. Consider the projection $\widetilde{X}_0 \hookrightarrow S^3 \times S^3 \xrightarrow{\mathrm{pr}_1} S^3$ onto the first S^3-factor. The restriction $\widetilde{X}_0 \to S^3$ is a fiber bundle. The fiber over a point $(z_1, z_2) \in S^3$ is

$$S^3 \setminus \big\{(\zeta^k z_1, \zeta^{qk} z_2) \mid k \in \mathbb{Z}/p\big\} = S^3 \setminus \{p \text{ points}\}.$$

A 3-sphere with p points removed is homotopy equivalent to a wedge of $(p-1)$ copies of S^2:

$$S^3 \setminus \{p \text{ points}\} \;\simeq\; \bigvee_{p-1} S^2.$$

Since L is parallelizable, the original bundle $X_0 \to L$ admits a section. Lifting to the universal cover, the bundle $\widetilde{X}_0 \to S^3$ also admits a section. Consider the Serre spectral sequence for the fibration

$$\bigvee_{p-1} S^2 \hookrightarrow \widetilde{X}_0 \to S^3.$$

The E_2-page is $E_2^{r,s} = H^r(S^3; H^s(\bigvee_{p-1} S^2))$. The nontrivial cohomology groups of the fiber are $H^0 = \mathbb{Z}$ and $H^2 = \mathbb{Z}^{p-1}$, and those of the base are $H^0 = \mathbb{Z}$ and $H^3 = \mathbb{Z}$. Hence the nonzero entries on E_2 are:

$$\begin{aligned} E_2^{0,0} &= \mathbb{Z}, & \qquad E_2^{0,2} &= \mathbb{Z}^{p-1}, \\ E_2^{3,0} &= \mathbb{Z}, & \qquad E_2^{3,2} &= \mathbb{Z}^{p-1}. \end{aligned}$$

There are no possible nontrivial differentials for degree reasons (the only potentially nontrivial differential would be $d_3 \colon E_3^{0,2} \to E_3^{3,0}$, but the existence of a section implies that the inclusion of the fiber splits the spectral sequence). Therefore, $E_2 = E_\infty$ and the cohomology of $\widetilde{X}_0$ is as stated.

The cup product statement follows because the spectral sequence is multiplicative and the section gives a tensor product decomposition:

$$H^*(\widetilde{X}_0) \cong H^*(S^3) \otimes H^*\Big(\bigvee_{p-1} S^2\Big)$$

as graded abelian groups. □

1.4.2 Explicit generators via Poincaré duality

Longoni and Salvatore provided explicit generators for $H^2(\widetilde{X}_0)$ using Poincaré–Lefschetz duality.

Definition 1.4.3 (The submanifolds A_k)**.** For each $k \in \mathbb{Z}/p$, define the submanifold

$$A_k = \Big\{\big((z_1, z_2), (\zeta^s z_1, \zeta^{qs} z_2)\big) \in S^3 \times S^3 \;\Big|\; s \in [k-1, k]\Big\} \;\subset\; S^3 \times S^3.$$

Each A_k is a compact 4-dimensional submanifold of $S^3 \times S^3$ with

$$\partial A_k = \Delta_k - \Delta_{k-1}.$$

That is, A_k is a cobordism between the two consecutive diagonal spheres Δ_{k-1} and Δ_k.

Definition 1.4.4 (Cohomology generators a_k)**.** Let $a_k \in H^2(\widetilde{X}_0)$ be the Poincaré dual of A_k relative to the ambient $S^3 \times S^3$ minus the diagonal copies. More precisely, a_k is defined via the Poincaré–Lefschetz duality isomorphism

$$H^2\Big(S^3 \times S^3 \setminus \bigcup \Delta_k\Big) \;\cong\; H_4\Big(S^3 \times S^3, \bigcup \Delta_k\Big).$$

Proposition 1.4.5 (Relations among generators)**.** *The generators $a_0, a_1, \dots, a_{p-1}$ satisfy the single relation*

$$a_0 + a_1 + \cdots + a_{p-1} = 0.$$

Consequently, $H^2(\widetilde{X}_0) \cong \mathbb{Z}^{p-1}$ is freely generated by any $p-1$ of the a_k's.

Proof. The relation comes from the fact that $\sum_k A_k$ represents the total cobordism from Δ_0 back to Δ_0 (indices mod p), and hence bounds the entire "annular" region. More precisely, $\partial(\sum_k A_k) = \sum_k(\Delta_k - \Delta_{k-1}) = 0$, and the corresponding class in relative homology is trivial since $\sum_k A_k$ fills the entire rotation cycle. □

1.4.3 The $\pi_1(X_0)$-module structure

Lemma 1.4.6. *The fundamental group $\pi_1(X_0) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts on the generators a_k of $H^2(\widetilde{X}_0)$ by*

$$\tau_{m,n}^*(a_k) = a_{k+m-n}, \qquad \textit{indices taken mod } p.$$

Proof. The deck transformation $\tau_{m,n}$ acts on the submanifolds Δ_k by $\tau_{m,n}(\Delta_k) = \Delta_{k+n-m}$, which follows directly from the definition of Δ_k and the action of $\tau_{m,n}$. Correspondingly, $\tau_{m,n}(A_k) = A_{k+n-m}$.

To pass to cohomology, recall the Poincaré duality diagram for an orientation-preserving diffeomorphism $f\colon W \to W$ of a closed oriented manifold W:

$$\begin{tikzcd}
H^k(W) \arrow[r, "\mathrm{PD}", "\sim"'] \arrow[d, "f^*"'] & H_{n-k}(W) \arrow[d, "f_*"] \\
H^k(W) \arrow[r, "\mathrm{PD}", "\sim"'] & H_{n-k}(W).
\end{tikzcd}$$

The diagram commutes when f^* is paired with $(f^{-1})_*$, since Poincaré duality intertwines pullback in cohomology with the inverse pushforward in homology. Applied to $f = \tau_{m,n}$, whose inverse is $\tau_{-m,-n}$, the relation $\tau_{m,n}(A_k) = A_{k+n-m}$ on the homology side becomes $\tau_{m,n}^*(a_k) = a_{k+m-n}$ on the cohomology side, where the sign change in the indices comes from passing to the inverse. □

Corollary 1.4.7 (Cyclotomic module structure)**.** *Let $t = \tau_{1,0}^*$ be the action of the generator $(1,0) \in \mathbb{Z}/p \oplus \mathbb{Z}/p$. The diagonal subgroup $\mathbb{Z}/p \subset \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts trivially on $H^2(\widetilde{X}_0)$, and the quotient group $\mathbb{Z}/p$ acts effectively*[1]*. The identification*

$$\varphi\colon H^2(\widetilde{X}_0) \xrightarrow{\sim} \mathbb{Z}[t]/(1 + t + \cdots + t^{p-1}), \qquad \varphi(a_{k+1}) = t^k,$$

is an isomorphism of $\mathbb{Z}[t]$-modules, where the right-hand side is the cyclotomic ring $\mathbb{Z}[\zeta_p] \cong \mathbb{Z}[t]/(N)$ with $N = 1 + t + \cdots + t^{p-1}$.

Proof. From Lemma 1.4.6, $\tau_{m,m}^*(a_k) = a_k$, so the diagonal acts trivially. The quotient $\mathbb{Z}/p$ acts by $t \cdot a_k = a_{k+1}$, hence $t \cdot \varphi(a_{k+1}) = t \cdot t^k = t^{k+1} = \varphi(a_{k+2}) = \varphi(t \cdot a_{k+1})$. The relation $\sum a_k = 0$ corresponds to $1 + t + \cdots + t^{p-1} = 0$. □

[1] An action of a group G on a set X is *effective* (or *faithful*) if the only element of G acting as the identity on X is the identity of G. Equivalently, the homomorphism $G \to \mathrm{Sym}(X)$ giving the action is injective.

1.5 Massey Products: Definitions and Properties

1.5.1 Definition of the triple Massey product

Definition 1.5.1 (Triple Massey product)**.** Let R be a commutative ring and let X be a topological space. Given cohomology classes $x, y, z \in H^*(X; R)$ with

$$x \cup y = 0 \quad \text{and} \quad y \cup z = 0,$$

The *triple Massey product* [Mas58] $\langle x, y, z\rangle$ is defined as follows.

Choose singular cocycle representatives $\bar{x}, \bar{y}, \bar{z}$ for x, y, z respectively. Since $x \cup y = 0$, there exists a cochain Z with $dZ = \bar{x} \cup \bar{y}$. Similarly, since $y \cup z = 0$, there exists a cochain X with $dX = \bar{y} \cup \bar{z}$.

The cochain

$$Z \cup \bar{z} - (-1)^{|x|}\bar{x} \cup X$$

is a cocycle. Its cohomology class is denoted $\langle x, y, z\rangle$.

The class $\langle x, y, z\rangle$ is well-defined only modulo the ideal

$$x \cup H^*(X; R) + H^*(X; R) \cup z,$$

because different choices of Z and X change the cocycle by terms of the form $x\cup\alpha+\beta\cup z$. Thus the Massey product is a coset in the quotient group $H^*(X; R)/(x \cup H^* + H^* \cup z)$.

Remark 1.5.2. In our application, $x, y, z \in H^2(\widetilde{X}_0)$ and all cup products $x \cup y$ vanish because $H^4(\widetilde{X}_0) = 0$. Hence the Massey product $\langle x, y, z\rangle$ is always defined and lies in $H^5(\widetilde{X}_0)$.

1.6 The Case $q = 1$: Formality of $\widetilde{X}_0(L(7, 1))$

1.6.1 Structure of the universal cover for $q = 1$

Proposition 1.6.1. *For the lens space $L(p, 1)$, the bundle $\widetilde{X}_0 \to S^3$ splits as a product:*

$$\widetilde{X}_0\big(L(p, 1)\big) \;\simeq\; \Big(\bigvee_{p-1} S^2\Big) \times S^3.$$

Lemma 1.6.2. *Identify S^3 with the group of unit quaternions* $\mathrm{Sp}(1)$*. The map*

$$\Phi\colon S^3 \times \big(S^3 \setminus \{\zeta^k\}_{k=0}^{p-1}\big) \longrightarrow \widetilde{X}_0\big(L(p, 1)\big), \qquad \Phi(g, h) = (g, gh),$$

is a homeomorphism, where $\zeta = e^{2\pi i/p}$ and quaternion multiplication is used in the second factor.

Proof. The map Φ is continuous and admits a continuous inverse given by $(x, y) \mapsto (x, x^{-1}y)$, so it is a homeomorphism between $S^3 \times S^3$ and itself. We must check that it restricts to a homeomorphism between the asserted domain and codomain.

Recall that $\widetilde{X}_0(L(p, 1))$ is the orbit configuration space, consisting of pairs $(x, y) \in S^3 \times S^3$ with $y \neq \zeta^k x$ for all $k = 0, \ldots, p-1$, where the $\mathbb{Z}/p$-action on S^3 is left multiplication by ζ. The condition $y \neq \zeta^k x$ translates, under $y = gh$ and $x = g$, to $gh \neq \zeta^k g$, that is, $h \neq g^{-1}\zeta^k g$. Since ζ is a complex number, hence a central element of the quaternion algebra, $g^{-1}\zeta^k g = \zeta^k$ for every $g \in \mathrm{Sp}(1)$. The condition therefore reduces to $h \neq \zeta^k$ for all k, which is independent of g. Hence Φ restricts to a homeomorphism between $S^3 \times (S^3 \setminus \{\zeta^k\}_{k=0}^{p-1})$ and $\widetilde{X}_0(L(p,1))$. □

Proof of Proposition 1.6.1. By Lemma 1.6.2, the projection $\widetilde{X}_0(L(p,1)) \to S^3$ onto the first factor is a trivial bundle with fiber $S^3 \setminus \{\zeta^k\}_{k=0}^{p-1}$. Removing p points from S^3 produces a space homotopy equivalent to $\bigvee_{p-1} S^2$: by stereographic projection, S^3 minus a point is $\mathbb{R}^3$, and $\mathbb{R}^3$ minus $p-1$ points deformation retracts onto a wedge of $p-1$ copies of S^2. Therefore

$$\widetilde{X}_0(L(p,1)) \;\simeq\; S^3 \times \Big(\bigvee_{p-1} S^2\Big). \qquad \square$$

Remark 1.6.3. For general q, this construction fails because the removed points $(\zeta^k z_1, \zeta^{qk} z_2)$ depend on the ratio z_1/z_2 when $q \neq 1$.

Theorem 1.6.4. *All Massey products in $H^*(\widetilde{X}_0(L(p,1)); R)$ vanish for any coefficient ring R. In particular, $\widetilde{X}_0(L(p,1))$ is a formal space.*

Proof. By Proposition 1.6.1, we have $\widetilde{X}_0 \simeq (\bigvee_{p-1} S^2) \times S^3$. Both factors of this product are formal: a wedge of spheres[2] is formal because its cohomology ring (with all cup products of positive-degree classes vanishing) is itself a model, and any sphere S^n is formal for the same reason. Formality is preserved by products, since the Künneth isomorphism identifies the cochain algebra of a product with the tensor product of the cochain algebras of the factors, and the tensor product of two formal CDGAs is formal. Hence $\widetilde{X}_0(L(p,1))$ is formal.

It remains to deduce the vanishing of all Massey products from formality. By definition, a formal space admits a quasi-isomorphism between its cochain algebra and its cohomology ring (regarded as a CDGA with zero differential). Massey products are invariants of the quasi-isomorphism type of the cochain algebra, and on a CDGA with zero differential every Massey product is automatically zero (there are no nontrivial bounding cochains to choose). Transporting this back along the quasi-isomorphism, every Massey product in $H^*(\widetilde{X}_0(L(p,1)); R)$ vanishes. □

1.7 The Case $q = 2$.

When $q \neq 1$, the fiber bundle $\widetilde{X}_0 \to S^3$ is *not* a product. This is because the p points removed from each fiber S^3 are now $\{(\zeta^k z_1, \zeta^{qk} z_2)\}$, and the relative positions of these points depend on the base point (z_1, z_2) when $q \neq 1$.

[2] More generally, a wedge of formal simply connected spaces of finite type is formal. The minimal model of a wedge $X \vee Y$ can be constructed from the minimal models of X and Y together with the splitting $\widetilde{H}^*(X \vee Y) \cong \widetilde{H}^*(X) \oplus \widetilde{H}^*(Y)$, and the formality quasi-isomorphisms on the factors combine to give a formality quasi-isomorphism on the wedge. One can look at [FHT01, §24] for the construction of minimal models of wedges and the verification that formality is preserved.

1.7.1 Intersection theory of the submanifolds A_k

We now specialize to $p = 7$, $q = 2$, i.e., the lens space $L(7, 2)$.

Longoni and Salvatore compute the Massey products using intersection theory on $S^3 \times S^3$. The key is to understand the intersections of the submanifolds A_k (Definition 1.4.3).

Lemma 1.7.1 (Transversality of A_k's). *For $L(7, 2)$, the submanifolds A_k and A_{k+3} intersect transversally in the interior of $\widetilde{X}_0$ for all $k \pmod 7$. Moreover, for $j \neq k$ with $j - k \not\equiv \pm 3 \pmod 7$, the submanifolds A_k and A_j have no interior intersection points.*

Proof. Recall that A_k consists of points $\big((z_1, z_2), (\zeta^s z_1, \zeta^{2s} z_2)\big)$ with $(z_1, z_2) \in S^3$ and $s \in [k - 1, k]$, where $\zeta = e^{2\pi i/7}$. An interior intersection of A_j and A_k is a point of $\widetilde{X}_0$ that lies on neither diagonal Δ_ℓ and admits two parameter representations: one with parameter $s \in (k - 1, k)$, one with parameter $t \in (j - 1, j)$, sharing the same first component (z_1, z_2) and the same second component $(\zeta^s z_1, \zeta^{2s} z_2) = (\zeta^t z_1, \zeta^{2t} z_2)$.

The two coordinate equations

$$\zeta^s z_1 = \zeta^t z_1, \qquad \zeta^{2s} z_2 = \zeta^{2t} z_2,$$

with $\zeta^s = e^{2\pi i s/7}$ extended to all $s \in \mathbb{R}$, must hold simultaneously. On the open dense locus where $z_1 z_2 \neq 0$ they reduce to $s \equiv t \pmod 7$ and $2s \equiv 2t \pmod 7$, the latter being equivalent to the former since $\gcd(2, 7) = 1$. On the codimension-2 loci $z_1 = 0$ and $z_2 = 0$, only one of the two equations is nontrivial, but a dimension count shows that the contribution to $A_j \cap A_k$ from these loci has codimension at least 4 in the 4-dimensional manifold $\widetilde{X}_0$ and so contributes only isolated points, which can be excluded by a small perturbation.

We are thus reduced to the condition $s - t \in 7\mathbb{Z}$ on the interiors of the parameter intervals. Working in the universal cover $\widetilde{X}_0$ requires choosing lifts of A_j and A_k, which shifts the equation to

$$s - t \equiv c_{jk} \pmod 7$$

for a definite $c_{jk} \in \mathbb{Z}/7$ depending on the lifts. The twisting by $q = 2$ in the second coordinate imposes an additional compatibility: the same relation must hold after multiplication by 2, that is, $2(s - t) \equiv 2c_{jk} \pmod 7$. Combined with the constraint $s - t \in (k - j - 1, k - j + 1)$, these two conditions are simultaneously solvable if and only if $k - j \equiv \pm 3 \pmod 7$, the values ± 3 being the unique classes for which $2(k - j) \equiv \mp 1 \pmod 7$. It remains to verify transversality. By the $\mathbb{Z}/7$-action permuting the A_k's cyclically, it suffices to treat a single pair, say A_1 and A_4. Each A_k is 4-dimensional in the 5-dimensional $\widetilde{X}_0$, and at an intersection point parametrized by $\big((z_1, z_2), (\zeta^s z_1, \zeta^{2s} z_2)\big)$ the tangent spaces are spanned by the tangent space to S^3 in the first factor together with the parameter direction

$$\partial_s\big(\zeta^s z_1, \zeta^{2s} z_2\big) = \tfrac{2\pi i}{7}\big(\zeta^s z_1,\ 2\zeta^{2s} z_2\big)$$

in the second factor (and the analogous direction $\partial_{s'}$ for A_4). The two parameter directions differ in their second coordinate by a factor of 2 coming from the $q = 2$ twisting, so their difference is a nonzero vector with nonzero component along the second coordinate and zero component along the first. Hence $TA_1 + TA_4 = T\widetilde{X}_0$ at the intersection point, and the intersection is transversal. □

1.7.2 The bounding cochains

For the Massey product computation, we need cochains that witness the vanishing of cup products of the a_k's.

Definition 1.7.2 (The submanifolds $D_{k,j}$)**.** For each pair (k, j) with $j - k \equiv 3 \pmod 7$, define the 5-dimensional submanifold $D_{k,j}$ of $S^3 \times S^3$ whose Poincaré dual serves as a bounding cochain for $a_k \cup a_j$. Geometrically, $D_{k,j}$ is constructed by "thickening" the intersection $A_k \cap A_j$.

The boundary of $D_{k,j}$ has three components:

(i) a component along $A_k \cap A_j$ at the interior intersection,

(ii) a component along Δ_{k-1},

(iii) a component along Δ_k.

The key geometric fact is:

Lemma 1.7.3. *The intersection $D_{k,k+3} \cap A_{k+1}$ is nonempty and represents a nontrivial homology class. Specifically*

$$[D_{k,k+3} \cap A_{k+1}] = a_{k+1} \cup \iota \quad \textit{in } H^5(\widetilde{X}_0),$$

where $\iota \in H^3(\widetilde{X}_0)$ is the generator dual to the sphere $S^3 \hookrightarrow S^3 \times S^3$, $x \mapsto (1, x)$.

Proof. By the cyclic $\mathbb{Z}/7$-symmetry it suffices to treat the case $k = 1$, that is, $D_{1,4} \cap A_2$. The bounding 3-manifold $D_{1,4}$ is parametrized as

$$D_{1,4} = \left\{ \big((r, x), (\zeta^{4t} r, \zeta^t x)\big) \ : \ (r, x) \in D^2,\ 0 \le t \le 1 \right\},$$

where $D^2 = \{(r, x) : 0 \le r \le 1,\ r^2 + |x|^2 = 1,\ x \in \mathbb{C}\}$. A point of $D_{1,4} \cap A_2$ must satisfy the system

$$\zeta^{4t} r = \zeta^{1+s} r, \qquad \zeta^t x = \zeta^{2+2s} x,$$

with $0 \le t \le 1$ and $0 \le s \le 1$. The second equation, read as a relation between exponents modulo 7, yields $t \equiv 2 + 2s \pmod 7$, which has no solution for $t \in [0, 1]$ unless $x = 0$. The constraint $x = 0$ then forces $r = 1$ from $r^2 + |x|^2 = 1$, and the first equation reduces to $4t \equiv 1 + s \pmod 7$, that is, $t = (1 + s)/4$. Hence

$$D_{1,4} \cap A_2 = \left\{ \big((1, 0), (\zeta^{1+s}, 0)\big) \ : \ 0 \le s \le 1 \right\},$$

which is a path connecting the diagonals Δ_1 and Δ_2. This path coincides with $A_2 \cap S^3$, where S^3 denotes the embedded sphere $\{(1, x) : x \in \mathbb{C},\ |x| = 1\} \subset S^3 \times S^3$ representing the generator $\iota \in H^3(\widetilde{X}_0)$.

Transversality at a point $((1, 0), (\zeta^{1+s}, 0)) = ((1, 0), (\zeta^{4t}, 0))$ is checked by exhibiting tangent vectors that span a six-dimensional space. The tangent space to A_2 at this point is generated by

$$\big((i, 0), (i\zeta^{1+s}, 0)\big),\ \big((0, 1), (0, \zeta^{2+2s})\big),\ \big((0, i), (0, i\zeta^{2+2s})\big),\ \big((0, 0), (i\zeta^{1+s}, 0)\big),$$

and the tangent space to $D_{1,4}$ at the same point is generated by

$$\big((0, 1), (0, \zeta^t)\big),\ \big((0, i), (0, i\zeta^t)\big),\ \big((0, 0), (i\zeta^{4t}, 0)\big).$$

A direct inspection shows that these seven vectors span a six-dimensional subspace, which is the tangent space to the ambient $\widetilde{X}_0$ at the point. Hence the intersection is transversal.

Under Poincaré duality, A_2 is dual to $a_2 \in H^2(\widetilde{X}_0)$ and the embedded sphere is dual to ι, so the transversal intersection $D_{1,4} \cap A_2 = A_2 \cap S^3$ is dual to the cup product $a_2 \cup \iota$. This class is nontrivial: in

$$H^5(\widetilde{X}_0) \;\cong\; \langle a_k \cup \iota \;:\; k = 0, \dots, 6\rangle \big/ \Big(\textstyle\sum_{k=0}^{6} a_k \cup \iota\Big),$$

the class $a_2 \cup \iota$ is independent of $a_4 \cup \iota$ and $(a_2 + a_6) \cup \iota$, which is precisely what is needed for the nontriviality of the Massey product $\langle a_4, a_1, a_2 + a_6\rangle$. The further fact that $D_{1,4} \cap A_6 = \emptyset$, also required for the Massey product calculation, follows from the same exponent analysis: the corresponding system has no solution with $0 \le t \le 1$ since the exponents in the two equations cannot be matched. □

1.7.3 The Massey product computation

Definition 1.7.4. Let $\alpha \in H^3(\widetilde{X}_0)$ denote the Poincaré dual of the fiber class, i.e., the generator of $H^3(\widetilde{X}_0) \cong \mathbb{Z}$ coming from the base S^3 of the fibration.

We now work with $\mathbb{Z}/2$ coefficients for the Massey product computation, to eliminate sign issues.

Theorem 1.7.5 (Longoni–Salvatore)**.** *For the lens space $L(7, 2)$, working with $\mathbb{Z}/2$ coefficients, the Massey product*

$$\langle a_{k+3}, a_k, a_{k+1} + a_{k+5}\rangle$$

contains the class $a_{k+1} \cup \alpha \in H^5(\widetilde{X}_0; \mathbb{Z}/2)$. This class is ***nonzero****.*

Proof. We first verify that the Massey product is defined. We need:

- $a_{k+3} \cup a_k = 0$: this holds because $H^4(\widetilde{X}_0) = 0$ (over $\mathbb{Z}$ and hence, by universal coefficients, also over $\mathbb{Z}/2$), so all cup products of degree-2 classes vanish.
- $a_k \cup (a_{k+1} + a_{k+5}) = 0$: same reason, by linearity.

Since all cup products in $H^2 \otimes H^2$ vanish, the bounding cochains Z and X in the Massey product definition correspond geometrically to the submanifolds $D_{k,k+3}$ and related objects. In particular:

- Choose Z to be the Poincaré dual of $D_{k,k+3}$ (which satisfies $dZ = \bar{a}_{k+3} \cup \bar{a}_k$).
- Choose X similarly for the pair $(a_k, a_{k+1} + a_{k+5})$.

The Massey product class is represented by

$$Z \cup (\bar{a}_{k+1} + \bar{a}_{k+5}) + \bar{a}_{k+3} \cup X.$$

After Poincaré duality, this corresponds to the intersection

$$D_{k,k+3} \cap (A_{k+1} \cup A_{k+5}) + A_{k+3} \cap D'_{k,k+1,k+5},$$

where D' denotes the bounding manifold for the second cup product. By Lemma 1.7.3, the contribution from the first term reduces to $D_{k,k+3} \cap A_{k+1}$, which gives the class $a_{k+1} \cup \alpha$. The intersection $D_{k,k+3} \cap A_{k+5}$ is empty, by the same exponent analysis used in Lemma 1.7.3: the corresponding system of equations (in the parameters t and s governing $D_{k,k+3}$ and A_{k+5}) has no solution with $0 \leq t \leq 1$, since the exponents arising from the second coordinate cannot be matched in this range.

The class $a_{k+1} \cup \alpha$ is nonzero in $H^5(\widetilde{X}_0; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{p-1}$ because:

- a_{k+1} is a nonzero element of H^2 (it is one of the generators).
- α is a generator of H^3.
- The cup product $H^2 \otimes H^3 \to H^5$ is an isomorphism by Lemma 1.4.2.

The indeterminacy of $\langle a_{k+3}, a_k, a_{k+1} + a_{k+5} \rangle$ is $a_{k+3} \cup H^3 + H^3 \cup (a_{k+1} + a_{k+5})$. Since $H^3 = \mathbb{Z} \cdot \alpha$ (or $\mathbb{Z}/2 \cdot \alpha$ over $\mathbb{Z}/2$), the indeterminacy is the $(\mathbb{Z}/2)$-span of

$$\{a_{k+3} \cup \alpha, \quad (a_{k+1} + a_{k+5}) \cup \alpha\}.$$

We must show that $a_{k+1} \cup \alpha$ does not lie in this span.

Working in the cyclotomic ring $R = (\mathbb{Z}/2)[t]/(1+t+\cdots+t^6)$ with the identification $a_{j+1} \leftrightarrow t^j$, and using the relation $\sum_{k=0}^{6} a_k \cup \alpha = 0$ in H^5, the question becomes whether

$$t^{k+1} \in \operatorname{span}_{\mathbb{Z}/2}\{t^{k+2},\ t^k + t^{k+4}\} \qquad \text{in } R,$$

where all exponents are read modulo 7. Reduce every monomial to the basis $\{1, t, t^2, t^3, t^4, t^5\}$ of R using the relation $t^6 = 1 + t + t^2 + t^3 + t^4 + t^5$. This reduction depends on which of the exponents k, $k+1$, $k+2$, $k+4$ (taken mod 7) equals 6. We treat the seven possible values of $k \in \mathbb{Z}/7$ in turn.

For each $k \in \{0, 1, \ldots, 6\}$, write the four exponents $k, k+1, k+2, k+4$ (mod 7), replace any occurrence of 6 by the expression $1 + t + t^2 + t^3 + t^4 + t^5$ (with t raised to the appropriate power), and check that the resulting linear system over $\mathbb{Z}/2$ has no solution. Concretely, in each case the candidate combination $\lambda t^{k+2} + \mu(t^k + t^{k+4})$ is a $\mathbb{Z}/2$-linear combination of two specific elements of R, while t^{k+1} is a third specific element. A direct enumeration shows that in every case, the three reduced expressions t^{k+1}, t^{k+2}, $t^k + t^{k+4}$ are linearly independent in the 6-dimensional $\mathbb{Z}/2$-vector space R. Hence $t^{k+1} \notin \operatorname{span}\{t^{k+2},\ t^k + t^{k+4}\}$ for every k.

Therefore $a_{k+1} \cup \alpha$ does not lie in the indeterminacy, and the Massey product is nontrivially represented by $a_{k+1} \cup \alpha$. □

Using the equivariance under the $\mathbb{Z}/7$-action, the full Massey product structure can be determined from the computation in Theorem 1.7.5 by the property:

$$\langle t^n x, t^n y, t^n z \rangle = t^n \cdot \langle x, y, z \rangle,$$

where t is the generator of the cyclotomic action.

1.8 Conclusion of the counterexample

Proof of Theorem 1.1.1. Suppose, for the sake of contradiction, that there exists a homotopy equivalence

$$f \colon F_2(L(7,2)) \xrightarrow{\simeq} F_2(L(7,1)).$$

Write $X_0 = F_2(L(7,1))$ and $X_0' = F_2(L(7,2))$.

Since f is a homotopy equivalence, it induces an isomorphism on fundamental groups:

$$f_*\colon \pi_1(X_0') \xrightarrow{\sim} \pi_1(X_0).$$

Both groups are $\mathbb{Z}/7 \oplus \mathbb{Z}/7$. By the theory of covering spaces, f lifts to a homotopy equivalence of universal covering spaces:

$$\widetilde{f}\colon \widetilde{X}_0' \xrightarrow{\simeq} \widetilde{X}_0.$$

The lift $\widetilde{f}$ is equivariant with respect to the deck transformation actions: for all $(m,n) \in \mathbb{Z}/7 \oplus \mathbb{Z}/7$,

$$\widetilde{f} \circ \tau'_{m,n} = \tau_{f_*(m,n)} \circ \widetilde{f}.$$

To see why the lift exists: a homotopy equivalence $f\colon X_0' \to X_0$ induces an isomorphism $f_*\colon \pi_1(X_0') \to \pi_1(X_0)$. The universality of the universal cover guarantees that the map f together with the isomorphism f_* on fundamental groups determines a unique (up to deck transformations) equivariant lift $\widetilde{f}$.

The homotopy equivalence $\widetilde{f}$ induces an isomorphism

$$\widetilde{f}^*\colon H^*(\widetilde{X}_0;\mathbb{Z}/2) \xrightarrow{\sim} H^*(\widetilde{X}_0';\mathbb{Z}/2).$$

This isomorphism is equivariant: it intertwines the $\pi_1(X_0)$-module structures on the two sides (up to the reparametrization given by f_*).

By the naturality of Massey products, for any classes $x, y, z \in H^2(\widetilde{X}_0;\mathbb{Z}/2)$:

$$\widetilde{f}^*\big(\langle x, y, z\rangle\big) = \langle \widetilde{f}^*(x), \widetilde{f}^*(y), \widetilde{f}^*(z)\rangle.$$

In particular, if a Massey product vanishes on $\widetilde{X}_0$, then the corresponding Massey product (via $\widetilde{f}^*$) must also vanish on $\widetilde{X}_0'$. Contrapositively, if a Massey product is nonzero on $\widetilde{X}_0'$, then the image Massey product on $\widetilde{X}_0$ must also be nonzero.

By Theorem 1.6.4, **all** Massey products in $H^*(\widetilde{X}_0;\mathbb{Z}/2)$ vanish (the universal cover of $F_2(L(7,1))$ is formal, being homotopy equivalent to $(\bigvee_6 S^2) \times S^3$). By naturality (Step 3), the pullback of any Massey product from $\widetilde{X}_0$ to $\widetilde{X}_0'$ must vanish. But more importantly, since $\widetilde{f}^*$ is an isomorphism, the *inverse* map $(\widetilde{f}^*)^{-1}$ sends nonzero Massey products on $\widetilde{X}_0'$ to nonzero Massey products on $\widetilde{X}_0$. By Theorem 1.7.5, the Massey product $\langle a_{k+3}, a_k, a_{k+1} + a_{k+5}\rangle$ in $H^*(\widetilde{X}_0';\mathbb{Z}/2)$ is **nonzero**. Its image under $(\widetilde{f}^*)^{-1}$ would be a nonzero Massey product on $\widetilde{X}_0$.

This contradicts the vanishing of all Massey products on $\widetilde{X}_0$. Therefore, no such homotopy equivalence f can exist, and we conclude

$$F_2\big(L(7,1)\big) \not\simeq F_2\big(L(7,2)\big).$$ □

Corollary 1.8.1. *For all $n \geq 2$, the configuration spaces $F_n(L(7,1))$ and $F_n(L(7,2))$ are not homotopy equivalent.*

Proof. The Fadell–Neuwirth theorem [FN62] states that the forgetful map

$$F_n(M) \to F_2(M), \qquad (x_1, \dots, x_n) \mapsto (x_1, x_2),$$

is a fiber bundle for any manifold M of dimension ≥ 2. If there existed a homotopy equivalence $F_n(L(7,1)) \simeq F_n(L(7,2))$, then by considering the long exact sequences of homotopy groups associated to the two fiber bundles and using induction, one could construct a homotopy equivalence $F_2(L(7,1)) \simeq F_2(L(7,2))$, contradicting Theorem 1.1.1. □

The counterexample of Longoni and Salvatore disprove the homotopy invariance conjecture.
However, the following conjecture remains open:

> **Conjecture** (Simply-connected case). If M and N are *simply connected* closed manifolds of the same homotopy type, then $F_n(M) \simeq F_n(N)$ for all n.

1.8.1 What survives of homotopy invariance

The Longoni–Salvatore counterexample shows that the homotopy invariance conjecture, in its strongest form, is false. We are led to ask which restricted forms of the conjecture do hold, and under what hypotheses on M the homotopy type of $F_n(M)$ is determined by that of M. Several positive results are known.

(i) **Real homotopy invariance.** For a smooth, simply connected, closed manifold M of dimension $n \geq 4$, the real homotopy type of $F_n(M)$ is determined by the real homotopy type of M. This is the theorem of Idrissi [Idr22], which forms the subject of Chapter 3. Both hypotheses are essential: simple connectedness rules out the Longoni–Salvatore phenomenon at the level of π_1, and the dimension bound enters through the partition function vanishing of Section 3.8.

(ii) **Manifolds with boundary.** For a compact manifold M with nonempty boundary, $F_n(M)$ is homotopy equivalent to $F_n(\operatorname{Int} M)$, since one can push points away from the boundary by an isotopy. A real homotopy invariance result analogous to (i) has been established by Campos–Idrissi–Lambrechts–Willwacher [CILW24] for simply connected compact manifolds with simply connected boundary, in dimension at least 4.

(iii) **Euclidean spaces.** The homotopy type of $F_n(\mathbb{R}^d)$ is completely understood: it is a $K(\pi, 1)$ for $d = 2$ with π the pure braid group, and highly connected with explicit cohomology ring for $d \geq 3$. The dependence is on d alone, as one would expect of the strongest possible homotopy invariance.

2

Commutative Differential Graded Algebras and Rational Homotopy Theory

This chapter collects the algebraic background in rational and real homotopy theory that the rest of the thesis needs. We work throughout over a field of characteristic zero, recall the basic language of commutative differential graded algebras and their homotopy theory, and describe Sullivan's minimal models together with the Bousfield–Gugenheim model structure on CDGAs. Two themes are given particular attention, because they are the ones used in later chapters: formality, which underlies the Longoni–Salvatore argument of Chapter 1, and Poincaré duality CDGAs together with the Lambrechts–Stanley model, which form the algebraic input to Idrissi's theorem (Chapter 3). The exposition is standard; our main references are Félix–Halperin–Thomas [FHT01], Bousfield–Gugenheim [BG76], Sullivan [Sul77], and Griffiths–Morgan [GM81].

2.1 Introduction to the graded setting

Throughout this chapter, $\Bbbk$ denotes a field of characteristic zero. In practice, $\Bbbk = \mathbb{Q}$ (rational homotopy theory) or $\Bbbk = \mathbb{R}$ (real homotopy theory, which we'll use for Idrissi's work).

2.1.1 Graded vector spaces

Definition 2.1.1 (Graded vector space)**.** A *graded vector space* (or simply *graded space*) over $\Bbbk$ is a family $V = \{V^n\}_{n\in\mathbb{Z}}$ of $\Bbbk$-vector spaces indexed by the integers. An element $v \in V^n$ is said to have *degree* $|v| = n$. We write $V^{\geq 0}$ for the subspace concentrated in nonnegative degrees, and similarly $V^{\geq k}$ for the subspace with $V^n = 0$ for $n < k$.

The graded space V is said to be of *finite type* if each V^n is finite-dimensional. It is *bounded below* (or *connective*) if $V^n = 0$ for n sufficiently negative.

Definition 2.1.2 (Degree shift)**.** Given a graded vector space V and an integer k, the *shifted* graded space $V[k]$ is defined by $(V[k])^n = V^{n+k}$.

Definition 2.1.3 (Tensor product)**.** The tensor product of two graded spaces V and W is the graded space $(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes_{\Bbbk} W^q$.

2.1.2 Graded algebras

Definition 2.1.4 (Graded algebra)**.** A *graded algebra* over $\Bbbk$ is a graded vector space A equipped with an associative multiplication $\mu\colon A \otimes A \to A$ of degree 0 (i.e., $A^p \cdot A^q \subset A^{p+q}$) and a unit $\eta\colon \Bbbk \to A^0$.

Definition 2.1.5 (Graded commutativity)**.** A graded algebra A is *graded commutative* (or *commutative* in the graded sense) if for all homogeneous elements $a, b \in A$,

$$a \cdot b = (-1)^{|a|\cdot|b|}\, b \cdot a.$$

This is the *Koszul sign rule*: transposing two elements of degrees p and q introduces a sign $(-1)^{pq}$.

Example 2.1.6 (The exterior algebra)**.** If V is concentrated in odd degrees, then ΛV (the free graded-commutative algebra on V) is an exterior algebra: $v^2 = 0$ for all $v \in V$ of odd degree, since $v \cdot v = (-1)^{|v|^2} v \cdot v = -v \cdot v$.

Example 2.1.7 (The polynomial algebra)**.** If V is concentrated in even degrees, then ΛV is a polynomial algebra: generators commute since $(-1)^{|v|\cdot|w|} = +1$ for even-degree elements.

Example 2.1.8 (Mixed case)**.** In general, for a graded space $V = V^{\text{even}} \oplus V^{\text{odd}}$, the free graded-commutative algebra is

$$\Lambda V \;=\; \Bbbk[V^{\text{even}}] \otimes \bigwedge V^{\text{odd}},$$

where the first factor is a polynomial algebra and the second is an exterior algebra.

2.1.3 Connectedness and augmentation

Definition 2.1.9. A graded algebra A is *connected* if $A^0 = \Bbbk$ (and $A^n = 0$ for $n < 0$). It is *simply connected* if in addition $A^1 = 0$.

An *augmentation* of a graded algebra A is an algebra morphism $\varepsilon\colon A \to \Bbbk$. The kernel $\bar{A} = \ker \varepsilon$ is the *augmentation ideal.* For a connected algebra, the augmentation is unique and $\bar{A} = A^{\geq 1}$.

2.2 Differential Graded Algebras

2.2.1 Cochain complexes

Definition 2.2.1 (Cochain complex)**.** A *cochain complex* over $\Bbbk$ is a graded vector space V equipped with a *differential* $d\colon V \to V$ of degree $+1$ satisfying $d^2 = 0$. The *cohomology* of (V, d) is

$$H^n(V, d) = \frac{\ker(d\colon V^n \to V^{n+1})}{\operatorname{im}(d\colon V^{n-1} \to V^n)}.$$

A morphism $f\colon (V, d) \to (W, d)$ of cochain complexes is a degree-0 linear map commuting with d. It is a *quasi-isomorphism* if the induced map $H^*(f)\colon H^*(V) \to H^*(W)$ is an isomorphism.

We denote by $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ the category of nonnegatively graded cochain complexes over $\Bbbk$.

2.2.2 Definition of a CDGA

Definition 2.2.2 (Commutative differential graded algebra)**.** A *commutative differential graded algebra* (CDGA) over $\Bbbk$ is a triple (A, d, μ) where:

(i) (A, μ) is a graded-commutative unital algebra over $\Bbbk$;

(ii) (A, d) is a cochain complex (so d has degree $+1$ and $d^2 = 0$);

(iii) the differential d satisfies the *graded Leibniz rule*: for all homogeneous $a, b \in A$,

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|}\, a \cdot (db).$$

In other words, d is a derivation of degree $+1$. A morphism of CDGAs is a degree-0 algebra map that commutes with the differentials. We denote the category of CDGAs over $\Bbbk$ by $\mathsf{CDGA}_{\Bbbk}$.

Example 2.2.3 (Cohomology ring)**.** For any topological space X, the singular cohomology $H^*(X; \Bbbk)$ is a CDGA with $d = 0$ and the cup product. This is the simplest CDGA one can associate to X, but it loses all higher-order homotopical information (such as Massey products).

Example 2.2.4 (De Rham complex)**.** For a smooth manifold M, the de Rham complex $(\Omega^*_{\mathrm{dR}}(M), d, \wedge)$ is a CDGA over $\mathbb{R}$. The algebra structure is given by the wedge product, the differential is the exterior derivative, and graded commutativity follows from $\alpha\wedge\beta = (-1)^{pq}\,\beta \wedge \alpha$ for p- and q-forms. De Rham's theorem provides a quasi-isomorphism $H^*(\Omega^*_{\mathrm{dR}}(M)) \cong H^*(M; \mathbb{R})$.

Example 2.2.5 (Polynomial forms)**.** Sullivan introduced the functor of *piecewise polynomial differential forms* $A_{\mathrm{PL}} \colon \mathsf{sSet} \to \mathsf{CDGA}_{\mathbb{Q}}$, which assigns to each simplicial set a CDGA of "polynomial de Rham forms" with rational coefficients. For a topological space X, one applies A_{PL} to the singular simplicial set $\mathrm{Sing}(X)$. We will come back to this in Section 2.6.

Example 2.2.6 (Free CDGAs)**.** Given a graded vector space V, the *free CDGA on* V is $(\Lambda V, d)$ where ΛV is the free graded-commutative algebra on V and d is specified by its values on generators (and extended by the Leibniz rule). The data of a free CDGA is thus a graded space V together with a degree-$+1$ linear map $d \colon V \to \Lambda V$ satisfying $d^2 = 0$ (checking $d^2 = 0$ only on generators suffices, since d is a derivation).

2.2.3 Homotopy category of CDGA

Definition 2.2.7. A morphism $f \colon (A, d) \to (B, d)$ of CDGAs is a *quasi-isomorphism* (written $f \colon A \xrightarrow{\sim} B$) if the induced map $H^*(f) \colon H^*(A) \to H^*(B)$ is an isomorphism of graded algebras.

Two CDGAs A and B are *weakly equivalent* (or have the same *rational/real homotopy type*) if they can be connected by a zigzag of quasi-isomorphisms:

$$A \xleftarrow{\sim} C_1 \xrightarrow{\sim} C_2 \xleftarrow{\sim} \cdots \xrightarrow{\sim} B.$$

2.3 Definition of Model categories

Before describing the model structure on $\mathsf{CDGA}_{\Bbbk}$, we recall the axioms of Quillen's model categories.

Definition 2.3.1 (Model category)**.** A *model category* is a category $\mathcal{C}$ equipped with three distinguished classes of morphisms—*weak equivalences*, *fibrations*, and *cofibrations*—satisfying the following axioms:

MC1. Limits and colimits. $\mathcal{C}$ has all finite limits and colimits.

MC2. Two-out-of-three. If f and g are composable morphisms and two of f, g, $g \circ f$ are weak equivalences, then so is the third.

MC3. Retracts. Each of the three classes is closed under retracts.

MC4. Lifting. Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

a lift h exists if either (a) i is an acyclic cofibration (= cofibration and weak equivalence) and p is a fibration, or (b) i is a cofibration and p is an acyclic fibration.

MC5. Factorization. Every morphism f can be factored as:

(a) $f = p \circ i$, where i is a cofibration and p is an acyclic fibration;

(b) $f = p \circ i$, where i is an acyclic cofibration and p is a fibration.

In most modern treatments, the factorization axiom **MC5** is strengthened to require that both factorizations be *functorial* in f, that is, given by functors on the arrow category of $\mathcal{C}$. The functoriality is automatic in the cofibrantly generated setting that covers all examples relevant to this thesis, including the Bousfield–Gugenheim model structure on CDGAs.

2.3.1 Cofibrant and fibrant objects

Definition 2.3.2. An object $X \in \mathcal{C}$ is *cofibrant* if the unique map $\varnothing \to X$ (from the initial object) is a cofibration. It is *fibrant* if the unique map $X \to *$ (to the terminal object) is a fibration.

A *cofibrant replacement* of X is a cofibrant object QX together with an acyclic fibration $QX \xrightarrow{\sim} X$. A *fibrant replacement* is defined dually.

2.3.2 The homotopy category

Definition 2.3.3. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is obtained by formally inverting all weak equivalences. Concretely, for cofibrant-fibrant objects X, Y, the morphisms in $\mathrm{Ho}(\mathcal{C})$ are the homotopy classes $[X, Y]$ of maps.

2.3.3 Quillen adjunctions and equivalences

Definition 2.3.4. A *Quillen adjunction* between model categories $\mathcal{C}$ and $\mathcal{D}$ is an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ where F preserves cofibrations and acyclic cofibrations (equivalently, G preserves fibrations and acyclic fibrations).

A Quillen adjunction is a *Quillen equivalence* if the derived adjunction $\mathbb{L}F\colon \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D})\colon \mathbb{R}G$ is an equivalence of categories.

The fundamental theorem of rational homotopy theory takes the form of a (partial) Quillen equivalence between simplicial sets and CDGAs.

2.4 Bousfield-Gugenheim Model Structure on CDGAs

Remark 2.4.1 (Standing convention). For the rest of this section we assume that all CDGAs are *nonnegatively graded*: $A^n = 0$ for $n < 0$. The model category we describe is therefore the one on $\mathsf{CDGA}_{\Bbbk}^{\geq 0}$. The same theory can be set up on unbounded CDGAs, but the cleaner classical statement of Bousfield–Gugenheim, and the construction of Sullivan models by induction on degree, both require boundedness from below. In the applications throughout the rest of the thesis the CDGAs that arise (cohomology rings, de Rham complexes, $A_{\mathrm{PL}}(X)$, Lambrechts–Stanley models) are all nonnegatively graded, so this assumption costs us nothing.

2.4.1 Statement of the theorem

Theorem 2.4.2 (Bousfield–Gugenheim [BG76])**.** *The category $\mathsf{CDGA}_{\Bbbk}^{\geq 0}$ of commutative differential graded algebras over a field $\Bbbk$ of characteristic zero admits a model category structure where:*

(i) ***Weak equivalences*** *are the quasi-isomorphisms.*

(ii) ***Fibrations*** *are the degreewise surjections (i.e., morphisms $f\colon A \to B$ such that $f^n\colon A^n \to B^n$ is surjective for all $n \geq 0$).*

(iii) ***Cofibrations*** *are determined by the lifting property: $i\colon A \to B$ is a cofibration if and only if it has the left lifting property with respect to all acyclic fibrations.*

Moreover, every object of $\mathsf{CDGA}_{\Bbbk}$ is fibrant.

Remark 2.4.3. The fact that every object is fibrant (since every CDGA maps surjectively to the terminal object $\Bbbk$ in each positive degree trivially) is a significant simplification: it means that cofibrant replacements are the only "resolution" one ever needs.

2.4.2 Generating cofibrations

The model structure is *cofibrantly generated.* The generating cofibrations and generating acyclic cofibrations are obtained by applying the free CDGA functor Λ to the standard generating (acyclic) cofibrations of cochain complexes.

Definition 2.4.4 (Standard spheres and disks)**.** For $n \geq 0$, define the following cochain complexes:

(i) $S^{n-1} = (\Bbbk$ concentrated in degree $n-1$, $d = 0)$ which we will call a sphere.

(ii) $D^n = (\Bbbk \xrightarrow{\text{id}} \Bbbk$ in degrees $n-1$ and n, $d = \text{id})$ which we will call a disk.

The inclusion $S^{n-1} \hookrightarrow D^n$ is the standard generating cofibration in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. The map $0 \to D^n$ is the standard generating acyclic cofibration.

Applying the free CDGA functor Λ:

(i) The generating cofibrations of $\mathsf{CDGA}_{\Bbbk}$ are the maps $\Lambda S^{n-1} \to \Lambda D^n$, which algebraically amount to "attaching a cell in degree n".

(ii) The generating acyclic cofibrations are the maps $\Bbbk \to \Lambda D^n$, which amount to "adjoining an acyclic pair" (a generator x in degree $n-1$ and a generator y in degree n with $dy = x$).

Proposition 2.4.5. *A morphism $i\colon A \to B$ in $\mathsf{CDGA}_{\Bbbk}$ is a cofibration if and only if it is a retract of a* relative Sullivan algebra*: a morphism of the form $A \hookrightarrow A \otimes \Lambda V = (A \otimes \Lambda V, d)$, where V is a graded vector space admitting a filtration*

$$0 = V(0) \subset V(1) \subset V(2) \subset \cdots \subset \bigcup_{k \geq 0} V(k) = V$$

such that $d(V(k)) \subset A \otimes \Lambda(V(k-1))$ for all k.

In particular, the cofibrant objects of $\mathsf{CDGA}_{\Bbbk}$ are precisely the retracts of Sullivan algebras (taking $A = \Bbbk$).

2.5 Sullivan Algebras and Minimal Models

2.5.1 Sullivan algebras

Definition 2.5.1 (Sullivan algebra)**.** A *Sullivan algebra* is a CDGA of the form $(\Lambda V, d)$ where:

(i) $V = \bigoplus_{n \geq 1} V^n$ is a graded vector space concentrated in positive degrees;

(ii) V admits a filtration $V(0) \subset V(1) \subset \cdots$ with $\bigcup V(k) = V$ such that $d(V(0)) = 0$ and $d(V(k)) \subset \Lambda(V(k-1))$ for $k \geq 1$.

A Sullivan algebra is *minimal* if in addition $d(V) \subset \Lambda^{\geq 2} V$.

Remark 2.5.2. As noted in Proposition 2.4.5, Sullivan algebras are precisely the cofibrant objects of the Bousfield–Gugenheim model structure (up to retract). The minimality condition is an additional "normalization" that makes Sullivan algebras unique up to isomorphism (not just quasi-isomorphism).

2.5.2 Minimal models

Definition 2.5.3 (Minimal model)**.** A *minimal model* of a CDGA (A, d) is a minimal Sullivan algebra $(\Lambda V, d)$ together with a quasi-isomorphism $\varphi\colon (\Lambda V, d) \xrightarrow{\sim} (A, d)$.

More generally, a *Sullivan model* of (A, d) is any Sullivan algebra with a quasi-isomorphism to (A, d) (not necessarily minimal).

Theorem 2.5.4 (Existence and uniqueness of minimal models)**.** *Let (A, d) be a connected CDGA (i.e., $H^0(A) = \Bbbk$) of finite type. Then:*

(i) (A, d) admits a minimal model $(\Lambda V, d) \xrightarrow{\sim} (A, d)$.

(ii) Any two minimal models are isomorphic (as CDGAs).

Proof. We follow [FHT01, §12], where the proof is given in full detail. We treat the existence and uniqueness statements separately.

Existence. The minimal model is built by induction on the degree of the generators. We construct an increasing sequence of Sullivan subalgebras

$$(\Lambda V(0), d) \subset (\Lambda V(1), d) \subset (\Lambda V(2), d) \subset \cdots$$

together with quasi isomorphisms up to degree k

$$\varphi_k \colon (\Lambda V(k), d) \longrightarrow (A, d)$$

satisfying two conditions: φ_k induces an isomorphism on H^i for $i \leq k$ and an injection on H^{k+1}, and the generators of V in degree $\leq k$ are precisely those in $V(k)$. The minimal model is then the colimit $(\Lambda V, d) = \bigcup_k (\Lambda V(k), d)$ with the induced map $\varphi \colon \Lambda V \to A$.

The base case $k = 0$ is immediate from connectivity: $V(0) = 0$ and $\varphi_0 \colon \Bbbk \hookrightarrow A$ is the unit, which induces an isomorphism on H^0.

For the inductive step, assume φ_k has been constructed. We must adjoin generators in degree $k+1$ to fix two failures: classes in $H^{k+1}(A)$ not yet in the image of φ_k (the *cokernel* on H^{k+1}), and classes in $H^{k+2}(\Lambda V(k))$ mapping to zero in $H^{k+2}(A)$ (the *kernel* on H^{k+2}). Choose a graded vector space W of degree $k+1$ together with:

- a basis of W' mapping bijectively under $\Lambda V(k) \otimes \Lambda W' \to A$ to a complement of $\operatorname{im}(\varphi_k^*) \subset H^{k+1}(A)$, with $dw' = 0$ for $w' \in W'$ and $\varphi_{k+1}(w')$ a chosen cocycle representative of the corresponding cohomology class;
- a basis of W'' in bijection with a basis of $\ker \varphi_k^* \subset H^{k+2}(\Lambda V(k))$, with $dw'' \in \Lambda V(k)$ a chosen cocycle representative of the corresponding class, and $\varphi_{k+1}(w'') \in A$ a chosen primitive of $\varphi_k(dw'')$, which exists because $\varphi_k(dw'')$ is exact in A by construction.

Set $V(k+1) = V(k) \oplus W' \oplus W''$. The resulting CDGA $(\Lambda V(k+1), d)$ extends $(\Lambda V(k), d)$, and a routine verification shows that φ_{k+1} induces an isomorphism on H^i for $i \leq k+1$ and an injection on H^{k+2}.

The minimality condition $d(V) \subset \Lambda^{\geq 2} V$ is maintained throughout: in the base case it holds vacuously, and at each inductive step the chosen primitives dw' and dw'' can be modified by elements of $\Lambda^{\geq 2} V(k)$ to remove any linear part, since the linear part lives in $V(k)$ and represents a cohomology class that is already accounted for at an earlier stage. This last point is the only nontrivial calculation; it is carried out in [FHT01, Prop. 12.2].

The colimit $(\Lambda V, d) = \bigcup_k (\Lambda V(k), d)$ with $\varphi = \bigcup_k \varphi_k$ is then a minimal Sullivan model of (A, d).

Uniqueness. Suppose $\varphi_1 \colon (\Lambda V_1, d) \to (A, d)$ and $\varphi_2 \colon (\Lambda V_2, d) \to (A, d)$ are two minimal models. By the lifting property of cofibrations in the model category of CDGAs (or [FHT01, Prop. 12.9]), the quasi-isomorphism φ_2 admits a lift through φ_1, producing a CDGA morphism $\psi\colon (\Lambda V_1, d) \to (\Lambda V_2, d)$ such that $\varphi_2 \circ \psi$ is homotopic to φ_1. Since φ_1 and φ_2 are quasi-isomorphisms, the two-out-of-three property implies ψ is a quasi-isomorphism.

It remains to show that any quasi-isomorphism $\psi\colon (\Lambda V_1, d) \to (\Lambda V_2, d)$ between minimal Sullivan algebras is an isomorphism. The argument is made degree by degree: at each degree k, the induced map $\psi^k\colon V_1^k \to V_2^k$ on generators fits into a commutative diagram with the cohomology and the indecomposables, and the minimality condition $d(V_i) \subset \Lambda^{\geq 2} V_i$ ensures that ψ^k is determined by its behavior on cohomology up to terms of higher word length. Since ψ is a quasi-isomorphism, ψ^k is a bijection on the relevant cohomology level, and an inductive argument on degree shows that ψ^k is itself a bijection. Hence ψ is an isomorphism. The full verification is in [FHT01, §12(c)]. □

2.5.3 The Lifting Lemma

Proposition 2.5.5 (Lifting Lemma)**.** *Let $(\Lambda V, d)$ be a Sullivan algebra and let $p\colon (B, d) \twoheadrightarrow (C, d)$ be a surjective quasi-isomorphism of CDGAs. Then for any CDGA map $f\colon (\Lambda V, d) \to (C, d)$, there exists a lift $\tilde{f}\colon (\Lambda V, d) \to (B, d)$ with $p \circ \tilde{f} = f$:*

$$\begin{tikzcd} & (B,d) \arrow[d, two heads, "p", "\sim"'] \\ (\Lambda V, d) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (C,d) \end{tikzcd}$$

Proof idea. The lift is constructed inductively on the filtration $V(k)$. At each stage, one needs to lift the image of one generator $v \in V(k)$. Since p is surjective, one can choose a preimage of $f(v)$. The condition $d\tilde{f}(v) = \tilde{f}(dv)$ (which is already determined by earlier choices) and the acyclicity of $\ker(p)$ ensure that the obstruction vanishes and a consistent choice exists. This is the algebraic analogue of the homotopy extension property for CW complexes. □

In the language of model categories, the Lifting Lemma is exactly the statement that Sullivan algebras are cofibrant and surjective quasi-isomorphisms are acyclic fibrations.

2.5.4 Explicit computations of minimal models

Example 2.5.6 (Spheres)**. Odd spheres.** The minimal model of S^{2k+1} (for $k \geq 1$) is $(\Lambda(x), d = 0)$ with $|x| = 2k+1$. The quasi-isomorphism $\Lambda(x) \to \Omega^*_{\mathrm{dR}}(S^{2k+1})$ sends x to the volume form. The cohomology is $H^*(\Lambda(x)) = \Bbbk \oplus \Bbbk \cdot x$, matching $H^*(S^{2k+1}; \Bbbk)$.

Even spheres. The minimal model of S^{2k} (for $k \geq 1$) is $(\Lambda(x, y), d)$ with $|x| = 2k$, $|y| = 4k - 1$, $dx = 0$, and $dy = x^2$. Here x represents the volume form, and the new generator y is needed to kill x^2 in cohomology (since $H^{4k}(S^{2k}) = 0$ but $x^2 \neq 0$ in $\Lambda(x)$).

Example 2.5.7 (Complex projective space)**.** The minimal model of $\mathbb{C}P^n$ is $(\Lambda(x, y), d)$ with $|x| = 2$, $|y| = 2n+1$, $dx = 0$, and $dy = x^{n+1}$. In cohomology, $x^{n+1} = 0$ (since $H^{2n+2}(\mathbb{C}P^n) = 0$), so y is needed to kill x^{n+1}. The cohomology $H^*(\Lambda(x, y), d) \cong \Bbbk[x]/(x^{n+1})$ reproduces $H^*(\mathbb{C}P^n; \Bbbk)$.

Example 2.5.8 (Products)**.** If $(\Lambda V, d)$ is a minimal model for X and $(\Lambda W, d)$ is a minimal model for Y, then $(\Lambda(V \oplus W), d)$ (with the product differential) is a minimal model for $X \times Y$.

Example 2.5.9 (Wedges of spheres)**.** We compute the minimal Sullivan model of the wedge $S^2 \vee S^2$. The cohomology ring is

$$H^*(S^2 \vee S^2; \mathbb{Q}) \;=\; \mathbb{Q}\langle 1, x_1, x_2\rangle, \qquad |x_1| = |x_2| = 2,$$

with all products of positive-degree classes equal to zero: in particular $x_1^2 = x_2^2 = x_1x_2 = 0$. The minimal model must realise this vanishing through generators in higher degrees whose differentials kill the unwanted products in $\Lambda(x_1, x_2)$.

At degree 3, the products x_1^2, x_1x_2, x_2^2 are nonzero in the free graded-commutative algebra $\Lambda(x_1, x_2)$ but must vanish in cohomology. We therefore introduce generators y_{11}, y_{12}, y_{22} of degree 3 with

$$dy_{11} = x_1^2, \qquad dy_{12} = x_1x_2, \qquad dy_{22} = x_2^2.$$

At the next stage, new products such as x_1y_{12}, x_2y_{12}, $y_{11}y_{12}$ become nonzero closed elements that need to be killed: each such class is the algebraic incarnation of a triple Massey product on the wedge, and each requires its own degree-5 (or higher) generator z with dz equal to the offending product. The pattern continues: at every stage one introduces generators killing the new closed elements that have appeared, indexed by multi-indices recording which products are being neutralised. The result is a free graded algebra

$$\big(\Lambda(x_1, x_2,\; y_{11}, y_{12}, y_{22},\; z_{111}, z_{112}, z_{122}, z_{222}, \ldots),\; d\big)$$

with infinitely many generators in increasing degrees, the differential of each new generator killing a specific previously unkilled closed element.

The fact that this iterative procedure terminates only at infinity reflects the fact that $S^2 \vee S^2$, although formal, has a minimal model that is not finitely generated. This is in sharp contrast to $S^2 \times S^2$, whose minimal model $\big(\Lambda(x_1, x_2, y_1, y_2),\, dy_i = x_i^2\big)$ has only four generators. The proliferation of generators in the wedge case is the rational-homotopical signature of the free Lie algebra structure on $\pi_*(\Omega(S^2 \vee S^2)) \otimes \mathbb{Q}$, which is the free graded Lie algebra on two generators in degree 1 by the theorem of Milnor and Moore; the standard reference is [FHT01, §24] for the general construction of minimal models of wedges and [FHT01, §13(e)] for the connection with free Lie algebras.

2.5.5 Rational homotopy groups via Sullivan models

Theorem 2.5.10 (Sullivan)**.** *Let X be a simply connected space of finite type, and let $(\Lambda V, d)$ be its minimal model. Then there is a natural isomorphism*

$$V^n \cong \mathrm{Hom}(\pi_n(X), \Bbbk) = \pi_n(X) \otimes \Bbbk$$

for all $n \geq 2$. In other words, the generators of the minimal model in degree n are dual to the rational (or real) homotopy groups in degree n.

Thus the minimal model simultaneously encodes:

- the rational cohomology ring (as the cohomology of ΛV);
- the rational homotopy groups (as the generators V);
- the Whitehead products (as the quadratic part of d);

Corollary 2.5.11 (Rational homotopy groups of spheres)**.** *From Example 2.5.6:*

(i) $\pi_n(S^{2k+1}) \otimes \mathbb{Q} = \mathbb{Q}$ *if* $n = 2k+1$, *and* 0 *otherwise.*

(ii) $\pi_n(S^{2k}) \otimes \mathbb{Q} = \mathbb{Q}$ *if* $n = 2k$ *or* $n = 4k-1$, *and* 0 *otherwise.*

2.6 Sullivan's Functor and the De Rham Theorem

2.6.1 The polynomial de Rham functor A_{PL}

Definition 2.6.1 (Polynomial forms on a simplex)**.** For each $n \geq 0$, define the CDGA of *polynomial differential forms on the standard n-simplex* as

$$(\Omega_n^*)_{\mathbb{Q}} = \frac{\mathbb{Q}[t_0, t_1, \ldots, t_n, dt_0, dt_1, \ldots, dt_n]}{(t_0 + t_1 + \cdots + t_n - 1,\ dt_0 + dt_1 + \cdots + dt_n)},$$

where $|t_i| = 0$ and $|dt_i| = 1$. These form a simplicial CDGA.

Definition 2.6.2 (PL forms on a simplicial set)**.** For a simplicial set K, the CDGA of *PL (piecewise linear) forms* is

$$A_{\mathrm{PL}}(K) = \mathsf{CDGA}_{\mathbb{Q}}(K, \Omega_\bullet^*),$$

the simplicial hom-set. Concretely, an element of $A_{\mathrm{PL}}(K)$ assigns to each n-simplex σ of K a polynomial form on Δ^n, compatibly with face and degeneracy maps. For a topological space X, one sets $A_{\mathrm{PL}}(X) = A_{\mathrm{PL}}(\mathrm{Sing}(X))$.

Theorem 2.6.3. *For a smooth manifold M, there is a natural zigzag of quasi-isomorphisms*

$$A_{\mathrm{PL}}(M) \otimes_{\mathbb{Q}} \mathbb{R} \xleftarrow{\sim} (\cdots) \xrightarrow{\sim} \Omega^*_{\mathrm{dR}}(M).$$

In particular, $A_{\mathrm{PL}}(M)$ computes $H^*(M;\mathbb{Q})$ and $\Omega^*_{\mathrm{dR}}(M)$ computes $H^*(M;\mathbb{R})$.

2.6.2 The Sullivan–de Rham adjunction

Definition 2.6.4 (Spatial realization)**.** For a CDGA A, its *spatial realization* (or *simplicial realization*) is the simplicial set $\langle A \rangle$ defined by

$$\langle A \rangle_n = \mathrm{Hom}_{\mathsf{CDGA}}(A, \Omega_n^*).$$

Theorem 2.6.5 (Sullivan–de Rham, Bousfield–Gugenheim)**.** *The functors*

$$A_{\mathrm{PL}} \colon \mathsf{sSet} \rightleftarrows \mathsf{CDGA}_{\mathbb{Q}}^{\mathrm{op}} \colon \langle - \rangle$$

form a Quillen adjunction. Moreover, this adjunction restricts to an equivalence of homotopy categories

$$\mathrm{Ho}(\mathsf{CDGA}_{\mathbb{Q}}^{\mathrm{op}})^{\mathrm{conn.,\,nilp.}}_{\mathrm{fin.\,type}} \ \simeq\ \mathrm{Ho}(\mathsf{sSet})^{\mathrm{nilp.,\,}\mathbb{Q}}_{\mathrm{fin.\,}\mathbb{Q}\text{-}type}$$

between the homotopy category of connected, cohomologically nilpotent CDGAs of finite type and the homotopy category of nilpotent rational simplicial sets of finite $\mathbb{Q}$-type.

2.7 Homotopy Theory of CDGAs

2.7.1 Homotopy of CDGA morphisms

Definition 2.7.1 (Path object)**.** The *path object* (or *cylinder object*) in $\mathsf{CDGA}_{\Bbbk}$ is the CDGA $\Bbbk[t, dt] = \Lambda(t, dt)$ with $|t| = 0$ and $|dt| = 1$, $d(t) = dt$. There are evaluation maps $\varepsilon_0, \varepsilon_1 \colon \Bbbk[t, dt] \to \Bbbk$ given by $t \mapsto 0$ and $t \mapsto 1$, respectively.

Definition 2.7.2 (Homotopy of CDGA maps)**.** Two CDGA morphisms $f, g\colon (\Lambda V, d) \to (A, d)$ from a Sullivan algebra are *homotopic* (written $f \simeq g$) if there exists a CDGA morphism

$$H\colon (\Lambda V, d) \to (A, d) \otimes \Bbbk[t, dt]$$

such that $(\mathrm{id}_A \otimes \varepsilon_0) \circ H = f$ and $(\mathrm{id}_A \otimes \varepsilon_1) \circ H = g$.

Proposition 2.7.3. *When the source is a Sullivan algebra and the target is any CDGA, homotopy is an equivalence relation on the set of CDGA morphisms. Moreover, quasi-isomorphic targets give isomorphic homotopy classes: if $A \xrightarrow{\sim} B$ is a quasi-isomorphism and ΛV is Sullivan, then $[\Lambda V, A] \cong [\Lambda V, B]$.*

2.7.2 The homotopy category of CDGAs

Corollary 2.7.4. *The homotopy category* $\mathrm{Ho}(\mathsf{CDGA}_\Bbbk)$ *has:*

- ***Objects:*** *CDGAs over $\Bbbk$ (since all objects are fibrant, one does not need fibrant replacement).*
- ***Morphisms:*** $\mathrm{Hom}_{\mathrm{Ho}}(A, B) = [QA, B]$*, where QA is a cofibrant (Sullivan) replacement of A and $[-, -]$ denotes homotopy classes.*

Two CDGAs A and B are isomorphic in $\mathrm{Ho}(\mathsf{CDGA}_\Bbbk)$ *if and only if they are connected by a zigzag of quasi-isomorphisms.*

2.8 Formality

2.8.1 Formal CDGAs and formal spaces

Definition 2.8.1 (Formal CDGA)**.** A CDGA (A, d) is *formal* if it is quasi-isomorphic to its cohomology algebra $(H^*(A), d = 0)$ regarded as a CDGA with zero differential. Equivalently, there is a zigzag of quasi-isomorphisms

$$(A, d) \xleftarrow{\sim} \cdots \xrightarrow{\sim} (H^*(A), 0).$$

Definition 2.8.2 (Formal space)**.** A topological space X is *formal* (over $\Bbbk$) if the CDGA $A_{\mathrm{PL}}(X)$ (or $\Omega^*_{\mathrm{dR}}(X)$ if $\Bbbk = \mathbb{R}$) is formal. Equivalently, the rational (or real) homotopy type of X is determined entirely by its cohomology ring $H^*(X; \Bbbk)$.

Remark 2.8.3. For a formal space, the minimal model can be read off directly from the cohomology ring. In particular, all Massey products vanish (as we used in Chapter 1 for the universal cover of $F_2(L(7, 1))$).

2.8.2 Examples of formal spaces

Theorem 2.8.4 (Examples of formal spaces)**.** *The following classes of spaces are formal:*

(i) ***Spheres*** S^n *(all $n \geq 1$).*

(ii) ***Complex projective spaces*** $\mathbb{C}P^n$*.*

(iii) ***Compact Kähler manifolds*** *(Deligne–Griffiths–Morgan–Sullivan [DGMS75]). This includes all smooth complex projective varieties.*

(iv) ***Compact symmetric spaces*** *G/K where G is a compact Lie group and K is the fixed subgroup of an involution.*

(v) ***Products and wedges*** *of formal spaces.*

(vi) ***H-spaces*** *(spaces with a homotopy-associative multiplication), including Lie groups.*

Proof. (i) For an odd sphere S^{2k+1}, the minimal model is $(\Lambda(x), 0)$ with $|x| = 2k + 1$, which coincides with its cohomology ring as a CDGA with zero differential. For an even sphere S^{2k}, the minimal model is $(\Lambda(x, y), dy = x^2)$ with $|x| = 2k$ and $|y| = 4k - 1$, and the projection sending x to the generator of $H^{2k}(S^{2k}; \mathbb{Q})$ and y to zero is a quasi-isomorphism onto $H^*(S^{2k}; \mathbb{Q})$.

(ii) The space $\mathbb{C}P^n$ has cohomology ring $\mathbb{Q}[x]/(x^{n+1})$ with $|x| = 2$, and minimal model $(\Lambda(x, y), dy = x^{n+1})$ with $|y| = 2n + 1$. The projection to $\mathbb{Q}[x]/(x^{n+1})$ sending y to zero is a quasi-isomorphism. See [FHT01, §12].

(iii) This is the theorem of Deligne, Griffiths, Morgan and Sullivan [DGMS75]: for a compact Kähler manifold M, the de Rham complex $\Omega^*(M)$ is connected by a zigzag of quasi-isomorphisms to its cohomology $H^*(M; \mathbb{R})$, both via the subcomplex of d^c-closed forms. The proof uses the dd^c-lemma, which holds on any compact Kähler manifold as a consequence of Hodge theory and the Kähler identities. It is a result that smooth complex projective varieties inherit a Kähler structure so the result applies.

(iv) For a compact symmetric space G/K, the subcomplex of G-invariant forms in $\Omega^*(G/K)$ has zero differential and is isomorphic, as a graded algebra, to $H^*(G/K; \mathbb{R})$. The inclusion of G-invariant forms into all forms is a quasi-isomorphism by averaging over G with respect to a Haar measure (using the compactness of G). This produces a zigzag of quasi-isomorphisms between $\Omega^*(G/K)$ and $H^*(G/K; \mathbb{R})$. See [FHT01, §15] or [GM81].

(v) Formality is preserved by products: quasi-isomorphisms $A_M \xrightarrow{\sim} H^*(M)$ and $A_N \xrightarrow{\sim} H^*(N)$ yield, by the Künneth theorem, a quasi-isomorphism $A_M \otimes A_N \xrightarrow{\sim} H^*(M) \otimes H^*(N) \cong H^*(M \times N)$. For wedges, formality is preserved through the identification $\widetilde{H}^*(M \vee N) \cong \widetilde{H}^*(M) \oplus \widetilde{H}^*(N)$ and a model construction that combines the minimal models of the factors compatibly with this splitting; see [FHT01, §24].

(vi) For an H-space X, the theorem of Hopf states that $H^*(X; \mathbb{Q})$ is a free graded-commutative algebra on odd-degree generators; see [FHT01, §12]. A free graded-commutative algebra on odd-degree generators coincides with its own minimal model under the zero differential, so X is formal. This applies in particular to compact connected Lie groups. □

2.8.3 Formality and Massey products

Proposition 2.8.5. *If (A, d) is a formal CDGA, then all Massey products in $H^*(A)$ vanish. The converse is false in general (vanishing of all triple Massey products does not imply formality), but it does hold in favorable situations.*

Proof. If A is formal, there is a zigzag of quasi-isomorphisms connecting A to $(H^*(A), 0)$. In $(H^*(A), 0)$, all cochains are cocycles and there are no nontrivial coboundaries (beyond degree 0). A Massey product $\langle [x], [y], [z] \rangle$ requires finding cochains u with $du =$

$x \cup y$; in $(H^*(A), 0)$ this would require $0 = [x] \cup [y]$, and one can take $u = 0$, giving a trivially zero Massey product. Since quasi-isomorphisms preserve Massey products (naturality), the original Massey products in $H^*(A)$ also vanish. $\square$

2.9 Poincaré Duality CDGAs and Configuration Space Models

In this section we give the algebraic setup for Idrissi's theorem on the real homotopy invariance of configuration spaces.

2.9.1 Poincaré duality CDGAs

Definition 2.9.1 (Poincaré duality CDGA)**.** A *Poincaré duality CDGA* of formal dimension n is a triple (A, d, ε) where:

(i) (A, d) is a connected CDGA of finite type with $A^k = 0$ for $k > n$;

(ii) $\varepsilon \colon A^n \to \Bbbk$ is a linear map (the "integration" or "orientation") satisfying $\varepsilon \circ d = 0$;

(iii) the induced pairing $A^k \otimes A^{n-k} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(a \cdot b)$, is non-degenerate for all k.

Remark 2.9.2. If M is a simply connected closed oriented manifold of dimension n, then $H^*(M; \Bbbk)$ with $\varepsilon =$ integration over the fundamental class $[M]$ is a Poincaré duality CDGA with $d = 0$. More generally, Lambrechts and Stanley [LS08] proved that any such M admits a Poincaré duality model: a CDGA (A, d, ε) with a quasi-isomorphism $A \xrightarrow{\sim} \Omega^*_{\mathrm{dR}}(M)$.

2.9.2 Introduction to the Lambrechts–Stanley model

We will now sketch a definition which we'll complete later, just to give an idea of what objects are we going to work.

Definition 2.9.3 (Lambrechts–Stanley CDGA)**.** Let (A, d, ε) be a Poincaré duality CDGA of dimension n, and let $r \geq 0$ be an integer. Define the CDGA $\mathsf{G}_A(r)$ as follows.

As an algebra, $\mathsf{G}_A(r)$ is generated by:

(i) r copies of A, with generators $\iota_i(a)$ for $a \in A$ and $1 \leq i \leq r$ ("labels");

(ii) elements ω_{ij} for $1 \leq i \neq j \leq r$ of degree $n-1$ ("propagators" connecting points i and j);

subject to relations encoding symmetry ($\omega_{ij} = \pm\omega_{ji}$), the Arnold relations, and the vanishing of products involving propagators and labels. The differential extends d on each copy of A and "splits edges": morally, $d\omega_{ij} = \Delta_A^{ij}$, where Δ_A is the diagonal class of A (the Poincaré dual of the diagonal in $M \times M$).

Remark 2.9.4. The elements ω_{ij} should be thought of as algebraic analogues of the "propagator forms" used in Kontsevich's construction of configuration space integrals which we can see in the following sections. The CDGA $\mathsf{G}_A(r)$ can also be understood as being built from graphs whose edges carry the propagator ω_{ij} and whose vertices are labeled by elements of A.

2.9.3 Idrissi's theorem

The following theorem, answering a conjecture of Lambrechts–Stanley, is the central result that motivates our study of CDGAs.

Theorem 2.9.5 (Idrissi, 2019)**.** *Let M be a smooth simply connected closed manifold of dimension n, and let (A, d, ε) be a Poincaré duality model of M. Then for all $r \geq 0$, the Lambrechts–Stanley CDGA $\mathsf{G}_A(r)$ is a real model of the configuration space $\mathrm{Conf}_r(M)$:*

$$\mathsf{G}_A(r) \;\simeq\; \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(M)).$$

In particular, the real homotopy type of $\mathrm{Conf}_r(M)$ depends only on the real homotopy type of M.

2.10 Applications of Rational Homotopy Theory

2.10.1 Rational homotopy groups

Beyond configuration spaces, the CDGA machinery has many applications.

Example 2.10.1 (Rational homotopy of Lie groups)**.** A compact, connected Lie group G has minimal model $(\Lambda(x_1, \ldots, x_r), d = 0)$ where $r = \mathrm{rank}(G)$ and the x_i have odd degrees. The differential is zero because G is an H-space and hence formal. This implies $\pi_*(G) \otimes \mathbb{Q} \cong \mathbb{Q}^r$ with each generator in the degree of the corresponding x_i.

Example 2.10.2 (Rational homotopy of $\mathbb{C}P^\infty$)**.** $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ has minimal model $(\Lambda(x), d = 0)$ with $|x| = 2$. Since $\Lambda(x) = \Bbbk[x]$ (polynomial algebra in one variable), the cohomology is $\Bbbk[x]$ and the only nontrivial rational homotopy group is $\pi_2 \otimes \mathbb{Q} = \mathbb{Q}$.

2.10.2 Mapping spaces

Sullivan models can be used to study mapping spaces.

Theorem 2.10.3 (Sullivan–Vigué-Poirrier)**.** *Let X and Y be simply connected finite CW complexes with $\dim X < \mathrm{conn}(Y)$ (where $\mathrm{conn}(Y)$ is the connectivity[1] of Y). Then the rational homotopy type of the mapping space $\mathrm{Map}(X, Y)$ can be computed from the Sullivan models of X and Y.*

2.10.3 String topology and free loop spaces

Example 2.10.4 (Free loop spaces)**.** For a simply connected manifold M, the free loop space $\mathcal{L}M = \mathrm{Map}(S^1, M)$ has a Sullivan model that can be computed from the Sullivan model of M using the "free loop construction." Specifically, if $(\Lambda V, d)$ is the minimal model of M, then the model of $\mathcal{L}M$ is $(\Lambda V \otimes \Lambda \bar{V}, D)$ where $\bar{V}$ is a copy of V shifted down by one degree and D extends d by $D(\bar{v}) = -\overline{dv}$ (the "contracting homotopy").

[1] The *connectivity* of a topological space Y is the largest integer k such that $\pi_i(Y) = 0$ for all $0 \leq i \leq k$; equivalently, Y is k-connected. For example, $\mathrm{conn}(S^n) = n - 1$, and a simply connected space has connectivity at least 1.

2.11 Open Questions and Further Directions

2.11.1 The integral homotopy invariance conjecture

> **Conjecture.** If M and N are simply connected closed manifolds with $M \simeq N$ (homotopy equivalence over $\mathbb{Z}$), then $\mathrm{Conf}_r(M) \simeq \mathrm{Conf}_r(N)$ for all $r \geq 0$.

This is related to the problem of extending rational homotopy theory to non-nilpotent spaces, which requires working with more general algebraic structures (e.g., L_∞-algebras over group algebras, or diagrams of CDGAs indexed by the fundamental groupoid).

2.11.2 Formality of configuration spaces

> **Question.** For which manifolds M is $\mathrm{Conf}_r(M)$ formal?

It is known that $\mathrm{Conf}_r(\mathbb{R}^n)$ is formal for all r and n (this is related to the formality of the little n-disks operad, proved by Kontsevich and Tamarkin). For closed manifolds, formality of $\mathrm{Conf}_r(M)$ is not automatic even when M itself is formal.

2.11.3 Rational homotopy of embedding spaces

> **Question.** Can the CDGA model of configuration spaces be used to study the rational homotopy type of spaces of embeddings $\mathrm{Emb}(M, N)$?

3

Real Models for Configuration Spaces of Closed Manifolds

This chapter gives a complete proof of Idrissi's theorem: for a smooth simply connected closed manifold M of dimension at least four, the Lambrechts–Stanley CDGA $\mathsf{G}_A(r)$ built from a Poincaré duality model of M is a real model for the configuration space $\mathrm{Conf}_r(M)$, and in particular the real homotopy type of $\mathrm{Conf}_r(M)$ depends only on the real homotopy type of M. We follow the papers of Idrissi [Idr19, Idr22]. The argument proceeds by introducing a labelled graph complex $\mathsf{Graphs}_R(r)$ that sits between $\mathsf{G}_A(r)$ and the piecewise semi-algebraic forms on the Fulton–MacPherson compactification of $\mathrm{Conf}_r(M)$, and by showing that the two natural maps out of it—the algebraic quotient p and the integration map I—are quasi-isomorphisms. The quotient p becomes a quasi-isomorphism after the partition function of vacuum graphs is shown to vanish, a degree-counting argument that uses simple connectedness and the dimension bound $n \geq 4$. The integration map I is a quasi-isomorphism by a comparison of two spectral sequences: a filtration by number of edges on the graph complex side, and the Cohen–Taylor–Bendersky–Gitler filtration by codimension of diagonals on the forms side. The two spectral sequences are matched at the E^1-page by the same quotient p studied earlier, and the conclusion follows from the standard comparison theorem for spectral sequences.

3.1 Proof of Idrissi's Theorem

The theorem we aim to prove is the following.

Theorem 3.1.1 (Idrissi, 2019)**.** *Let M be a smooth, simply connected, closed manifold of dimension n, and let (A, d_A, ε_A) be a Poincaré duality model of M over $\mathbb{R}$. Then for every integer $r \geq 0$, the Lambrechts–Stanley CDGA $\mathsf{G}_A(r)$ is a real model for the configuration space $\mathrm{Conf}_r(M)$.*

Corollary 3.1.2. *If two smooth, simply connected, closed manifolds have the same real homotopy type, then so do their configuration spaces of any number of points.*

Before describing the many ingredients and tools that go into the proof, let us give a sketch. The central idea is to connect two objects through an intermediate combinatorial object built from graphs. On the algebraic side, we have the Lambrechts–Stanley CDGA $\mathsf{G}_A(r)$, which is explicit, small, and depends only on a Poincaré duality model A of the manifold M. On the geometric side, we have the CDGA of piecewise semi-algebraic forms $\Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$ on the Fulton–MacPherson compactification of the configuration space, which faithfully captures the real homotopy type of $\mathrm{Conf}_r(M)$. The connection between these two objects is mediated by a *graph complex* $\mathsf{Graphs}_R(r)$, which sits between them in a zigzag of quasi-isomorphisms:

$$\mathsf{G}_A(r) \xleftarrow{\ p\ } \mathsf{Graphs}_R(r) \xrightarrow{\ I\ } \Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r)).$$

The map p on the left is a purely algebraic quotient map: it collapses all graphs that contain at least one *internal vertex* to zero, and sends the remaining "purely external" graphs to elements of $\mathsf{G}_A(r)$. The map I on the right is geometric: it sends each graph Γ to a differential form on $\mathrm{FM}_M(r)$ defined by integrating a product of *propagator* forms over the positions of the internal vertices.

The heart of the proof consists in showing that both p and I are quasi-isomorphisms. For I, this requires the Stokes formula on manifolds with corners (to show I is a chain map) and a spectral sequence argument (to show it induces an isomorphism on cohomology). For p, the key input is the *vanishing of the partition function*—a result that follows from a degree-counting argument when M is simply connected and $\dim M \geq 4$. This vanishing implies that the subcomplex of graphs with internal vertices is acyclic, making p a quasi-isomorphism.

We will see these in turn.

3.2 The Lambrechts–Stanley model

3.2.1 Poincaré duality models and the diagonal class

The starting point of the construction is the observation that a simply connected closed manifold M of dimension n satisfies Poincaré duality, and that this duality can be realized at the cochain level.

Definition 3.2.1 (Poincaré duality model)**.** A *Poincaré duality model* for a closed oriented manifold M of dimension n is a connected CDGA (A, d_A) of finite type, concentrated in degrees 0 through n, together with:

(i) a quasi-isomorphism $\rho_A \colon A \xrightarrow{\sim} \Omega^*_{\mathrm{dR}}(M)$;

(ii) a linear functional $\varepsilon_A \colon A^n \to \mathbb{R}$ (the *orientation*) satisfying $\varepsilon_A \circ d_A = 0$ and such that the pairing

$$A^k \otimes A^{n-k} \longrightarrow \mathbb{R}, \qquad a \otimes b \longmapsto \varepsilon_A(ab),$$

is non-degenerate for each k.

Theorem 3.2.2 (Lambrechts–Stanley [LS08])**.** *Every simply connected closed manifold M admits a Poincaré duality model in the sense of Definition 3.2.1.*

Proof sketch. We follow [LS08]. Start from any CDGA model (A, d) of M whose cohomology $H^*(A, d) = H^*(M; \mathbb{R})$ is a simply connected Poincaré duality algebra in dimension n. We may assume A is of finite type with $A^0 = \mathbb{R}$, $A^1 = 0$, and $A^2 \subset \ker d$ by passing to a minimal Sullivan model. The ground orientation $\varepsilon \colon A^n \to \mathbb{R}$ is chosen as a chain map inducing the canonical surjection on H^n.

The pairing $\varepsilon(a \cdot b)$ on A may fail to be non-degenerate even though it is non-degenerate on cohomology. The defect is measured by the *ideal of orphans*

$$O \;=\; \{a \in A \;:\; \varepsilon(a \cdot b) = 0 \text{ for all } b \in A\}.$$

The quotient $\bar{A} = A/O$ inherits an orientation $\bar{\varepsilon}$ for which the pairing is non-degenerate by construction, so $\bar{A}$ is a Poincaré duality algebra. The remaining issue is that the projection $\pi \colon A \to \bar{A}$ need not be a quasi-isomorphism: it fails to be one precisely when $H^*(O) \neq 0$.

The strategy is therefore to enlarge A, by adding generators in a controlled way, until the orphan ideal of the enlarged algebra becomes acyclic. The enlargement is performed by induction on a single integer parameter k, the *range of half-acyclicity*: the orphan ideal O of A is called k-half-acyclic if $H^i(O) = 0$ for $n/2 + 1 \leq i \leq k$. The inductive step takes an oriented CDGA whose orphan ideal is $(k-1)$-half-acyclic and produces a quasi-isomorphic oriented CDGA $\hat{A}$ whose orphan ideal is k-half-acyclic, by adjoining freely chosen generators c_i, w_i (and, in positive characteristic, further generators $u_{i,j}, v_{i,j}$) whose differentials are designed to kill the obstructing classes in $H^k(O)$. The construction is explicit and preserves the conditions $A^0 = \mathbb{R}$, $A^1 = 0$, $A^2 \subset \ker d$.

Iterating this construction $\lceil n/2 \rceil$ times produces a quasi-isomorphic oriented CDGA whose orphan ideal is $(n+1)$-half-acyclic, hence acyclic. The quotient by orphans is then a Poincaré duality CDGA quasi-isomorphic to the original A, and composing with the quasi-isomorphism to $\Omega^*_{\mathrm{dR}}(M)$ gives the desired Poincaré duality model.

The case $n \leq 6$ does not require the orphan construction: by the theorem of Miller and Neisendorfer, every simply connected closed manifold of dimension at most 6 is formal, and one may take $A = H^*(M; \mathbb{R})$ directly.

The full argument can be found in [LS08, §3–6]. □

Example 3.2.3 (The formal case)**.** When M is formal, one may take $A = H^*(M; \mathbb{R})$ with $d_A = 0$ and ε_A equal to integration against the fundamental class. The orphan ideal vanishes already, and no enlargement is needed.

Remark 3.2.4. In general, A has a non-trivial differential, but it remains a small model: finite-dimensional in each degree, concentrated in degrees 0 through n.

Given a Poincaré duality model, the duality pairing produces a distinguished element of $(A \otimes A)^n$ that is central to the entire construction.

Definition 3.2.5 (Diagonal class)**.** Let (A, d_A, ε_A) be a Poincaré duality model. Choose a homogeneous basis $\{e_\alpha\}$ of A, and let $\{e^\alpha\}$ be the Poincaré dual basis, defined by $\varepsilon_A(e_\alpha \cdot e^\beta) = \delta_\alpha^\beta$. The *diagonal class* is the element

$$\Delta_A = \sum_\alpha (-1)^{|e_\alpha|}\, e_\alpha \otimes e^\alpha \;\in\; (A \otimes A)^n.$$

Lemma 3.2.6. *The diagonal class Δ_A has the following properties:*

(i) it is independent of the choice of homogeneous basis $\{e_\alpha\}$;

(ii) geometrically, it represents the Poincaré dual of the diagonal $\Delta \subset M \times M$;

(iii) it satisfies the symmetry $\tau(\Delta_A) = (-1)^n \Delta_A$, where τ is the graded swap of tensor factors.

Proof. (i) Let $\{e'_\alpha\}$ be another homogeneous basis, related to $\{e_\alpha\}$ by $e'_\alpha = \sum_\beta C^\beta_\alpha e_\beta$ for an invertible matrix C. The dual basis $\{e'^\alpha\}$ is related to $\{e^\alpha\}$ by the transpose-inverse: $e'^\alpha = \sum_\beta (C^{-1})^\alpha_\beta e^\beta$. Substituting into the definition,

$$\begin{aligned}\sum_\alpha (-1)^{|e'_\alpha|} e'_\alpha \otimes e'^\alpha &= \sum_\alpha (-1)^{|e_\alpha|} \Big(\sum_\beta C^\beta_\alpha e_\beta\Big) \otimes \Big(\sum_\gamma (C^{-1})^\alpha_\gamma e^\gamma\Big) \\ &= \sum_{\beta,\gamma} (-1)^{|e_\beta|} \Big(\sum_\alpha C^\beta_\alpha (C^{-1})^\alpha_\gamma\Big) e_\beta \otimes e^\gamma \\ &= \sum_{\beta,\gamma} (-1)^{|e_\beta|} \delta^\beta_\gamma \, e_\beta \otimes e^\gamma = \sum_\beta (-1)^{|e_\beta|} e_\beta \otimes e^\beta = \Delta_A,\end{aligned}$$

where in the second line we used that C is degree-preserving, so $|e'_\alpha| = |e_\beta|$ whenever $C^\beta_\alpha \neq 0$.

(ii) Geometrically, the Poincaré dual of the diagonal $\Delta \subset M \times M$ is characterized as the unique class in $H^n(M \times M)$ whose pullback under any inclusion $M \cong M \times \{x\} \hookrightarrow M \times M$ is the Poincaré dual of the point $\{x\}$. Under the Künneth isomorphism $H^*(M \times M) \cong H^*(M) \otimes H^*(M)$, the dual basis property $\varepsilon_A(e_\alpha \cdot e^\beta) = \delta^\beta_\alpha$ is exactly the statement that $\sum_\alpha \pm e_\alpha \otimes e^\alpha$ acts on $H^*(M)$ by the identity under the slant product. The signs $(-1)^{|e_\alpha|}$ are the Koszul signs forced by graded commutativity. Hence Δ_A realizes the diagonal class. Another approach via a verification at the form level is given in [LS08, §4].

(iii) The graded swap τ acts on a tensor product of homogeneous elements by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$. Applying this,

$$\tau(\Delta_A) = \sum_\alpha (-1)^{|e_\alpha|} (-1)^{|e_\alpha| \cdot |e^\alpha|} e^\alpha \otimes e_\alpha.$$

Now use $|e^\alpha| = n - |e_\alpha|$ from the duality pairing, which gives $|e_\alpha| \cdot |e^\alpha| = |e_\alpha|(n - |e_\alpha|) \equiv n \cdot |e_\alpha| - |e_\alpha|^2 \pmod 2$. The term $|e_\alpha|^2$ has the same parity as $|e_\alpha|$, so this reduces to $|e_\alpha|(n-1) \pmod 2$. Reindexing by setting $\alpha' = \alpha$ but renaming the basis (using that $\{e^\alpha\}$ is also a basis of A with dual basis $\{(-1)^{|e^\alpha| \cdot |e_\alpha|} e_\alpha\}$, by the symmetry of the duality pairing up to the global sign $(-1)^n$), one finds

$$\tau(\Delta_A) = (-1)^n \sum_\alpha (-1)^{|e_\alpha|} e_\alpha \otimes e^\alpha = (-1)^n \Delta_A.$$

□

3.2.2 Definition of the CDGA $\mathsf{G}_A(r)$

We now define the Lambrechts–Stanley CDGA. Its construction is modeled on the presentation of the cohomology algebra of $\mathrm{Conf}_r(\mathbb{R}^n)$ due to Arnold [Arn69] and Cohen [Coh76], augmented with the data of a Poincaré duality model A.

Definition 3.2.7 (Generators of $\mathsf{G}_A(r)$)**.** The graded-commutative algebra $\mathsf{G}_A(r)$ is generated by two families of generators:

(i) for each $1 \leq i \leq r$ and each $a \in A$, a generator $\iota_i(a)$ of degree $|a|$, with the requirement that $\iota_i \colon A \to \mathsf{G}_A(r)$ be an algebra morphism;

(ii) for each pair $1 \leq i \neq j \leq r$, a generator ω_{ij} of degree $n-1$.

Remark 3.2.8. Geometrically, $\iota_i(a)$ represents the pullback of $\rho_A(a)$ via the i-th projection $p_i \colon M^r \to M$, while ω_{ij} represents a cocycle Poincaré dual to the diagonal submanifold $\Delta_{ij} = \{x \in M^r : x_i = x_j\}$ removed from M^r to form $\mathrm{Conf}_r(M)$.

Definition 3.2.9 (Relations in $\mathsf{G}_A(r)$)**.** The generators of $\mathsf{G}_A(r)$ are subject to four families of relations:

(R1) graded symmetry: $\omega_{ji} = (-1)^n \omega_{ij}$;

(R2) nilpotency: $\omega_{ij}^2 = 0$;

(R3) the Arnold relation: for any triple of distinct indices i, j, k,

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0;$$

(R4) the diagonal relation: for all $a \in A$ and all distinct i, j,

$$\iota_i(a) \cdot \omega_{ij} = \iota_j(a) \cdot \omega_{ij}.$$

Remark 3.2.10. Relation (R3) reflects the geometric fact that the three pairwise diagonals $\Delta_{ij}, \Delta_{jk}, \Delta_{ki}$ in M^3 satisfy $\Delta_{ij} \cap \Delta_{jk} = \Delta_{jk} \cap \Delta_{ki} = \Delta_{ki} \cap \Delta_{ij}$. Relation (R4) reflects the fact that on the diagonal Δ_{ij}, where $x_i = x_j$, the projections p_i and p_j agree, so pulling back a form via either projection gives the same result when multiplied by ω_{ij}.

Definition 3.2.11 (Differential on $\mathsf{G}_A(r)$)**.** The differential on $\mathsf{G}_A(r)$ is defined on the two families of generators by

$$d\big(\iota_i(a)\big) = \iota_i(d_A a), \qquad d(\omega_{ij}) = \iota_{ij}(\Delta_A) := \sum_\alpha (-1)^{|e_\alpha|}\, \iota_i(e_\alpha) \cdot \iota_j(e^\alpha),$$

and extended to all of $\mathsf{G}_A(r)$ by the graded Leibniz rule.

Proposition 3.2.12. *The differential of Definition 3.2.11 satisfies $d^2 = 0$ and is compatible with the four relations (R1)–(R4) of Definition 3.2.9. Hence $(\mathsf{G}_A(r), d)$ is a well-defined CDGA.*

Proof. We are following [Idr22, §3] but one can follow the original paper [LS04], where one can also find the verification in the operadic language.
We verify $d^2 = 0$ on each type of generator. On $\iota_i(a)$,

$$d^2 \iota_i(a) = d\iota_i(d_A a) = \iota_i(d_A^2 a) = 0,$$

using that ι_i is an algebra morphism and $d_A^2 = 0$ in A. On ω_{ij},

$$d^2\omega_{ij} = d\big(\iota_{ij}(\Delta_A)\big) = \sum_\alpha (-1)^{|e_\alpha|}\big(\iota_i(d_A e_\alpha) \cdot \iota_j(e^\alpha) + (-1)^{|e_\alpha|}\iota_i(e_\alpha) \cdot \iota_j(d_A e^\alpha)\big),$$

which equals $\iota_{ij}(d_{A\otimes A}\Delta_A)$ by the Leibniz rule. Since Δ_A is a cocycle in $A \otimes A$, a fact that follows from the compatibility of ε_A with d_A and the dual basis identity, the right-hand side vanishes.

Compatibility with (R1) follows from the symmetry $\tau(\Delta_A) = (-1)^n\Delta_A$ established in Lemma 3.2.6: applying d to both sides of $\omega_{ji} = (-1)^n\omega_{ij}$ gives $\iota_{ji}(\Delta_A) = (-1)^n\iota_{ij}(\Delta_A)$, which is exactly $\tau(\Delta_A) = (-1)^n\Delta_A$ pulled back along ι_{ij}. Compatibility with (R2) is automatic since d acts by the Leibniz rule and $\omega_{ij}^2 = 0$ holds in the underlying graded algebra. Compatibility with (R3) is the most delicate verification: applying d to the Arnold sum and using $d\omega_{ij} = \iota_{ij}(\Delta_A)$ produces a sum of six quadratic terms in the ι generators that cancel pairwise after using the symmetry of Δ_A and the diagonal relation (R4). Compatibility with (R4) reduces to the identity $\iota_i(a) \cdot \iota_{ij}(\Delta_A) = \iota_j(a) \cdot \iota_{ij}(\Delta_A)$, which follows from $(a \otimes 1) \cdot \Delta_A = (1 \otimes a) \cdot \Delta_A$ in $A \otimes A$, a standard property of the diagonal class in Poincaré duality algebras.

□

Remark 3.2.13. The formula $d\omega_{ij} = \iota_{ij}(\Delta_A)$ is the algebraic incarnation of the geometric fact that ω_{ij}, viewed as a form on $\mathrm{Conf}_r(M) \subset M^r$, is a primitive for the closed form representing the diagonal class, which becomes exact once the diagonal is removed.

Example 3.2.14 (The case $M = \mathbb{R}^n$)**.** When $M = \mathbb{R}^n$, we have $A = \mathbb{R}$, $\Delta_A = 0$, and the differential of $\mathsf{G}_{\mathbb{R}}(r)$ vanishes entirely. The resulting algebra is the Arnold–Cohen cohomology algebra $H^*(\mathrm{Conf}_r(\mathbb{R}^n); \mathbb{R})$, with its presentation by generators ω_{ij} of degree $n-1$ and the Arnold relation (R3). For a general manifold, the additional data of the labels $\iota_i(a)$, the diagonal relation (R4), and the non-trivial differential account for the global topology of M.

3.3 The Fulton–MacPherson compactification

The configuration space $\mathrm{Conf}_r(M)$ is an open submanifold of M^r. To integrate differential forms over fibers of projections from higher to lower configuration spaces, we need the fibers to be compact, which motivates the introduction of a compactification.

Definition 3.3.1 (Fulton–MacPherson compactification)**.** Fix an embedding $M \hookrightarrow \mathbb{R}^N$, which exists by Whitney's theorem. For a configuration $x = (x_1, \dots, x_r) \in \mathrm{Conf}_r(M)$ and distinct indices $i \neq j$, define the unit direction vector

$$\theta_{ij}(x) = \frac{x_j - x_i}{|x_j - x_i|} \ \in \ S^{N-1},$$

and for distinct i, j, k, define the distance ratio

$$\delta_{ijk}(x) = \frac{|x_i - x_j|}{|x_i - x_k|} \ \in \ (0, \infty).$$

The *Fulton–MacPherson compactification* $\mathrm{FM}_M(r)$ is the closure of the image of

$$\mathrm{Conf}_r(M) \longrightarrow M^r \times \prod_{i \neq j} S^{N-1} \times \prod_{\{i,j,k\}} [0, \infty), \qquad x \longmapsto \big(x,\ (\theta_{ij})_{i\neq j},\ (\delta_{ijk})_{\{i,j,k\}}\big).$$

Remark 3.3.2 (Geometric interpretation of boundary points). A boundary point of $\mathrm{FM}_M(r)$ records a virtual configuration in which groups of points are allowed to collide, but the directions and relative speeds of approach are remembered. When two points x_i and x_j approach each other, the direction θ_{ij} survives in the limit as a well-defined unit vector in the unit sphere of the tangent space $T_{x_i}M \cong \mathbb{R}^n$, that is, in S^{n-1}. When an entire cluster of k points collides simultaneously, the limiting data includes a full infinitesimal configuration of k points in $T_{x_0}M \cong \mathbb{R}^n$, considered up to translation and positive dilation. This infinitesimal configuration is itself a point in the Fulton–MacPherson compactification $\mathrm{FM}_{\mathbb{R}^n}(k)$ of $\mathrm{Conf}_k(\mathbb{R}^n)$, which parametrizes configurations in Euclidean space modulo translation and positive scaling.

The space $\mathrm{FM}_M(r)$ has a number of structural properties that will be used throughout the proof.

Theorem 3.3.3 (Fulton–MacPherson, Axelrod–Singer, Sinha)**.** *The space* $\mathrm{FM}_M(r)$ *defined in Definition 3.3.1 is a compact smooth manifold with corners of dimension* nr*, and satisfies:*

(i) the inclusion $\mathrm{Conf}_r(M) \hookrightarrow \mathrm{FM}_M(r)$ *is a homotopy equivalence;*

(ii) for every subset $S \subset \{1, \dots, r\}$*, the forgetful projection* $\pi_S \colon \mathrm{Conf}_r(M) \to \mathrm{Conf}_S(M)$ *extends to a map* $\pi_S \colon \mathrm{FM}_M(r) \to \mathrm{FM}_M(S)$ *which is a semi-algebraic fiber bundle;*

(iii) the boundary strata of $\mathrm{FM}_M(r)$ *are indexed by trees whose vertices correspond to clusters of colliding points and whose edges record the nesting of these clusters.*

Proof. We treat the four statements in order.

The original construction of Fulton and MacPherson [FM94] realizes $\mathrm{FM}_M(r)$ as an iterated blowup. One starts from M^r and blows up the small diagonals $\Delta_S = \{x \in M^r : x_i = x_j \text{ for all } i, j \in S\}$ in order of increasing dimension, that is, starting from the deepest diagonal $\Delta_{\{1,\dots,r\}}$ and proceeding upward. Each blowup replaces a smooth submanifold of codimension c by a $\mathbb{R}\mathbf{P}^{c-1}$-bundle over it, or, in the smooth setting of Axelrod and Singer [AS94], by a sphere bundle, which produces a boundary face. Because the diagonals are smooth submanifolds and one blows them up in a nested order, the result is a smooth manifold with corners. The dimension is preserved at each blowup, so $\dim \mathrm{FM}_M(r) = \dim M^r = nr$. Compactness is automatic since each blowup of a compact space along a compact submanifold is compact, and M^r is compact when M is. A self-contained smooth-manifolds-with-corners account is given in [Sin04, §4], where one can find the precise verification of the smooth structure at the corners.

For (i), we use the following general fact, which is the only piece of differential topology we need.

Lemma 3.3.4. *Let* W *be a compact smooth manifold with corners. Then the inclusion of the interior* $\mathrm{int}(W)$ *into* W *is a homotopy equivalence.*

Proof. A manifold with corners W admits a collar of its boundary by the corner-collar theorem of Douady, which one can find in [Sin04, §3] and, for the corner-free case, in [Lee03, Thm. 16.3]. The extension to corners is proved by induction on the codimension of the corner strata: there exists an open neighborhood U of ∂W in W

and a diffeomorphism $\Phi\colon \partial W \times [0,1) \to U$ fixing $\partial W = \partial W \times \{0\}$. Using Φ, define the homotopy $H\colon W \times [0,1] \to W$ by

$$H(x,t) = \begin{cases} \Phi(y,\, s + t(1-s)) & \text{if } x = \Phi(y,s) \in U, \\ x & \text{if } x \notin U, \end{cases}$$

suitably smoothed near the boundary of U. Then $H_0 = \mathrm{id}_W$, and H_1 maps W into $\mathrm{int}(W)$. Hence the inclusion $\mathrm{int}(W) \hookrightarrow W$ admits a homotopy inverse, namely H_1, so it is a homotopy equivalence. □

Applying Lemma 3.3.4 to $W = \mathrm{FM}_M(r)$, which is a compact manifold with corners by the construction above, gives the desired homotopy equivalence. The interior of $\mathrm{FM}_M(r)$ is exactly $\mathrm{Conf}_r(M)$, since the points where no diagonal has been blown up correspond to usual configurations of r distinct points.

For (ii), the forgetful map $\pi_S\colon M^r \to M^S$, defined on the level of full Cartesian products, restricts to $\pi_S\colon \mathrm{Conf}_r(M) \to \mathrm{Conf}_S(M)$ on configuration spaces. The extension to a map between Fulton–MacPherson compactifications is constructed by compatibility with the iterated blowup: forgetting the points indexed outside S corresponds, on the level of blowup data, to forgetting the components of the direction-vector data θ_{ij} and the distance-ratio data δ_{ijk} that involve indices outside S. The defining map of Definition 3.3.1 for $\mathrm{FM}_M(r)$ projects componentwise onto its analog for $\mathrm{FM}_M(S)$, and this componentwise projection extends continuously to the closure. The extension is a polynomial map in local semi-algebraic charts, hence semi-algebraic.

To see that the extended π_S is a locally trivial bundle, fix a configuration $\bar{x} \in \mathrm{FM}_M(S)$. The fiber over $\bar{x}$ is the space of ways of inserting the points indexed by $\{1, \ldots, r\} \setminus S$ into a configuration whose S-points are $\bar{x}$. This fiber is itself a Fulton–MacPherson compactification, namely $\mathrm{FM}_M(r)/\mathrm{FM}_M(S)$ in the notation of [Sin04, §4], and varies continuously with $\bar{x}$ in the semi-algebraic category. Local triviality of π_S follows from the existence of local sections of the forgetful map at the level of compactifications, which are produced by choosing local coordinates on M near each point of $\bar{x}$ and inserting the additional points along these coordinates. This arguement is present in full details in the references [Sin04, §4] and [Idr22, §5.1].

For (iii), a boundary point of $\mathrm{FM}_M(r)$ corresponds to a configuration in which some subset of points have collided. The combinatorial type of the collision is recorded by a rooted tree as follows. Each cluster of points that collide simultaneously is recorded as an internal node of the tree, with leaves labeled by the indices in the cluster. Clusters that collide at different rates are nested: a cluster that collides faster sits below a cluster that contains it but collides more slowly, producing a parent-child relation in the tree. The root of the tree corresponds to the global ambient M, and the leaves are labeled $1, \ldots, r$. The codimension of the corresponding boundary stratum equals the number of internal edges of the tree.

□

Remark 3.3.5. The semi-algebraic nature of the projection extensions is crucial. The extended maps π_S are not smooth submersions at the boundary. This means that the standard de Rham theory of integration along fibers does not apply. This is the motivation for introducing piecewise semi-algebraic forms in the next section.

3.4 Piecewise semi-algebraic forms

The theory of *piecewise semi-algebraic (PA) forms*, developed by Kontsevich–Soibelman [KS00] and successively by Hardt, Lambrechts, Turchin and Volić [HLTV11], provides the framework that Idrissi used and that we will need.

Definition 3.4.1 (Semi-algebraic set)**.** A *semi-algebraic set* is a subset of $\mathbb{R}^N$ defined by a finite Boolean combination of polynomial equalities and inequalities.

Theorem 3.4.2 (Nash–Tognoli)**.** *Every closed smooth manifold admits a semi-algebraic structure. In particular, the Fulton–MacPherson compactifications* $\mathrm{FM}_M(r)$ *are semi-algebraic sets.*

Proof sketch. The first statement is the theorem of Nash [Nas52] (in a weaker form) and Tognoli [Tog73]: every closed smooth manifold M is diffeomorphic to a real algebraic variety, hence in particular to a semi-algebraic subset of $\mathbb{R}^N$ for some N. A modern reference for the full proof is [BCR98, Ch. 14]. The second statement, that $\mathrm{FM}_M(r)$ inherits a semi-algebraic structure, follows because the iterated blowup construction of Theorem 3.3.3 is performed along smooth algebraic submanifolds (the diagonals $\Delta_S \subset M^r$), and each blowup of a real algebraic variety along a smooth algebraic subvariety is again a real algebraic variety. □

Remark 3.4.3. The Nash–Tognoli theorem refines Whitney's embedding theorem by producing a polynomial, rather than smooth, embedding into $\mathbb{R}^N$. We will not need its proof, only the existence of the semi-algebraic structure on $\mathrm{FM}_M(r)$.

Definition 3.4.4 (CDGA of PA forms)**.** The CDGA of *piecewise semi-algebraic forms* $\Omega^*_{\mathrm{PA}}(X)$ on a compact semi-algebraic set X is an analog of the de Rham complex, adapted to the semi-algebraic category. Its elements are generated by minimal forms of the type

$$f_0\, df_1 \wedge \cdots \wedge df_k,$$

where the $f_i \colon X \to \mathbb{R}$ are semi-algebraic functions[1].

The features of this theory that we will use are summarized in the following theorem.

Theorem 3.4.5 (Hardt–Lambrechts–Turchin–Volić [HLTV11])**.** *The CDGA of PA forms enjoys the following three properties.*

(i) There is a well-defined integration

$$\int_X \colon \Omega^{\dim X}_{\mathrm{PA}}(X) \longrightarrow \mathbb{R}$$

satisfying Stokes' theorem: for every form $\omega \in \Omega^{\dim X-1}_{\mathrm{PA}}(X)$*, one has* $\int_X d\omega = \int_{\partial X} \omega$.

(ii) For any SA bundle $\pi \colon E \to B$ *with compact fibers of dimension* d*, there is a well-defined* pushforward

$$\pi_* \colon \Omega^{k+d}_{\mathrm{PA}}(E) \longrightarrow \Omega^k_{\mathrm{PA}}(B),$$

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the integration along fibers, satisfying the projection formula

$$\pi_*\big((\pi^*\alpha)\wedge\beta\big)=\alpha\wedge\pi_*(\beta)$$

and the Stokes formula for fiber integration

$$d\circ\pi_*=\pi_*\circ d+(\pi|_\partial)_*.$$

(iii) The functor Ω^*_{PA} *computes real cohomology: there is a natural quasi-isomorphism*

$$\Omega^*_{\mathrm{PA}}(X)\xrightarrow{\sim}\Omega^*_{\mathrm{dR}}(X)$$

whenever X *is a smooth manifold.*

Remark 3.4.6. The Stokes formula for the pushforward, item (ii) of the theorem, deserves special attention since it is the engine of the entire proof. When we compute $d\circ\pi_*(\omega)$ for a form ω on $\mathrm{FM}_M(r+s)$, the formula gives two types of terms: a "bulk" term $\pi_*(d\omega)$, and "boundary" terms $(\pi|_\partial)_*(\omega|_\partial)$ coming from the boundary of $\mathrm{FM}_M(r+s)$. These boundary terms correspond exactly to the contraction differential in the graph complex, and this correspondence is what makes the integration map I a chain map.

3.5 The propagator

The propagator is the geometric form that will be assigned to the edges of graphs. Before defining it, we fix a cofibrant model.

Definition 3.5.1 (Cofibrant model R)**.** Let (R,d_R) be a cofibrant CDGA equipped with a quasi-isomorphism $\rho\colon R\xrightarrow{\sim}\Omega^*_{\mathrm{PA}}(M)$.

Remark 3.5.2. The reason for introducing R rather than working directly with the Poincaré duality model A is that we need a cofibrant source for the CDGA map into forms, which ensures good homotopy-theoretic behavior. Since every CDGA is fibrant in the Bousfield–Gugenheim model structure, a cofibrant replacement $R\xrightarrow{\sim}A$ composed with $A\xrightarrow{\sim}\Omega^*_{\mathrm{PA}}(M)$ provides ρ.

The motivation for the propagator is the following. By Poincaré duality on M, the diagonal class $[\Delta_M]\in H^n(M\times M;\mathbb{R})$ becomes exact when restricted to $\mathrm{Conf}_2(M)=M\times M\setminus\Delta$. A representative primitive for this class on the compactification $\mathrm{FM}_M(2)$ is what we call a propagator.

Definition 3.5.3 (Propagator)**.** A propagator is a PA form $\varphi\in\Omega^{n-1}_{\mathrm{PA}}(\mathrm{FM}_M(2))$ satisfying the following three conditions.

(P1) $d\varphi=\pi_{12}^*(\rho(\Delta_R))$, where $\pi_{12}\colon\mathrm{FM}_M(2)\to M\times M$ is the projection and $\Delta_R\in(R\otimes R)^n$ is the diagonal class of R, which maps to the Poincaré dual of the diagonal in $\Omega^n_{\mathrm{PA}}(M\times M)$.

(P2) $\sigma^*\varphi=(-1)^n\varphi$, where σ swaps the two points.

(P3) The restriction of φ to the boundary $\partial\mathrm{FM}_M(2)$, which fibers over M with fiber S^{n-1} representing the collision boundary, satisfies $p_*(\varphi|_\partial)=\mathrm{vol}_{n-1}$, the volume form on S^{n-1} normalized to integrate to 1.

Proposition 3.5.4 (Existence of a propagator)**.** *A propagator φ in the sense of Definition 3.5.3 exists.*

Proof. We follow the strategy of [Idr19, §4], with one elementary modification at the rescaling step. We construct φ in three stages, each adjusting it to satisfy one of the three conditions.

For (P1), we observe that $\pi_{12}^*(\rho(\Delta_R))$ is a closed form of degree n on $\mathrm{FM}_M(2)$, and we must show that it is exact. The closedness follows from $d_R(\Delta_R) = 0$, which is Poincaré duality, together with the fact that ρ and π_{12}^* are chain maps. For exactness, we compute the cohomology class $[\pi_{12}^*(\rho(\Delta_R))] \in H^n(\mathrm{FM}_M(2); \mathbb{R})$. Since $\mathrm{FM}_M(2)$ is homotopy equivalent to $\mathrm{Conf}_2(M) = M \times M \setminus \Delta$ by Theorem 3.3.3, we may compute the class in $H^n(\mathrm{Conf}_2(M); \mathbb{R})$. The long exact sequence of the pair $(M \times M, \mathrm{Conf}_2(M))$ gives

$$H^{n-1}(\mathrm{Conf}_2(M)) \longrightarrow H^n(M \times M, \mathrm{Conf}_2(M)) \longrightarrow H^n(M \times M) \longrightarrow H^n(\mathrm{Conf}_2(M)),$$

and the diagonal class $[\Delta_M]$ in $H^n(M \times M)$ is the image of the Thom class of the normal bundle of the diagonal under the second arrow. The image of $[\Delta_M]$ in $H^n(\mathrm{Conf}_2(M))$ is zero, since the diagonal has been removed. Hence $\pi_{12}^*(\rho(\Delta_R))$ is exact on $\mathrm{FM}_M(2)$, and there exists $\varphi_0 \in \Omega_{\mathrm{PA}}^{n-1}(\mathrm{FM}_M(2))$ with $d\varphi_0 = \pi_{12}^*(\rho(\Delta_R))$.

For (P2), replace φ_0 by its symmetrization

$$\varphi_1 = \tfrac{1}{2}\big(\varphi_0 + (-1)^n \sigma^* \varphi_0\big).$$

Since Δ_R satisfies $\tau(\Delta_R) = (-1)^n \Delta_R$ by Lemma 3.2.6, the form $\pi_{12}^*(\rho(\Delta_R))$ inherits the same symmetry under σ, and hence $d\varphi_1 = \pi_{12}^*(\rho(\Delta_R))$ remains valid. By construction $\sigma^*\varphi_1 = (-1)^n \varphi_1$, so (P2) is satisfied.

For (P3), the boundary $\partial \mathrm{FM}_M(2)$ is the unit sphere bundle STM of the tangent bundle of M, fibered over M by a projection $p\colon \partial \mathrm{FM}_M(2) \to M$ whose fibers are spheres S^{n-1}. We consider the fiberwise integral

$$c \;=\; (p|_\partial)_*(\varphi_1|_\partial) \;\in\; \Omega_{\mathrm{PA}}^0(M),$$

which is a PA function on M. We claim that c is locally constant, hence constant since M is connected. To see this, apply the Stokes formula for the pushforward (Theorem 3.4.5, item (ii)) to the projection $p\colon \mathrm{FM}_M(2) \to M$ onto the first factor and the form φ_1 of degree $n-1$:

$$d\big(p_*\varphi_1\big) \;=\; p_*\big(d\varphi_1\big) \;-\; (p|_\partial)_*\big(\varphi_1|_\partial\big).$$

The left-hand side is the differential of an $(n-1)$-form on M, which is an n-form. The first term on the right, $p_*(d\varphi_1) = p_*(\pi_{12}^*(\rho(\Delta_R)))$, is also an n-form on M, equal by the projection formula to a multiple of the volume form on M. The second term, $(p|_\partial)_*(\varphi_1|_\partial) = c$, is a function on M. Comparing degrees, the identity above takes place in degree n on M, and splitting it into the locally constant component (degree 0) and top-degree component is what shows that c is locally constant on M. Hence c is a real number depending only on φ_1.

We now distinguish two cases according to whether c is zero or not.

If $c \neq 0$, rescale φ_1 by $1/c$ to obtain $\varphi = \varphi_1/c$. Conditions (P1) and (P2) are linear in φ_1 and so are preserved by the rescaling, while (P3) is satisfied by construction.

If $c = 0$, the rescaling argument fails, and we correct φ_1 additively. Choose any PA form $\eta \in \Omega_{\mathrm{PA}}^{n-1}(\mathrm{FM}_M(2))$ satisfying:

(a) $d\eta = 0$,

(b) $\sigma^*\eta = (-1)^n\eta$,

(c) $(p|_\partial)_*(\eta|_\partial) = 1$.

Such an η exists: pick a PA representative η_∂ of the volume form on S^{n-1} in the boundary $\partial\mathrm{FM}_M(2) \cong STM$, normalized so that $(p|_\partial)_*(\eta_\partial) = 1$, and extend it by a collar of the boundary to a closed PA form η on all of $\mathrm{FM}_M(2)$ using the corner-collar argument of Lemma 3.3.4. The symmetry under σ is ensured by replacing η by its σ-symmetrization, which preserves both closedness and the boundary integral since the boundary $\partial\mathrm{FM}_M(2)$ is preserved by σ and the volume form pulls back to itself up to the sign $(-1)^n$ arising from the orientation behaviour of S^{n-1} under the antipodal map.

With such η in hand, set $\varphi = \varphi_1 + \eta$. Then $d\varphi = d\varphi_1 + d\eta = \pi_{12}^*(\rho(\Delta_R))$, so (P1) holds. The symmetry $\sigma^*\varphi = (-1)^n\varphi$ holds because both φ_1 and η satisfy it. Finally, $(p|_\partial)_*(\varphi|_\partial) = c + 1 = 1$, so (P3) holds in the form $(p|_\partial)_*(\varphi|_\partial) = \mathrm{vol}_{n-1}$, since the right-hand side is the constant function 1 on M, matching the normalization of the volume form on S^{n-1}.

In either case, the form φ obtained at the end of these three stages satisfies all three conditions of Definition 3.5.3, and is therefore a propagator. □

Remark 3.5.5. A propagator is not unique. Different choices differ by a closed form of degree $n-1$ on $\mathrm{FM}_M(2)$. The final model $\mathsf{G}_A(r)$ produced by the construction is, however, independent of the choice.

3.6 The labeled graph complex

We now come to the combinatorial object of the proof: the graph complex $\mathsf{Graphs}_R(r)$. This is a CDGA whose elements are formal linear combinations of labeled graphs, and whose differential encodes both the algebraic data of R and the geometric data of the propagator.

3.6.1 The module of graphs

Definition 3.6.1 (Labeled graph). A *labeled graph* Γ consists of the following data:

(i) a set of *external vertices* $E = \{1, 2, \ldots, r\}$, which are numbered and ordered, and represent the r labeled points of the configuration;

(ii) a finite set I of *internal vertices*, which are unlabeled and represent auxiliary points that will eventually be integrated over M;

(iii) a set of edges, each connecting two distinct vertices, with edges allowed of all three types: external-external, external-internal, and internal-internal, but with multiple edges between the same pair of vertices and loops at a single vertex not allowed;

(iv) for each vertex v in $E \cup I$, a *label* $\ell(v) \in R$ that is a homogeneous element of the cofibrant model.

Definition 3.6.2 (The module $\mathsf{Graphs}_R(r)$)**.** The graded $\mathbb{R}$-module $\mathsf{Graphs}_R(r)$ is the free $\mathbb{R}$-vector space on isomorphism classes of labeled graphs in the sense of Definition 3.6.1, where graphs are identified up to isomorphisms permuting internal vertices while fixing external vertices, and where Koszul signs account for the reordering of edges and the degrees of labels.

3.6.2 Grading

Definition 3.6.3 (Grading on $\mathsf{Graphs}_R(r)$)**.** The degree of a labeled graph Γ is

$$|\Gamma| = \sum_{v \in V(\Gamma)} |\ell(v)| + (n-1) \cdot |E(\Gamma)| - n \cdot |I(\Gamma)|.$$

Remark 3.6.4. Each label contributes its own cohomological degree. Each edge contributes $n-1$, since it will eventually be replaced by the propagator form φ of degree $n-1$. Each internal vertex subtracts n, since integrating over M lowers the degree of a form by $\dim M = n$.

3.6.3 Algebra structure

Definition 3.6.5 (Multiplication on $\mathsf{Graphs}_R(r)$)**.** Given two labeled graphs Γ_1, Γ_2 sharing the same set of external vertices but having disjoint sets of internal vertices and edges, their product $\Gamma_1 \cdot \Gamma_2$ is the labeled graph obtained by taking the union of their edge and internal vertex sets, and multiplying the labels on each shared external vertex. The unit is the empty graph: no edges, no internal vertices, all external labels equal to 1_R.

Proposition 3.6.6 (Freeness of $\mathsf{Graphs}_R(r)$)**.** *The graded-commutative algebra $\mathsf{Graphs}_R(r)$, equipped with the multiplication of Definition 3.6.5, is the free graded-commutative algebra on the set of connected labeled graphs (where "connected" means that the underlying graph, after adding edges between all external vertices, is connected).*

Proof. We follow the standard construction of graph complexes as in [LV14, §6] and [Idr22, §5].

Every labeled graph Γ admits a unique decomposition into connected components $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, where the components are taken with respect to the augmented graph in which all external vertices are first identified into a single "virtual" vertex (so two components are identified if they share an external vertex). Each Γ_i is itself a labeled graph in the sense of Definition 3.6.1, containing some subset of the external vertices and some subset of the internal vertices, and the multiplication of Definition 3.6.5 satisfies

$$\Gamma = \Gamma_1 \cdot \Gamma_2 \cdots \Gamma_m.$$

This decomposition is unique up to reordering of the components.

Conversely, any product of connected labeled graphs (sharing external vertices but with disjoint internal vertices and edges) yields a labeled graph that decomposes back into the original factors. The map from the symmetric algebra on connected graphs to $\mathsf{Graphs}_R(r)$ sending a formal product to the corresponding labeled graph is therefore a bijection on basis elements, and respects the grading and the Koszul sign conventions. Hence $\mathsf{Graphs}_R(r) \cong \mathrm{Sym}\big(\mathsf{Graphs}_R^{\mathrm{conn}}(r)\big)$ as graded-commutative algebras, where $\mathsf{Graphs}_R^{\mathrm{conn}}(r)$ denotes the submodule of connected graphs. $\square$

Remark 3.6.7. The freeness of $\mathsf{Graphs}_R(r)$ is one of the main advantages of the graph complex framework: to define a CDGA map out of $\mathsf{Graphs}_R(r)$, it suffices to specify its values on connected graphs and extend multiplicatively. This is the principle on which the integration map I of Section 3.7 will be constructed.

3.6.4 The differential

The differential on $\mathsf{Graphs}_R(r)$ is a sum of three terms, each with a distinct geometric meaning:

$$d = d_R + d_{\mathrm{split}} + d_{\mathrm{contr}}.$$

We define each term separately and then verify that their sum squares to zero.

Definition 3.6.8 (Internal differential)**.** The *internal differential* d_R applies the differential of the model R to each label, one vertex at a time:

$$d_R(\Gamma) = \sum_{v \in V(\Gamma)} \pm \Gamma\big|_{\ell(v) \mapsto d_R(\ell(v))},$$

with appropriate Koszul signs.

Remark 3.6.9. Geometrically, d_R corresponds to applying the exterior derivative to the label forms inside the integrand defining the integration map I.

Definition 3.6.10 (Edge-splitting differential)**.** The *edge-splitting differential* d_{split} removes one edge at a time and replaces it with the diagonal class. If $e = (v_i, v_j)$ is an edge of Γ and $\Delta_R = \sum_\alpha \pm e_\alpha \otimes e^\alpha$, then d_{split} acts on Γ by removing e and multiplying the labels at v_i and v_j by e_α and e^α respectively, summing over α with the appropriate Koszul signs.

Remark 3.6.11. The graph $d_{\mathrm{split}}(\Gamma)$ has one fewer edge than Γ but possibly more complicated labels. Geometrically, this corresponds to the bulk part of the Stokes formula: applying d to a propagator gives $d\varphi = \pi_{12}^*(\rho(\Delta_R))$, which replaces an edge by the diagonal class pulled back to its two endpoints.

Definition 3.6.12 (Contraction differential)**.** The *contraction differential* d_{contr} contracts edges incident to at least one internal vertex. Contracting an edge $e = (v_i, v_j)$ means:

(i) merge v_i and v_j into a single vertex, which is internal if both endpoints were internal, and external otherwise;

(ii) multiply the labels $\ell(v_i) \cdot \ell(v_j)$ at the merged vertex;

(iii) redirect all edges incident to v_i or v_j to the merged vertex, deleting any loops or double edges that result, with the appropriate Koszul signs.

Remark 3.6.13. Geometrically, d_{contr} corresponds to the boundary terms in the Stokes formula. When an internal vertex collides with another vertex, which happens at a boundary of $\mathrm{FM}_M(r+s)$, the integral picks up a boundary contribution equal to the integral of the contracted graph.

Theorem 3.6.14 ($\mathsf{Graphs}_R(r)$ is a CDGA)**.** *The total differential $d = d_R + d_{\mathrm{split}} + d_{\mathrm{contr}}$ satisfies $d^2 = 0$. Hence $(\mathsf{Graphs}_R(r), d)$ is a CDGA.*

Proof. We follow the verification given in [LV14, §6] and [Idr22, §5]. Expanding d^2 gives nine terms grouped into three families:

$$d^2 = d_R^2 + d_{\text{split}}^2 + d_{\text{contr}}^2 + \{d_R, d_{\text{split}}\} + \{d_R, d_{\text{contr}}\} + \{d_{\text{split}}, d_{\text{contr}}\},$$

where $\{X, Y\} = X \circ Y + Y \circ X$ denotes the graded anticommutator.

The first three terms vanish individually. Indeed $d_R^2 = 0$ because d_R acts vertex-wise by d_R in the model and $d_R^2 = 0$ in R. For d_{split}^2, removing two edges in succession produces a sum over ordered pairs of edges; the antisymmetry of the diagonal class under the swap and the Koszul signs cancel the two orderings against each other. For d_{contr}^2, contracting two edges in succession produces a sum over ordered pairs of edges sharing or not sharing a vertex, and the same antisymmetry argument gives cancellation, with the additional ingredient that contracting a triangle to a point yields zero by the Koszul signs.

The mixed terms vanish for the following reasons. The anticommutator $\{d_R, d_{\text{split}}\}$ vanishes because d_R acts on labels while d_{split} removes edges, and the two operations commute up to sign except for a contribution from $d_R(\Delta_R)$, which vanishes since Δ_R is a cocycle in $R \otimes R$. The anticommutator $\{d_R, d_{\text{contr}}\}$ vanishes because d_R commutes with merging vertices and multiplying labels, the Leibniz rule for d_R on the product of labels takes care of the sign. The most delicate term is $\{d_{\text{split}}, d_{\text{contr}}\}$, in which one operation removes an edge and the other contracts an edge: when both operations act on the same edge, the contributions cancel because the diagonal class produced by d_{split} and the merged labels produced by d_{contr} combine into the same expression with opposite signs. When they act on different edges, the operations commute up to the Koszul sign and again cancel pairwise.

□

3.6.5 The quotient map $p\colon \mathsf{Graphs}_R(r) \to \mathsf{G}_A(r)$

Definition 3.6.15 (The quotient map p)**.** Define $p\colon \mathsf{Graphs}_R(r) \to \mathsf{G}_A(r)$ on a labeled graph Γ as follows:

(i) if Γ contains at least one internal vertex, set $p(\Gamma) = 0$;

(ii) if Γ has no internal vertices, send each label $\ell(v_i) \in R$ to its image in A under the quasi-isomorphism $R \xrightarrow{\sim} A$, and send each edge (i, j) to the generator ω_{ij} of $\mathsf{G}_A(r)$.

Extend by $\mathbb{R}$-linearity.

Proposition 3.6.16. *The map p of Definition 3.6.15 is a surjective CDGA morphism: it respects the four relations (R1)–(R4) of Definition 3.2.9 and distributes well over the differentials.*

Proof. Surjectivity is immediate, since every generator ω_{ij} of $\mathsf{G}_A(r)$ is the image of the labeled graph with two external vertices i, j and a single edge between them, and every $\iota_i(a)$ is the image of the graph with no edges and label a on the i-th external vertex.

Compatibility with the relations is verified relation by relation. Relation (R1), the graded symmetry $\omega_{ji} = (-1)^n \omega_{ij}$, follows from the orientation conventions on edges, which assign the sign $(-1)^n$ to the reversal of an edge. Relation (R2), $\omega_{ij}^2 = 0$, follows from the prohibition on multiple edges between the same pair of vertices in

Definition 3.6.1. Relation (R4), the diagonal relation $\iota_i(a) \cdot \omega_{ij} = \iota_j(a) \cdot \omega_{ij}$, holds modulo the differential: the image under p of the difference $\iota_i(a) \cdot \omega_{ij} - \iota_j(a) \cdot \omega_{ij}$ corresponds to a graph that is d_{contr}-exact and therefore vanishes in $\mathsf{G}_A(r)$.

The verification of (R3), the Arnold relation, requires more care and is the content of Lemma 3.6.17 below.

For compatibility with the differentials, applying d_{G_A} to the image of an edge graph gives $\iota_{ij}(\Delta_A)$, which matches the image under p of d_{split} applied to that edge. The contraction differential vanishes under p because every graph with an internal vertex maps to zero, and the splitting differential preserves the property of having no internal vertices. □

Lemma 3.6.17 (Arnold relation up to homotopy)**.** *In $\mathsf{Graphs}_R(r)$, the triangle graph T_{ijk} on external vertices i, j, k with three edges $\omega_{ij}, \omega_{jk}, \omega_{ki}$ is not zero, but is d_{contr}-exact: there exists a graph S_{ijk} with one internal vertex connected to each of i, j, k such that $d_{\text{contr}}(S_{ijk}) = T_{ijk}$ up to sign and lower-order terms. Consequently, the Arnold relation holds in cohomology, and the quotient map p imposes it on the nose.*

Proof. Let S_{ijk} denote the graph with three external vertices i, j, k, one internal vertex v, and three edges (i, v), (j, v), (k, v), with all labels equal to 1_R. Applying d_{contr} to S_{ijk} contracts each of the three edges in turn, producing three terms corresponding to the three pairs of external vertices that get identified with v. After the contractions, each term is a graph with three external vertices i, j, k and the three remaining edges, that is, a triangle. Summing the three contractions with the appropriate Koszul signs yields the cyclic combination

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}$$

up to a global sign. Hence T_{ijk} is d_{contr}-exact, and the Arnold relation holds in $H^*(\mathsf{Graphs}_R(r))$. Such computation can also be found in [Idr22, §6.2]. □

3.7 The integration map

3.7.1 Construction

The integration map

$$I\colon \mathsf{Graphs}_R(r) \longrightarrow \Omega^*_{\text{PA}}(\text{FM}_M(r))$$

assigns to each graph Γ a differential form on $\text{FM}_M(r)$. The construction proceeds in two stages.

Definition 3.7.1 (The form $\omega(\Gamma)$)**.** Let Γ be a labeled graph with external vertices $\{1, \ldots, r\}$ and internal vertices $I = \{r+1, \ldots, r+s\}$. For each vertex v, let $p_v\colon \text{FM}_M(r+s) \to M$ be the projection remembering the v-th point. For each edge $e = (v_i, v_j)$, let $p_{v_iv_j}\colon \text{FM}_M(r+s) \to \text{FM}_M(2)$ be the projection remembering the two endpoints. The PA form $\omega(\Gamma) \in \Omega^*_{\text{PA}}(\text{FM}_M(r+s))$ is

$$\omega(\Gamma) = \Big(\prod_{v \in V(\Gamma)} p_v^*(\rho(\ell(v))) \Big) \cdot \Big(\prod_{e=(v_i,v_j) \in E(\Gamma)} p_{v_iv_j}^*(\varphi) \Big).$$

Definition 3.7.2 (The integration map I)**.** Let $\pi\colon \mathrm{FM}_M(r+s) \to \mathrm{FM}_M(r)$ be the projection forgetting the internal points $r+1,\dots,r+s$. This is an SA bundle whose fibers are compact. Define

$$I(\Gamma) = \pi_*\big(\omega(\Gamma)\big) \;\in\; \Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r)),$$

where π_* is the pushforward of Theorem 3.4.5. Extend I to all of $\mathsf{Graphs}_R(r)$ by $\mathbb{R}$-linearity and multiplicativity, using the freeness from Proposition 3.6.6.

Lemma 3.7.3 (Degree of $I(\Gamma)$)**.** *The form $I(\Gamma)$ has degree $|\Gamma|$ as defined in Definition 3.6.3.*

Proof. The pushforward π_* lowers degree by ns, the dimension of the fiber. The form $\omega(\Gamma)$ has degree $\sum_v |\ell(v)| + (n-1)|E(\Gamma)|$ by construction. Hence

$$|I(\Gamma)| = \sum_v |\ell(v)| + (n-1)|E(\Gamma)| - ns = |\Gamma|,$$

where the last equality uses $|I(\Gamma)| = s$ in Definition 3.6.3. □

3.7.2 I is a CDGA morphism

Theorem 3.7.4 (I is a CDGA morphism)**.** *The integration map $I\colon \mathsf{Graphs}_R(r) \to \Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$ of Definition 3.7.2 is a morphism of CDGAs.*

Proof. We follow the proof from [Idr19, §4] and [Idr22, §6].

We must verify two things: that I commutes with differentials, and that I is multiplicative.

For the chain-map property, apply the Stokes formula for the pushforward (Theorem 3.4.5, item (ii)) to the projection π and the form $\omega(\Gamma)$:

$$d \circ I(\Gamma) = d \circ \pi_*\big(\omega(\Gamma)\big) = \pi_*\big(d\omega(\Gamma)\big) + (\pi|_\partial)_*\big(\omega(\Gamma)|_\partial\big).$$

The first term, $\pi_*(d\omega(\Gamma))$, decomposes by the Leibniz rule into two contributions according to whether d falls on a label form or on a propagator. Applying d to the label forms gives terms corresponding to $I(d_R(\Gamma))$, since $d\rho(\ell(v)) = \rho(d_R\ell(v))$ because ρ is a chain map. Applying d to the propagator forms gives terms corresponding to $I(d_{\mathrm{split}}(\Gamma))$, using the propagator property $d\varphi = \pi_{12}^*(\rho(\Delta_R))$, which replaces an edge by the diagonal class pulled back to its two endpoints.

The second term, the boundary contribution $(\pi|_\partial)_*(\omega(\Gamma)|_\partial)$, corresponds to $I(d_{\mathrm{contr}}(\Gamma))$. The boundary of $\mathrm{FM}_M(r+s)$ where internal vertices collide with one another or with external vertices produces precisely the contracted graphs. The contributions from boundary strata where two non-adjacent vertices collide vanish by a degree-counting argument: the propagator integral over a sphere of dimension less than $n-1$ is zero. Hence only the strata corresponding to contractions of edges incident to an internal vertex survive, which is exactly the content of d_{contr}.

Combining the three contributions,

$$d \circ I(\Gamma) = I(d_R(\Gamma)) + I(d_{\mathrm{split}}(\Gamma)) + I(d_{\mathrm{contr}}(\Gamma)) = I(d\Gamma),$$

which is the chain-map property.

For multiplicativity, the product $\Gamma_1 \cdot \Gamma_2$ in $\mathsf{Graphs}_R(r)$ is defined by disjoint union of internal vertices and edges, and the corresponding forms $\omega(\Gamma_1)$ and $\omega(\Gamma_2)$ are pulled back from disjoint sets of factors in $\mathrm{FM}_M(r+s_1+s_2)$. Their product $\omega(\Gamma_1)\cdot\omega(\Gamma_2)$ equals $\omega(\Gamma_1 \cdot \Gamma_2)$, and the pushforward along π commutes with the wedge product because the fibers of π factor as the product of the fibers integrating out the two disjoint sets of internal vertices. Hence $I(\Gamma_1 \cdot \Gamma_2) = I(\Gamma_1) \cdot I(\Gamma_2)$. □

3.8 The partition function and its vanishing

3.8.1 The partition function

The quotient map p sends all graphs with internal vertices to zero. To analyze the kernel of p we extract a single scalar invariant from the graph complex with no external vertices.

Definition 3.8.1 (The partition function)**.** Consider the CDGA $\mathsf{Graphs}_R(0)$, the graph complex with no external vertices, only internal ones. The *partition function* is the element

$$Z = \sum_{\substack{\Gamma \text{ connected,} \\ \text{no ext. vertices}}} \frac{I(\Gamma)}{|\mathrm{Aut}(\Gamma)|} \in \mathbb{R},$$

where the sum runs over all connected graphs with only internal vertices, and the automorphisms account for the symmetry of the internal vertices.

Remark 3.8.2. The partition function Z is a Maurer–Cartan element in a completed graded Lie algebra associated with vacuum graphs. Its vanishing or non-vanishing determines whether the graph complex can be simplified by removing the internal vertices: if $Z \neq 0$, one must twist the differential of $\mathsf{G}_A(r)$ by Z, replacing d by $d+[Z,-]$; if $Z = 0$, no twisting is needed and the quotient map p is a quasi-isomorphism. The Maurer–Cartan formalism is developed in detail in [Idr22, §5.3].

3.8.2 The vanishing theorem

Theorem 3.8.3. *If M is simply connected, closed, and of dimension $n \geq 4$, then $Z = 0$.*

Proof. We follow the elementary degree-counting argument of [Idr19, §5]. We show that no connected graph Γ with $s \geq 1$ internal vertices and no external vertices can contribute a nonzero term to Z.

For $I(\Gamma)$ to be a nonzero real number, that is, a 0-form, the degree of $\omega(\Gamma)$ must equal ns, the dimension of the fiber over which we integrate. Equivalently,

$$\sum_{v \in V(\Gamma)} |\ell(v)| + (n-1)e \;=\; ns, \qquad \text{or} \qquad \sum_{v} |\ell(v)| = ns - (n-1)e. \tag{3.1}$$

Two facts constrain the contributing graphs.

Lemma 3.8.4 (Label-degree constraint)**.** *Since M is simply connected, the model R satisfies $R^0 = \mathbb{R}$ and $R^1 = 0$. Hence every label has degree 0, in which case it is a scalar that can be absorbed into the coefficient, or degree at least 2.*

Proof. Simple connectedness implies $H^0(M;\mathbb{R}) = \mathbb{R}$ and $H^1(M;\mathbb{R}) = 0$ by Hurewicz. The cofibrant model R may be chosen with the same property in low degrees, so $R^0 = \mathbb{R}$ and $R^1 = 0$. $\square$

Lemma 3.8.5 (Valence constraint)**.** *For a graph to contribute a nonzero term in* Graphs_R *modulo the dead-end relations, every internal vertex with a scalar label must have valence at least* 3*. Vertices of valence* 1 *or* 2 *with scalar labels contribute zero, by the normalization of the propagator and an explicit computation.*

Proof. The argument is given in [Idr19, §5] and in [Idr22, §5.4]. A vertex of valence 1 with a scalar label produces a factor that is the integral of the propagator along a fiber, which vanishes by the propagator normalization (P3). A vertex of valence 2 with a scalar label produces a factor that is the convolution of two propagators along a fiber, which vanishes by the symmetry property (P2) and a degree-counting argument identical to the one in the main proof. $\square$

We now combine the two lemmas in two cases.

In the first case, every internal vertex has a scalar label. Then by Lemma 3.8.5 every vertex has valence at least 3, and the handshaking lemma gives $2e = \sum_v \mathrm{val}(v) \geq 3s$, that is, $e \geq 3s/2$. Substituting into (3.1),

$$\sum_v |\ell(v)| = ns - (n-1)e \leq ns - (n-1) \cdot \tfrac{3s}{2} = s \cdot \tfrac{3-n}{2}.$$

For $n \geq 4$, the right-hand side is strictly negative, while the left-hand side is non-negative. The inequality is impossible, so no such graph contributes.

In the second case, some vertices carry non-scalar labels. Let k be the number of vertices with labels of degree at least 2, and $s-k$ the number of vertices with scalar labels. By Lemma 3.8.4, $\sum_v |\ell(v)| \geq 2k$. The scalar-labeled vertices must have valence at least 3 by Lemma 3.8.5, while the non-scalar-labeled vertices may have valence as low as 1, the minimum needed for connectivity. The handshaking lemma now gives $2e \geq 3(s-k) + k = 3s - 2k$, that is, $e \geq (3s-2k)/2$. Combined with (3.1),

$$2k \;\leq\; \sum_v |\ell(v)| \;=\; ns - (n-1)e \;\leq\; ns - (n-1) \cdot \tfrac{3s-2k}{2}.$$

Rearranging, $4k \leq s(3-n) + 2k(n-1)$, hence $2k(3-n) \leq s(3-n)$. For $n \geq 4$, both factors $(3-n)$ are negative; dividing reverses the inequality and yields $2k \geq s$. But then $\sum |\ell(v)| \geq 2k \geq s > s(3-n)/2$, which contradicts the upper bound from (3.1). Again no graph contributes.

In both cases, no graph Γ contributes a nonzero term to Z, and the partition function vanishes. $\square$

Remark 3.8.6 (The cases $n=2$ and $n=3$). When $n=2$ or $n=3$, the degree-counting argument above fails: the bound $\sum |\ell(v)| \leq s(3-n)/2$ gives $s/2$ or 0, which does not lead to a contradiction. However, the only simply connected closed manifolds in these dimensions are spheres, namely S^2 and S^3, the latter by the Poincaré conjecture. For spheres, the result is proved by separate arguments: in the case of S^2, one uses the formality of the framed little 2-disks operad, and in the case of S^3, one uses the formality of odd spheres together with the explicit model of $\mathrm{Conf}_r(S^3)$. One can find the details in [Idr19].

3.9 The spectral sequence comparison and conclusion of the proof

We have now assembled everything. On one side stands the algebraic, finite-dimensional Lambrechts–Stanley CDGA $\mathsf{G}_A(r)$ of Section 3.2; on the other, the much larger geometric CDGA of piecewise semi-algebraic forms $\Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$ on the Fulton–MacPherson compactification. Joining them is the labelled graph complex $\mathsf{Graphs}_R(r)$ of Section 3.6, with its quotient map p to the left and its integration map I to the right. What we want to do now is to prove that both p and I are quasi-isomorphisms. For the quotient map p, the vanishing $Z = 0$ established in Theorem 3.8.3 reduces the problem to a relatively easy acyclicity statement about a subcomplex. For the integration map I, the argument is necessarily deeper: it amounts to showing that the graph complex $\mathsf{Graphs}_R(r)$ really does compute the cohomology of $\mathrm{FM}_M(r)$, and the proof goes via a comparison of two spectral sequences, one on the graph side and one on the form side, that are matched, term by term, by I itself.

The proof here has a classical pattern in homotopy theory: one constructs an intermediate free or cofibrant object placed between the algebraic and the geometric model, and one shows that the legs of the resulting zigzag are quasi-isomorphisms. The graph complex $\mathsf{Graphs}_R(r)$ is such an intermediate object, with the additional property that its quasi-isomorphism class is governed by a single computable scalar, the partition function Z.

We treat the two maps in turn, p first and I second.

3.9.1 The quotient map p is a quasi-isomorphism

Theorem 3.9.1. *Let M be a smooth, simply connected, closed manifold of dimension $n \geq 4$. Then the natural CDGA projection*

$$p\colon \mathsf{Graphs}_R(r) \longrightarrow \mathsf{G}_A(r)$$

is a quasi-isomorphism for every $r \geq 0$.

The proof relies on a structural decomposition of $\mathsf{Graphs}_R(r)$ into external and vacuum components, which we record as a separate lemma since it will be used in the discussion of the partition function below.

Lemma 3.9.2 (Vacuum decomposition)**.** *There is a tensor-product splitting at the level of graded vector spaces,*

$$\mathsf{Graphs}_R(r) \;\cong\; \mathsf{Graphs}^{\mathrm{ext}}_R(r) \;\otimes\; \mathrm{Sym}\big(\mathsf{fGC}_R\big),$$

where $\mathsf{Graphs}^{\mathrm{ext}}_R(r)$ is the subcomplex of graphs whose every connected component contains at least one external vertex, and fGC_R is the reduced connected graph complex of vacuum graphs, those with internal vertices and no external ones. The total differential of $\mathsf{Graphs}_R(r)$ is compatible with this splitting after the identification that the $\mathrm{Sym}(\mathsf{fGC}_R)$ factor inherits the Chevalley–Eilenberg differential of fGC_R twisted by the Maurer–Cartan element

$$Z = \sum_{\Gamma} \frac{I(\Gamma)}{|\mathrm{Aut}(\Gamma)|}\, \Gamma \;\in\; \mathsf{fGC}_R,$$

the sum running over connected vacuum graphs.

Proof. Every graph $\Gamma \in \mathsf{Graphs}_R(r)$ admits a canonical decomposition into connected components: some components contain at least one external vertex (the external components), the remaining components contain only internal vertices (the vacuum components). Because the multiplication on $\mathsf{Graphs}_R(r)$ is given by disjoint union, this decomposition produces the tensor-product splitting at the level of graded vector spaces.

The verification that Z is Maurer–Cartan, that is, $dZ + \frac{1}{2}[Z, Z] = 0$ in the natural Lie algebra structure on fGC_R, is a consequence of the Stokes formula on FM_M, and is carried out in [Idr19, §6] and [Idr22, §5.3]. The construction of the splitting and the twist has its origins in perturbative quantum field theory [Kon94, AS94, Kon99]. $\square$

Proof of Theorem 3.9.1. We follow [Idr19, §7]. We show that the kernel of p has trivial cohomology; the result then follows from the long exact sequence in cohomology associated with the short exact sequence $0 \to \ker p \to \mathsf{Graphs}_R(r) \xrightarrow{p} \mathsf{G}_A(r) \to 0$. By construction, $\ker p$ is the subcomplex of $\mathsf{Graphs}_R(r)$ spanned by graphs containing at least one internal vertex.

By Lemma 3.9.2, the kernel decomposes into external and vacuum components, with the vacuum factor $\mathrm{Sym}(\mathsf{fGC}_R)$ carrying the Chevalley–Eilenberg differential twisted by the Maurer–Cartan element Z. Invoking Theorem 3.8.3, the partition function Z vanishes when M is simply connected, closed, of dimension $n \geq 4$. Substituting $Z = 0$ in the twisted Chevalley–Eilenberg differential reduces it to the untwisted one, and the symmetric algebra of an acyclic complex is itself acyclic in positive degrees. Hence the $\mathrm{Sym}(\mathsf{fGC}_R)$ factor contributes only its constant part to the cohomology, and we are reduced to computing $H^*(\mathsf{Graphs}_R^{\mathrm{ext}}(r) \cap \ker p)$.

To finish, filter $\mathsf{Graphs}_R^{\mathrm{ext}}(r) \cap \ker p$ by the number of internal vertices. The internal differential d_R preserves this number, the splitting differential d_{split} also preserves it (it removes an edge but does not touch the vertex set), and only the contraction differential d_{contr} decreases it by one. On the associated graded, the total differential simplifies to $d_R + d_{\mathrm{split}}$. On this associated graded, an explicit chain homotopy is available: contract any chosen edge from an internal vertex of maximal valence to its neighbour, as constructed in [Idr19, §7]. This provides a contracting homotopy on each fixed-internal-vertex layer, and induction on the number of internal vertices collapses the entire filtration. The conclusion is $H^*(\ker p) = 0$, as desired. $\square$

3.9.2 The integration map I is a quasi-isomorphism

We turn now to the second part of the proof. The strategy, due to Kontsevich [Kon99] and used for closed manifolds by Idrissi [Idr19, §7], [Idr22, §6], is to compare two spectral sequences: one obtained by filtering the graph complex by the number of edges, the other obtained from the geometry of $\mathrm{FM}_M(r)$ via the codimension of diagonal strata in M^r. The map I will be shown to respect both filtrations, and the induced map at the E^1-page will be exactly the projection p studied above. By the comparison theorem for spectral sequences, I itself is a quasi-isomorphism on cohomology.

We need a particular spectral sequence on the form side, due in its original form to Cohen and Taylor [CT78] and extended to closed manifolds by Bendersky and Gitler [BG91] and later by Totaro [Tot96].

Definition 3.9.3 (CTBG spectral sequence)**.** The *Cohen–Taylor–Bendersky–Gitler spectral sequence*, abbreviated CTBG, arises from the inclusion $\mathrm{Conf}_r(M) \hookrightarrow M^r$ as the complement of the union of the pairwise diagonals $\Delta_{ij} = \{x \in M^r : x_i = x_j\}$, each of which is a closed submanifold of codimension n. The codimension filtration of M^r by these diagonals[2] pulls back to a filtration of $\Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(M))$, and the associated spectral sequence has E_2-page generated, as an algebra over $H^*(M^r) = A^{\otimes r}$, by classes ω_{ij} of degree $n-1$ subject to the Arnold relations [Arn69, Coh76].

Remark 3.9.4. For formal M, the CTBG spectral sequence collapses at E_2 and yields $H^*(\mathsf{G}_A(r))$. For general M, it has a non-trivial differential $d_2(\omega_{ij}) = \iota_{ij}(\Delta_A)$ encoding the Poincaré dual of the diagonal class, after which it collapses [LS04]. In all cases, the abutment is $H^*(\mathrm{Conf}_r(M);\mathbb{R})$.

Theorem 3.9.5. *Under the hypotheses of Theorem 3.1.1, the integration map*

$$I\colon \mathsf{Graphs}_R(r) \longrightarrow \Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$$

is a quasi-isomorphism of CDGAs.

Proof. We follow the proof from [Idr19, §7] and [Idr22, §6]. The argument is a comparison of two spectral sequences, set up in three steps: the filtrations on each side, the compatibility of I with the filtrations, the identification of the E^1-page with the projection p.

On the graph side, define an increasing filtration by

$$F_p\mathsf{Graphs}_R(r) = \big\{\Gamma \ :\ |E(\Gamma)| \le p\big\},$$

the subspace spanned by graphs with at most p edges. Of the three components of the total differential, the internal differential d_R preserves the edge count, while d_{split} and d_{contr} each remove a single edge. Hence $d \cdot F_p \subseteq F_p$, the filtration is respected by d, and we obtain a homologically well-defined spectral sequence. The cohomological degree of a graph is $|\Gamma| = \sum_v |\ell(v)| + (n-1)|E(\Gamma)| - n|I(\Gamma)|$, in which the edge count enters with positive coefficient $n-1 \ge 3$; for fixed total degree, $|E(\Gamma)|$ is therefore bounded above, the filtration is bounded in each total degree, and the spectral sequence converges [McC01, Ch. 3].

On the form side, pull back the codimension filtration of M^r along the natural projection $\mathrm{FM}_M(r) \to M^r$, and consider the induced increasing filtration on $\Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$ by polynomial degree in the diagonal classes. This is precisely the CTBG filtration of Definition 3.9.3, in its dual presentation. This filtration is also bounded and the associated spectral sequence converges to $H^*(\mathrm{Conf}_r(M);\mathbb{R})$.

Compatibility of I with the filtrations. The integration map I sends a graph Γ with $|E(\Gamma)|$ edges to a form built from $|E(\Gamma)|$ pullbacks of the propagator φ, integrated along the fibre of the projection $\mathrm{FM}_M(r+|I|) \to \mathrm{FM}_M(r)$. Each propagator contributes one factor that is, on the boundary, dual to the codimension-n pairwise diagonal in $M \times M$ (this is condition (P3) of Definition 3.5.3, together with the cohomological identity $d\varphi = \pi^*_{12}\rho(\Delta_R)$). Consequently $I(F_p) \subseteq F_p^{\mathrm{CTBG}}$, where F_p^{CTBG} denotes the p-th step of the CTBG filtration on the form side. We thus obtain a well-defined morphism of spectral sequences induced by I.

The graph-side E^1. On the associated graded $\mathrm{Gr}_p\mathsf{Graphs}_R(r) = F_p/F_{p-1}$, the only piece of the total differential that survives is d_R, since both d_{split} and d_{contr} strictly

[2]The codimension filtration is the filtration of M^r by closed subsets indexed by how many independent diagonal coincidences are imposed: the locus of p simultaneous coincidences has codimension at least np.

decrease the edge count and so vanish on the associated graded. Now d_R acts on each label $\ell(v) \in R$ independently, so its cohomology is computed label by label. Recall that R was chosen as a cofibrant CDGA together with a quasi-isomorphism $R \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$, and that A is the Poincaré duality model with another quasi-isomorphism $A \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$. By two-out-of-three for quasi-isomorphisms in the model category of CDGAs ([Hir03, §11], [Hov99]), $H^*(R) \cong A$ as graded algebras. We thus identify the E^1-page on the graph side as

$$E^1_{\text{graphs}} = \mathsf{Graphs}_A(r),$$

the graph complex with labels in A rather than R, equipped with the residual differential $d_{\text{split}} + d_{\text{contr}}$.

The form-side E^1. The E^1-page of the CTBG spectral sequence is the cohomology of the associated graded. By the Thom isomorphism applied to the codimension-n pairwise diagonals $\Delta_{ij} \subset M^r$, the contribution at filtration level p is freely generated, as an $H^*(M^r)$-module, by the products $\omega_{i_1j_1}\omega_{i_2j_2}\cdots\omega_{i_pj_p}$ subject to the Arnold relations [Coh76, FN62]. In other words, the form-side E^1 is exactly the subcomplex of $\mathsf{Graphs}_A(r)$ spanned by graphs with no internal vertices, modulo Arnold relations:

$$E^1_{\text{forms}} \cong \mathsf{Graphs}^{\text{ext}}_A(r),$$

which we recognise as the image of the projection $p\colon \mathsf{Graphs}_A(r) \to \mathsf{G}_A(r)$.

Comparison at E^1 via the partition function. The two E^1-pages differ by exactly the same content that distinguishes $\mathsf{Graphs}_R(r)$ from $\mathsf{G}_A(r)$: the graph-side E^1 contains internal vertices, the form-side E^1 does not. The comparison map $E^1(I)\colon E^1_{\text{graphs}} \to E^1_{\text{forms}}$ induced by I on the E^1-pages coincides, after the identifications above, with the projection p now applied with A-labels in place of R-labels. By Theorem 3.9.1, whose proof depends only on the partition function Z and not on whether the labels live in R or A, this projection is a quasi-isomorphism whenever $Z = 0$. Hence $E^1(I)$ is an isomorphism on cohomology, and so is the induced map on E^2.

Conclusion. Both spectral sequences are bounded and converge: the graph side as already observed, the form side by convergence of the Cohen–Taylor–Bendersky–Gitler construction [CT78, BG91, Tot96]. Since I is a filtration-preserving chain map of bounded spectral sequences and induces an isomorphism on E^1-pages, the comparison theorem for spectral sequences [McC01, Ch. 5] implies that I induces an isomorphism on the abutments. Hence I is a quasi-isomorphism of CDGAs.

We conclude that $H^*(I)$ is an isomorphism, that is, I is a quasi-isomorphism of CDGAs. □

3.9.3 Combining the zigzag

We have shown that both maps in the zigzag

$$\mathsf{G}_A(r) \underset{\sim}{\xleftarrow{p}} \mathsf{Graphs}_R(r) \underset{\sim}{\xrightarrow{I}} \Omega^*_{\mathrm{PA}}(\mathrm{FM}_M(r))$$

are quasi-isomorphisms of CDGAs. By the Hardt–Lambrechts–Turchin–Volić comparison theorem [HLTV11], $\Omega^*_{\mathrm{PA}}(X)$ is quasi-isomorphic to $\Omega^*_{\mathrm{dR}}(X)$ for every smooth semi-algebraic X, in a manner natural with respect to SA maps. Combined with the homotopy equivalence $\mathrm{Conf}_r(M) \simeq \mathrm{FM}_M(r)$ proved in [Sin04, §4], we deduce a quasi-isomorphism

$$\mathsf{G}_A(r) \simeq \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(M))$$

in the homotopy category of CDGAs. In Sullivan's sense [Sul77, FHT01], this exhibits $\mathsf{G}_A(r)$ as a real model for $\mathrm{Conf}_r(M)$, completing the proof of Theorem 3.1.1.

Corollary 3.9.6. *If M and N are two smooth, simply connected, closed manifolds with the same real homotopy type, a common Poincaré duality model A may be chosen for both, as one can look at in [LS08], and the CDGAs $\mathsf{G}_A(r)$ produced from A are by their very definition identical. Applying the zigzag separately to M and to N yields*

$$\Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(M)) \simeq \mathsf{G}_A(r) \simeq \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(N)),$$

hence $\mathrm{Conf}_r(M)$ and $\mathrm{Conf}_r(N)$ have the same real homotopy type.

3.9.4 Independence of the auxiliary data

A delicate point concerns the dependence of the construction on various non-canonical data: the cofibrant model R, the propagator φ, and the embedding $M \hookrightarrow \mathbb{R}^N$ used to define FM_M. None of these choices is canonical, but the resulting CDGA $\mathsf{G}_A(r)$ is.

Proposition 3.9.7. *The CDGA $\mathsf{G}_A(r)$, viewed up to quasi-isomorphism, is independent of the choices of cofibrant model R, propagator φ, and embedding $M \hookrightarrow \mathbb{R}^N$.*

Proof. We follow [Idr19, §6.4].

The projection p depends only on A, not on R, since R is collapsed onto A by the chosen quasi-isomorphism $R \xrightarrow{\sim} A$, which exists because cofibrant replacements are unique up to homotopy in the Bousfield–Gugenheim model structure ([BG76], [Hir03, §11]).

Two different propagators φ and φ' differ by a closed PA form of degree $n-1$ on $\mathrm{FM}_M(2)$, and a homotopy of integration maps is constructed by integrating along the interpolating family $\varphi_t = (1-t)\varphi + t\varphi'$, which remains a valid propagator for every $t \in [0,1]$. This homotopy is a CDGA homotopy in the standard sense of model categories [Hov99] and lifts to a quasi-isomorphism between the resulting graph complexes; the full verification is in [Idr19, §6.4].

Independence from the embedding $M \hookrightarrow \mathbb{R}^N$ is more transparent: passing to a larger embedding induces an inclusion of Fulton–MacPherson spaces that is a homeomorphism on the interior, and I is functorial under such inclusions. □

Putting all of this together, we have proved that for every smooth, simply connected, closed manifold M of dimension $n \geq 4$, the CDGA $\mathsf{G}_A(r)$ depends only on the real homotopy type of M and is a real model for $\mathrm{Conf}_r(M)$. This is the full statement of Theorem 3.1.1, and with it Idrissi's theorem on the real homotopy invariance of configuration spaces is established.

Still open question

Given two simply connected closed manifolds M and N with $M \simeq N$ (homotopy equivalent), is $F_k(M) \simeq F_k(N)$ for all $k \geq 2$?

This is open for all $k \geq 2$ in full generality.

Positive results

(P1) The loop space is invariant (Levitt, 1995). For any closed M, the based loop space $\Omega F_k(M)$ is a homotopy invariant of M.

(P2) Stable invariance (Aouina–Klein, 2004; Knudsen; Malin, 2022). A suitable iterated suspension $\Sigma^j F_k(M)$ is a homotopy invariant. The number of suspensions j depends on k, $\dim M$, and the connectivity of M. The suspension spectrum $\Sigma^\infty_+ F_k(M)$ is always a homotopy invariant. In particular, $\Sigma F_2(M)$ is a homotopy invariant whenever M is simply connected.

(P3) Two-connected case ($k = 2$): Levitt 1995. If M is 2-connected and closed, then $F_2(M)$ is a homotopy invariant of M. The proof uses a general-position argument: the diagonal has codimension $\geq n$, and 2-connectedness gives enough room to deform homotopy equivalences to embeddings transverse to the diagonal.

(P4) Rational invariance for $F_2(M)$, even-dimensional case (Cordova Bulens, 2015). When M is simply connected, closed, even-dimensional, the rational homotopy type of $F_2(M)$ depends only on the rational homotopy type of M. The odd-dimensional case is more subtle: rational invariance for $F_2(M)$ depends on a single class $x \in H^{n-2}(M;\mathbb{Q})$, and is not yet fully resolved.

(P5) Rational invariance under 2-connectedness, all k (Lambrechts–Stanley). If M is 2-connected closed, then the rational homotopy type of $F_k(M)$ depends only on the rational homotopy type of M for all k, via an explicit Sullivan model.

(P6) Real invariance, simply connected, dim ≥ 4 **(Idrissi 2019/2022).** Over $\mathbb{R}$, the result holds with only simple connectedness and the dimension bound $n \geq 4$. The dimensions $n = 2, 3$ are spheres and handled separately.

(P7) Manifolds with boundary, real version (Campos–Idrissi–Lambrechts–Willwacher 2018). Same conclusion as Idrissi for compact manifolds with simply connected boundary.

(P8) Smooth complex projective varieties, all k (Kriz, then Totaro, then Lambrechts–Stanley). For smooth complex projective M, $F_k(M)$ has a rational model coming from $H^*(M;\mathbb{Q})$, so rational invariance holds.

(P9) Knudsen's stable result. The equivariant stable homotopy type of $F_k(M)$ is a proper homotopy invariant of M.

Counterexamples and obstructions

(N1) Without simple connectedness: Longoni–Salvatore 2005. $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent but $F_2(L_{7,1}) \not\simeq F_2(L_{7,2})$, detected by Massey products on the universal cover.

(N2) Evans-Lee (2015). Provides further evidence that any pair of homotopy equivalent but non-homeomorphic lens spaces gives a counterexample.

(N3) Simple homotopy invariance is also open (Raptis–Salvatore 2016). If we restrict to *simple* homotopy equivalence, no counterexample is known. Multiplying by S^1 produces simple-homotopy equivalences from homotopy equivalences. Raptis–Salvatore prove an invariance result for $F_2(M \times S^1)$ in this setting. The lens space counterexample exploits exactly the failure of simple-homotopy equivalence, since the Whitehead torsion is nonzero.

Constraints on a simply-connected counterexample

A simply connected counterexample (M, N) with $M \simeq N$ closed and $F_k(M) \not\simeq F_k(N)$ would hopefully satisfy the following conditions and at the same times imply the following constraint on a possible proof of counterexample:

- $\pi_2(M) = \pi_2(N) \neq 0$, else (P3) gives invariance for $k = 2$, otherwise one should look at higher configurations.
- M and N have the same real homotopy type and rational homotopy type, so the obstruction must be torsion-detected, by (P6) and (P5/P8 in good cases).
- Iterated enough suspensions $\Sigma^i F_k(M)$ and $\Sigma^i F_k(N)$ must be equivalent (P2). Massey products are killed by suspension, so they cannot detect a simply connected counterexample.
- Any obstruction must come from torsion Massey products or other torsion-sensitive secondary operations.
- The loop space $\Omega F_k(M) \simeq \Omega F_k(N)$ (P1). The difference must be detected at the level of F_k itself, not its loop space, so by something like a k-invariant or higher Postnikov data.
- In the simply connected case, $\mathrm{Wh}(\pi_1) = \mathrm{Wh}(1) = 0$, so all homotopy equivalences are automatically simple. The Raptis–Salvatore observation that no counterexample is known to simple-homotopy invariance therefore applies in full force.

Counterexample candidates

(C1) Simply connected closed 4-manifolds. By Freedman's classification, simply connected closed topological 4-manifolds are classified up to homeomorphism by their intersection form together with the Kirby–Siebenmann invariant. Two simply connected closed smooth 4-manifolds with the same intersection form are homotopy equivalent, and there are pairs that are homotopy equivalent but not diffeomorphic (exotic smooth structures). $\pi_2 \neq 0$ for any interesting simply connected 4-manifold, satisfying constraint (P3).

In dimension 4, homotopy-equivalent-but-not-homeomorphic simply connected closed manifolds are rare in the smooth category, since simply-connected-with-same-intersection-form often forces homeomorphism (Freedman). The closest candidates are pairs that are homeomorphic but not diffeomorphic, which makes F_k automatically equal.

(C2) Higher-dimensional homotopy-equivalent simply connected manifolds with different smooth/PL structures. In dimensions ≥ 5, surgery theory produces many homotopy equivalent simply connected closed manifolds. Milnor exotic spheres are homotopy equivalent to S^7 but not diffeomorphic; however S^7 is 6-connected so (P3) and stronger applies.

Wall classified simply connected closed 5- and 6-manifolds; there are pairs that are homotopy equivalent without being diffeomorphic. In dimension ≥ 5, one can often find homotopy-equivalent manifolds distinguished only by Pontryagin classes or by Whitehead torsion-like data, and these are precisely the kinds of invariants that suspension would kill.

(C3) The simple homotopy version. Raptis–Salvatore's result suggests that the conjecture might be true for simple homotopy equivalences. A counterexample, if it exists, would have to be a homotopy equivalence that is not a simple homotopy equivalence. In the simply connected case, $\mathrm{Wh}(\pi_1) = 0$, so all homotopy equivalences are automatically simple, leaving no room for this kind of counterexample.

3.10 Candidate counterexamples and computations

We work through several candidate pairs of homotopy equivalent simply connected closed manifolds, in increasing dimension, and test whether their configuration spaces can be distinguished by standard invariants. The goal is twofold: to confirm invariance where possible, and to identify which invariants survive (or fail to survive) to F_k.

3.10.1 First attempt: dimension 5, Smale manifolds

Smale (1962) classified simply connected closed 5-manifolds. The simply connected closed *spin* 5-manifolds are determined up to diffeomorphism by their second homology $H_2(M;\mathbb{Z})$, so two such manifolds with the same H_2 are diffeomorphic, and the question of homotopy invariance of F_k is trivially answered. To get homotopy-equivalent-but-not-diffeomorphic pairs in dimension 5, one would need to consider non-spin simply connected 5-manifolds, or move up in dimension.

3.10.2 Dimension 6, Wall manifolds

Wall (1966) classified simply connected closed smooth 6-manifolds with torsion-free homology. The classification invariants are:

- $H = H_2(M;\mathbb{Z})$, a finitely generated free abelian group;
- a trilinear form $\mu\colon H \otimes H \otimes H \to \mathbb{Z}$ (the cup product);
- a linear form $p_1\colon H \to \mathbb{Z}$ (the first Pontryagin class, in $H^4 \cong H^*$ via Poincaré duality);
- a linear form $w_2\colon H \to \mathbb{Z}/2$ (the second Stiefel–Whitney class).

Two such manifolds are diffeomorphic iff there is an isomorphism $H \to H'$ identifying all four invariants. Two such manifolds are *homotopy equivalent* iff there is an isomorphism $H \to H'$ identifying μ and w_2, but possibly not p_1. Hence p_1 is a diffeomorphism invariant but *not* a homotopy invariant, and varying p_1 produces homotopy-equivalent-but-not-diffeomorphic pairs.

3.10.3 The $\mathbb{CP}^3$ family

Take $M_0 = \mathbb{CP}^3$. Its Wall invariants are:

- $H = \mathbb{Z}$, generator $x \in H^2$;

- $\mu(x,x,x) = x^3 =$ generator of H^6, so $\mu = 1$;
- $w_2 = 0$, since $\mathbb{CP}^n$ is spin iff n is odd;
- $p_1 = 4x^2$, from the standard formula $c(T\mathbb{CP}^n) = (1+x)^{n+1}$ which gives $c_1 = 4x$, $c_2 = 6x^2$, hence $p_1 = c_1^2 - 2c_2 = 4x^2$.

By Wall's theorem, any simply connected closed smooth 6-manifold with torsion-free homology, $H = \mathbb{Z}$, $\mu(x,x,x) = 1$, $w_2 = 0$ is homotopy equivalent to $\mathbb{CP}^3$. Varying p_1 over the multiples of $4x^2$ realizable by smooth manifolds with the given invariants gives an infinite family $\{M_k\}_{k\in\mathbb{Z}}$ with $p_1(M_k) = (4+24k)x^2$, all pairwise non-diffeomorphic but all homotopy equivalent to $\mathbb{CP}^3$.

This is our test pair: $M_0 = \mathbb{CP}^3$ and M_1 with $p_1 = 28x^2$.

3.10.4 The Lambrechts–Stanley model is the same for M_0 and M_1

The Poincaré duality model A depends only on the rational cohomology ring as a Poincaré duality algebra. For both M_0 and M_1,

$$H^*(M_k;\mathbb{Q}) = \mathbb{Q}[x]/(x^4), \qquad |x| = 2, \qquad \int x^3 = 1.$$

The diagonal class is

$$\Delta_A = 1\otimes x^3 + x\otimes x^2 + x^2\otimes x + x^3\otimes 1 \in (A\otimes A)^6.$$

The Lambrechts–Stanley model

$$\mathsf{G}_A(2) = (A\otimes A)\otimes\Lambda(\omega_{12})/(\iota_1(a)\omega_{12} = \iota_2(a)\omega_{12}), \qquad |\omega_{12}| = 5, \qquad d\omega_{12} = \Delta_A,$$

is therefore identical for M_0 and M_1. Hence $F_2(M_0)$ and $F_2(M_1)$ have the same rational homotopy type by Lambrechts–Stanley, and the same real homotopy type by Idrissi's theorem. This was already guaranteed by the general theorem.

3.10.5 Integral cohomology of F_2 via CTBG

The integral cohomology ring of $F_2(M)$ for a Poincaré duality manifold M is computed by the integral Cohen–Taylor–Bendersky–Gitler spectral sequence. For our pair, the input data on the E_2 page (which is $H^*(M^2;\mathbb{Z})\otimes H^*(F_2(\mathbb{R}^6);\mathbb{Z})$) is identical, since the integral cohomology rings of M_0 and M_1 are isomorphic. The differential $d_6(\omega_{12}) = \Delta_M$ is also the same, since the diagonal class depends only on the integral cohomology ring as a Poincaré duality algebra.

Consequently

$$H^*(F_2(M_0);\mathbb{Z}) \cong H^*(F_2(M_1);\mathbb{Z})$$

as graded rings.

3.10.6 What survives, what does not

The Pontryagin class p_1, which distinguishes M_0 and M_1 as smooth manifolds, is invisible to the integral cohomology of F_2. If a counterexample distinguishing M_0 and M_1 exists at the level of F_2, it must come from finer invariants. The candidates are:

(1) **Higher Massey products** in $H^*(F_2(M_k);\mathbb{Z})$. By formality of $\mathbb{CP}^3$ over $\mathbb{Q}$ (Kähler) and the fact that M_1 has the same rational homotopy type, all rational Massey products vanish on both. Torsion Massey products are a separate question, but M_k have torsion-free cohomology.

(2) **Characteristic classes of $F_2(M_k)$ as a smooth manifold.** The configuration space $F_2(M)$ is itself an open smooth manifold of dimension 12, with tangent bundle $TM \times TM$ restricted to $F_2(M)$. So

$$p_1(F_2(M_k)) = p_1(M_k) \otimes 1 + 1 \otimes p_1(M_k),$$

which differs between M_0 and M_1. Hence $F_2(M_0)$ and $F_2(M_1)$ are not diffeomorphic as smooth open 12-manifolds. This does not imply they are not homotopy equivalent, since Pontryagin classes are not homotopy invariants.

(3) **Postnikov k-invariants of $F_2(M_k)$.** This is the most plausible obstruction. The first nontrivial Postnikov invariant of a simply connected space lives in $H^{n+2}(K(\pi_n, n); \pi_{n+1})$ for the lowest n with $\pi_{n+1} \neq 0$. For configuration spaces these get complicated quickly and we have not computed them.

3.10.7 Conclusion of the computation

For the Wall pair $M_0 = \mathbb{CP}^3$ and M_1:

- $F_2(M_0)$ and $F_2(M_1)$ have identical integral cohomology rings (CTBG argument).
- They have the same rational and real homotopy type (Lambrechts–Stanley + Idrissi).
- They have homotopy equivalent suspensions ΣF_2 (Aouina–Klein).
- They have homotopy equivalent loop spaces ΩF_2 (Levitt).
- Their tangent bundles differ (p_1 differs), so they are not diffeomorphic as open 12-manifolds, but this is irrelevant for the homotopy question.

All standard invariants give the same answer. No standard invariant detects a difference. This is a meaningful negative result: the Wall pair, despite differing by a non-homotopy-invariant (p_1) of the underlying manifold, is invisible to all cohomological and standard homotopy-theoretic invariants of F_2. If there is a difference at the level of homotopy type, it must come from very subtle Postnikov data.

This is consistent with the conjecture being true for simply connected closed manifolds: the obstruction data (p_1) gets washed out at the level of F_2 because the tangent bundle structure does not appear in the integral cohomology of the configuration space.

3.10.8 Where to look next

The Wall computation suggests that for a simply connected counterexample, one would need:

- Manifolds with isomorphic integral cohomology rings as Poincaré duality algebras (forced by the CTBG argument).
- Differing in some secondary or tertiary invariant that survives to F_2.

The most promising direction is **Wall 6-manifolds with torsion in** H_2. Wall's full classification includes the case where H_2 has torsion. Two such manifolds with the same μ, p_1, w_2 but different torsion linking forms could differ in subtler ways.

3.11 An explicit computation: $F_2(\mathbb{HP}^2)$ and the Eells–Kuiper family

We now turn to an explicit candidate pair of homotopy-equivalent simply connected closed smooth manifolds whose configuration spaces we can compare in detail. The candidates we consider are the quaternionic projective plane $\mathbb{HP}^2$ together with the infinite family of exotic smooth manifolds homotopy equivalent to it.

3.11.1 The manifolds under consideration

The quaternionic projective plane $\mathbb{HP}^2$ is the simplest non-trivial example of a closed simply connected manifold whose homotopy type, despite being completely determined by its rational cohomology ring, admits multiple inequivalent smooth structures. Its basic invariants are:

- dimension 8;
- integral cohomology ring $H^*(\mathbb{HP}^2;\mathbb{Z}) = \mathbb{Z}[u]/(u^3)$ with $|u| = 4$;
- simply connected, with $\pi_2 = 0$, $\pi_3 = 0$, $\pi_4 = \mathbb{Z}$;
- Pontryagin class $p_1(\mathbb{HP}^2) = 2u$;
- second Stiefel–Whitney class $w_2 = 0$ (it is a spin manifold).

By a classical theorem of Eells and Kuiper, there exist infinitely many pairwise non-diffeomorphic simply connected closed smooth 8-manifolds, all homotopy equivalent to $\mathbb{HP}^2$, distinguished by their first Pontryagin class. Concretely, for each integer $k \in \mathbb{Z}$, one can construct a smooth manifold M_k homotopy equivalent to $\mathbb{HP}^2$ with $p_1(M_k) = (2 + 24k)u$. The case $k = 0$ recovers the standard $\mathbb{HP}^2$. These manifolds form an infinite family of homotopy-equivalent but non-diffeomorphic candidates, in close analogy with the Wall family of homotopy-$\mathbb{CP}^3$ manifolds in dimension 6.

The Eells–Kuiper family satisfies all the constraints that a counterexample to homotopy invariance of configuration spaces must satisfy: simple connectedness, non-trivial π_2 (in this case trivial, so we should expect even stronger invariance results since $\mathbb{HP}^2$ is in fact 3-connected), torsion-free cohomology, and identical real homotopy type by Idrissi's theorem. Indeed, $\mathbb{HP}^2$ is 3-connected, so by Levitt's theorem the F_2 invariance holds at the level of homotopy type for $k = 2$, making this a less-than-ideal candidate for a counterexample but an ideal candidate for an explicit computation against which to test our techniques.

3.11.2 The Lambrechts–Stanley model $\mathsf{G}_A(2)$ for $\mathbb{HP}^2$

We compute $\mathsf{G}_A(2)$ explicitly. The Poincaré duality CDGA model is

$$A = \mathbb{Q}[u]/(u^3), \quad |u| = 4, \quad \int u^2 = 1.$$

The diagonal class is

$$\Delta_A = 1 \otimes u^2 + u \otimes u + u^2 \otimes 1 \ \in \ (A \otimes A)^8.$$

The Lambrechts–Stanley CDGA $\mathsf{G}_A(2)$ is generated by:

- even-degree generators $A \otimes A$ with basis $\{u_1^i u_2^j : 0 \le i, j \le 2\}$, where $u_1 = u \otimes 1$ and $u_2 = 1 \otimes u$;
- one odd-degree generator $\omega := \omega_{12}$ of degree $|\omega| = 7$, satisfying $d\omega = \Delta_A$.

The relations are: graded commutativity, $\omega^2 = 0$, and the diagonal relation $\iota_1(a)\omega = \iota_2(a)\omega$ for all $a \in A$.

3.11.3 The diagonal relation collapses the odd part

The diagonal relation forces, in the ω-part of $\mathsf{G}_A(2)$,

$$(a \otimes 1)\omega = (1 \otimes a)\omega \quad \text{for all } a \in A.$$

This means that the multiplication on the ω-part of $\mathsf{G}_A(2)$ factors through the multiplication map $\mu\colon A \otimes A \to A$. Concretely, the odd-degree part $(A \otimes A)\omega/\sim$ is isomorphic, as a graded vector space, to $A \cdot \omega$, with one generator per element of A:

$$(A \otimes A)\omega/(\iota_1(a)\omega = \iota_2(a)\omega) \ \cong \ A \cdot \omega.$$

For $A = \mathbb{Q}[u]/(u^3)$, this gives a 3-dimensional odd-degree part spanned by ω, $u\omega$, and $u^2\omega$, in degrees 7, 11, and 15 respectively.

Lemma 3.11.1. *For $M = \mathbb{HP}^2$ with $A = \mathbb{Q}[u]/(u^3)$, the Lambrechts–Stanley CDGA $\mathsf{G}_A(2)$ has total dimension 12 over $\mathbb{Q}$, with*

$$\dim \mathsf{G}_A(2) = \dim(A \otimes A) + \dim A = 9 + 3 = 12.$$

The Euler characteristic of $\mathsf{G}_A(2)$ as a complex is

$$\chi(\mathsf{G}_A(2)) = \dim(A \otimes A) - \dim A = 9 - 3 = 6,$$

which matches $\chi(F_2(\mathbb{HP}^2)) = \chi(\mathbb{HP}^2)^2 - \chi(\mathbb{HP}^2) = 9 - 3 = 6$.

3.11.4 Differential

The differential d on $\mathsf{G}_A(2)$ is determined by $d\omega = \Delta_A$ and the Leibniz rule. We compute:

$$\begin{aligned} d\omega &= 1 \otimes u^2 + u \otimes u + u^2 \otimes 1 \\ &= u_1^2 + u_1u_2 + u_2^2, \\ d(u\omega) &= u_1 \cdot \Delta_A = u_1 \otimes u^2 + u_1^2 \otimes u + u_1^3 \otimes 1 = u \otimes u^2 + u^2 \otimes u = u_1u_2^2 + u_1^2u_2, \\ d(u^2\omega) &= u_1^2 \cdot \Delta_A = u_1^2 \otimes u^2 + u_1^3 \otimes u + u_1^4 \otimes 1 = u^2 \otimes u^2 = u_1^2u_2^2. \end{aligned}$$

The verification that $u_1 \cdot \Delta_A = u_2 \cdot \Delta_A$ (consistency of the diagonal relation under d) follows from the identity $(a \otimes 1)\Delta_A = (1 \otimes a)\Delta_A$ for all $a \in A$.

3.11.5 Cohomology

We organise $\mathsf{G}_A(2)$ by cohomological degree and apply d:

Degree	Generators	dim	Differential
0	1	1	—
4	u_1, u_2	2	0
7	ω	1	$d\omega = u_1^2 + u_1u_2 + u_2^2$
8	u_1^2, u_1u_2, u_2^2	3	0
11	$u\omega$	1	$d(u\omega) = u_1^2u_2 + u_1u_2^2$
12	$u_1^2u_2, u_1u_2^2$	2	0
15	$u^2\omega$	1	$d(u^2\omega) = u_1^2u_2^2$
16	$u_1^2u_2^2$	1	0

Theorem 3.11.2. *The rational cohomology of $F_2(\mathbb{HP}^2)$ is*

$$H^*(F_2(\mathbb{HP}^2); \mathbb{Q}) \;=\; \mathbb{Q} \;\oplus\; \mathbb{Q}^2[4] \;\oplus\; \mathbb{Q}^2[8] \;\oplus\; \mathbb{Q}[12].$$

That is, the cohomology is concentrated in degrees $0, 4, 8, 12$ *with ranks* $(1, 2, 2, 1)$. *All other degrees have trivial cohomology.*

Proof. We compute $H^*(\mathsf{G}_A(2)) = \ker d/\operatorname{im} d$ degree by degree, using the table above.

- **Degree** 0. $\ker d^0 = \mathbb{Q}\{1\}$. No differential lands in degree 0. $H^0 = \mathbb{Q}$.
- **Degree** 4. $\ker d^4 = \mathbb{Q}\{u_1, u_2\}$, since d raises degree by 1 and degree 5 is empty. No differential lands in degree 4. $H^4 = \mathbb{Q}^2$.
- **Degree** 7. $\ker d^7 = \{c\omega : c\,d\omega = 0\}$. Since $d\omega = u_1^2 + u_1u_2 + u_2^2 \neq 0$ in H^8, we have $\ker d^7 = 0$. $H^7 = 0$.
- **Degree** 8. $\ker d^8 = \mathbb{Q}^3$ (degree 9 is empty). Image of d^7 is $\mathbb{Q}\{u_1^2 + u_1u_2 + u_2^2\}$, rank 1. $H^8 = \mathbb{Q}^3/\mathbb{Q}\{u_1^2 + u_1u_2 + u_2^2\} = \mathbb{Q}^2$.
- **Degree** 11. $\ker d^{11} = 0$ since $d(u\omega) = u_1^2u_2 + u_1u_2^2 \neq 0$. $H^{11} = 0$.
- **Degree** 12. $\ker d^{12} = \mathbb{Q}\{u_1^2u_2, u_1u_2^2\}$. Image of d^{11} is $\mathbb{Q}\{u_1^2u_2 + u_1u_2^2\}$, rank 1. $H^{12} = \mathbb{Q}^2/\mathbb{Q}\{u_1^2u_2 + u_1u_2^2\} = \mathbb{Q}$.

- **Degree** 15. $\ker d^{15} = 0$ since $d(u^2\omega) = u_1^2u_2^2 \neq 0$. $H^{15} = 0$.
- **Degree** 16. $\ker d^{16} = \mathbb{Q}\{u_1^2u_2^2\}$. Image of d^{15} is $\mathbb{Q}\{u_1^2u_2^2\}$, rank 1. $H^{16} = 0$.

This gives the stated ranks. □

Corollary 3.11.3. $\chi(F_2(\mathbb{HP}^2)) = 1 + 2 + 2 + 1 = 6$, *in agreement with the direct computation* $\chi(M^2) - \chi(M) = 9 - 3 = 6$.

3.11.6 Consequences for the homotopy invariance question

Corollary 3.11.4. *For every* $k \in \mathbb{Z}$*, the manifold* M_k *in the Eells–Kuiper family satisfies*

$$H^*(F_2(M_k); \mathbb{Q}) \cong H^*(F_2(\mathbb{HP}^2); \mathbb{Q})$$

with the explicit ranks given in Theorem 3.11.2. Moreover, $F_2(M_k)$ *has the same real homotopy type as* $F_2(\mathbb{HP}^2)$ *by Idrissi's theorem.*

Proof. The Lambrechts–Stanley model $\mathsf{G}_A(2)$ depends only on the rational Poincaré duality CDGA structure on $A = H^*(M_k; \mathbb{Q})$, which is the same for all k since these manifolds are homotopy equivalent. Hence $H^*(F_2(M_k); \mathbb{Q})$ is computed by the same $\mathsf{G}_A(2)$ as $H^*(F_2(\mathbb{HP}^2); \mathbb{Q})$.

The integral cohomology of $F_2(M_k)$ is similarly computed by the integral CTBG spectral sequence, whose E_2-page and differential $d_8(\omega) = \Delta_M$ depend only on the integral cohomology ring as a Poincaré duality algebra. This is identical for all M_k, hence

$$H^*(F_2(M_k); \mathbb{Z}) \cong H^*(F_2(\mathbb{HP}^2); \mathbb{Z})$$

as graded rings. □

Corollary 3.11.5. *The space* $F_2(\mathbb{HP}^2)$ *has rationally formal cohomology concentrated in even degrees, and all rational Massey products in* $H^*(F_2(\mathbb{HP}^2); \mathbb{Q})$ *vanish.*

Proof. By Theorem 3.11.2, $H^*(F_2(\mathbb{HP}^2); \mathbb{Q})$ is concentrated in degrees $0, 4, 8, 12$, all of which are even. Any triple Massey product $\langle a, b, c\rangle$ with a, b, c of even degree lies in degree $|a| + |b| + |c| - 1$, which is odd. Since the cohomology is zero in odd degrees, all triple Massey products vanish. The same argument applies to higher Massey products. □

3.11.7 Conclusion: invisibility of p_1 to F_2

The first Pontryagin class p_1 distinguishes the Eells–Kuiper manifolds M_k as smooth 8-manifolds, but is invisible to the configuration space $F_2(M_k)$ at the level of:

- rational cohomology (Theorem 3.11.2),
- integral cohomology as a graded ring (Corollary 3.11.4),
- rational and real homotopy type (Idrissi's theorem, Corollary 3.11.4),
- rational Massey products (Corollary 3.11.5),
- the suspension $\Sigma F_2(M_k)$ (Aouina–Klein, since $\mathbb{HP}^2$ is simply connected),

- the loop space $\Omega F_2(M_k)$ (Levitt).

As $\mathbb{HP}^2$ is in fact 3-connected, Levitt's theorem already implies that F_2 is a homotopy invariant in this case, so no counterexample exists within this family.

The interest of the computation is therefore not the discovery of a counterexample, but the explicit determination of $H^*(F_2(\mathbb{HP}^2);\mathbb{Q})$ in low degrees, together with the observation that the Lambrechts–Stanley machinery washes out the Pontryagin class p_1 at every cohomological level. To our knowledge, this explicit computation has not appeared in the literature, although it follows in principle from the general results of [LS04, LS08, Idr19].

4

Beyond Idrissi's theorem: the state of the art and a computation

The theorem of Idrissi proved in Chapter 3 settles the question of real homotopy invariance of configuration spaces for simply connected closed manifolds of dimension at least four. It does not settle, however, the original conjecture, namely the question of whether the homotopy type of $F_r(M)$ is determined by the homotopy type of M. The Longoni–Salvatore counterexample of Chapter 1 settles this question in the negative when M is allowed to be non-simply connected, and the Idrissi theorem settles a real-coefficient analog in the simply connected case, but the integral statement for closed simply connected manifolds, in its full generality, remains open.

This chapter is dedicated to surveying what is known about this open question, and to presenting one explicit computation. The survey gathers in one place the positive results that constrain where a counterexample could live, together with the obstructions that make the question hard. The computation is a concrete worked example: the rational homology of the unordered configuration spaces $B_2(\mathbb{HP}^2)$ and $B_3(\mathbb{HP}^2)$, derived via Knudsen's factorization-homology formula. To our knowledge, these explicit Betti numbers do not appear in the literature in this form, although they follow in principle from Knudsen's general result [Knu17]. The computation serves a double purpose: it illustrates how Knudsen's formalism reduces a geometric question to a tractable algebraic one, and it provides a sanity check between two distinct approaches to the rational homotopy type of configuration spaces, namely the Lambrechts–Stanley approach of Chapter 3 and the factorization-homology approach.

4.1 What is known

The central open question can be stated as follows.

Conjecture 4.1.1. Let M and N be simply connected closed manifolds. If M and N have the same homotopy type, then $F_r(M)$ and $F_r(N)$ have the same homotopy type for every $r \geq 0$.

The conjecture is open. What follows is an attempt to organize the known results around it, with the goal of describing where a counterexample could plausibly live.

4.1.1 Positive results

A first set of theorems provides invariance of various weakened forms of the configuration space. Each of these theorems is non-trivial, but each falls short of the full integral homotopy type.

Theorem 4.1.2 (Levitt [Lev95]). *For M a closed manifold, the based loop space $\Omega F_r(M)$ is a homotopy invariant of M.*

Theorem 4.1.3 (Aouina–Klein [AK04], Malin [Mal23], Knudsen). *For M and N homotopy equivalent simply connected closed manifolds, there is an integer j depending on r, the dimension of M, and the connectivity of M, such that $\Sigma^j F_r(M) \simeq \Sigma^j F_r(N)$. In particular, the suspension spectra $\Sigma^\infty_+ F_r(M)$ and $\Sigma^\infty_+ F_r(N)$ are equivalent.*

Theorem 4.1.4 (Levitt [Lev95]). *If M is a 2-connected closed manifold, then $F_2(M)$ is a homotopy invariant of M.*

The result of Levitt for F_2 uses general position: the diagonal in $M \times M$ has codimension $n = \dim M$, and 2-connectedness gives enough room to deform any homotopy equivalence $M \to N$ to one that is transverse to the diagonal, producing a homotopy equivalence of complements. The same argument fails for $r \geq 3$ because the configuration of points lives in a higher-dimensional ambient space and the relevant connectivity bound becomes more restrictive.

The next layer of positive results concerns rational and real homotopy type. Idrissi's theorem of Chapter 3 gives the most general real-coefficient statement; over $\mathbb{Q}$, the known results are more restrictive.

Theorem 4.1.5 (Idrissi [Idr19, Idr22]). *For M and N homotopy equivalent simply connected closed manifolds of dimension at least 4, the configuration spaces $F_r(M)$ and $F_r(N)$ have the same real homotopy type for every r.*

Theorem 4.1.6 (Lambrechts–Stanley [LS04], Cordova Bulens [CB15]). *For M a closed simply connected manifold:*

- *if M is 2-connected, the rational homotopy type of $F_r(M)$ depends only on the rational homotopy type of M for every r;*
- *if M is simply connected and of even dimension, the rational homotopy type of $F_2(M)$ depends only on the rational homotopy type of M.*

Theorem 4.1.7 (Knudsen [Knu17]). *For M a manifold, the rational homology of the unordered configuration spaces $B_k(M) = F_k(M)/\Sigma_k$ depends only on the rational cohomology ring of M together with its Poincaré duality structure. Explicitly, there is an isomorphism of bigraded vector spaces*

$$\bigoplus_{k\geq 0} H_*(B_k(M);\mathbb{Q}) \;\cong\; H_*^{\mathrm{Lie}}\big(H_c^*(M;\mathbb{Q}_w)\otimes L(\mathbb{Q}_w[n-1])\big),$$

where the right-hand side is the Lie algebra homology of the indicated graded Lie algebra, L denotes the free Lie algebra functor, and $\mathbb{Q}_w$ is the orientation sheaf.

Knudsen's theorem deserves emphasis: it provides an explicit combinatorial object, the Chevalley–Eilenberg complex of a small graded Lie algebra, whose homology computes that of every $B_k(M)$ at once. This is the engine of the explicit computation in Section 4.3 below.

Finally, there is an observation of Raptis and Salvatore that constrains the search space even further.

Remark 4.1.8 (Raptis–Salvatore [RS18]). No counterexample to the homotopy invariance of $F_r(M)$ is known for simple homotopy equivalences. The Longoni–Salvatore counterexample with the lens spaces $L_{7,1}$ and $L_{7,2}$ relies crucially on the failure of these manifolds to be simply homotopy equivalent; their Whitehead torsion is non-zero.

Two simply connected closed manifolds with the same homotopy type are automatically *simply* homotopy equivalent, since the Whitehead group $\mathrm{Wh}(\pi_1) = \mathrm{Wh}(1)$ vanishes for simply connected spaces. The Raptis–Salvatore observation thus applies to Conjecture 4.1.1, providing evidence in favor of its truth.

4.1.2 Constraints on a hypothetical counterexample

Putting these positive results together, any pair (M, N) of simply connected closed manifolds witnessing a failure of Conjecture 4.1.1 would have to satisfy a big list of constraints which we already briefly discussed prior to here.

First, M and N that are homotopically equivalent will have isomorphic real and rational homotopy types and so by Theorem 4.1.5 any difference between $F_r(M)$ and $F_r(N)$ must therefore be torsion in nature.

Second, M and N should potentially have $\pi_2(M) = \pi_2(N) \neq 0$, by Theorem 4.1.4: otherwise M is 2-connected and the conjecture holds for $r = 2$.

Third, M and N will have homotopy equivalent loop spaces $\Omega F_r(M) \simeq \Omega F_r(N)$ by Theorem 4.1.2, so any difference between the configuration spaces must be detected at the level of the configuration spaces themselves, not their loop spaces: all stable invariants (such as homotopy groups and their Whitehead products) are not useful.

Forth, by Theorem 4.1.8 and the vanishing of $\mathrm{Wh}(1)$, the pair (M, N) is automatically simply homotopy equivalent. The technique that generates the Longoni–Salvatore counterexample, namely non-trivial Whitehead torsion, is unavailable in the simply connected setting.

Any counterexample would have to be a pair of simply homotopy equivalent, rationally and stably indistinguishable simply connected closed manifolds whose configuration spaces differ unstably and at the level of integral torsion. No known mechanism produces such a pair.

4.1.3 Where a counterexample could plausibly live

The most natural candidates are pairs of homotopy equivalent but not homeomorphic simply connected closed manifolds, where the homeo/diffeomorphism obstruction comes from a characteristic class that is not a homotopy invariant. The classical example of such a phenomenon is the family of Wall manifolds.

Example 4.1.9 (The Wall family of $\mathbb{C}P^3$-like manifolds)**.** By Wall's classification of simply connected closed smooth 6-manifolds with torsion-free homology, there exists an infinite family $\{M_k\}_{k \in \mathbb{Z}}$ of pairwise non-diffeomorphic simply connected closed smooth manifolds, all homotopy equivalent to $\mathbb{C}P^3$, with $p_1(M_k) = (4+24k)x^2$ for some choice of generator $x \in H^2(M_k;\mathbb{Z})$. The case $k = 0$ recovers the standard $\mathbb{C}P^3$.

Example 4.1.10 (The Eells–Kuiper family of $\mathbb{HP}^2$-like manifolds)**.** By a theorem of Eells and Kuiper, there exists an infinite family of pairwise non-diffeomorphic simply connected closed smooth 8-manifolds homotopy equivalent to $\mathbb{HP}^2$, distinguished by their first Pontryagin class. Amongst these family, there are finitely many different homeomorphism type, which I want to really stress are what we should really care about (and not reallu different diffeomorphism type but same homeomorphism one): the (homeomorphism classes of) configurations spaces are topological invariant.

In both examples, the manifolds in question are homotopy equivalent but distinguished as smooth manifolds by a Pontryagin class. The question is whether this distinction is visible to the configuration spaces. We will show in Section 4.3 that, at least at the level of rational homology of unordered configuration spaces, the answer is no: $H_*(B_k(M_j);\mathbb{Q})$ is the same for all members of the Eells–Kuiper family. This is consistent with, and a small strengthening of, the conclusion of Idrissi's theorem.

4.2 The factorization-homology approach

The computation that follows uses Knudsen's Theorem 4.1.7. We recall the precise form of the result and unpack the input that goes into it for our manifolds of interest.

4.2.1 The Chevalley–Eilenberg complex

Let M be a closed oriented manifold of dimension n, with n even. By Theorem 4.1.7, the rational homology of $\bigsqcup_k B_k(M)$ is computed by the Chevalley–Eilenberg complex of the graded Lie algebra

$$\mathfrak{g}_M = H^*_c(M;\mathbb{Q}) \otimes L(\mathbb{Q}[n-1]),$$

which, since M is closed orientable, has the following structure. The free Lie algebra $L(\mathbb{Q}[n-1])$ for $n-1$ odd has $L^1 = \mathbb{Q}[n-1]$ and $L^2 = \mathbb{Q}[2n-2]$ (the latter generated by the self-bracket $[v,v]$ of the weight-1 generator v, which is non-zero because $|v|$ is odd), with $L^{\geq 3} = 0$.

The Lie algebra $\mathfrak{g}_M$ has therefore:

- a weight-1 part $H^*(M;\mathbb{Q}) \cdot v$ in homological degree $-|*|+n-1$, where $|*|$ denotes the cohomological degree of the relevant class in $H^*(M)$;
- a weight-2 part $H^*(M;\mathbb{Q}) \cdot w$, where $w = \frac{1}{2}[v,v]$, in homological degree $-|*|+2n-2$;
- the bracket $[\alpha v, \beta v] = (\alpha \cup \beta)w$ for $\alpha, \beta \in H^*(M)$, with $\alpha \cup \beta$ denoting the cup product.

The Chevalley–Eilenberg complex is the graded-commutative algebra $\mathrm{CE}(\mathfrak{g}_M) = \mathrm{Sym}(\mathfrak{g}_M[1])$ equipped with a differential D of homological degree -1. The shift [1] moves the weight-1 part to homological degree $-|*|+n$ and the weight-2 part to $-|*|+2n-1$, so for n even the weight-1 generators have even homological degree (they are polynomial in the symmetric algebra) and the weight-2 generators have odd homological degree (they are exterior powers).

The differential D is determined by the bracket of $\mathfrak{g}_M$ as a coderivation: on a product $a_\alpha \wedge a_\beta$ of two weight-1 generators corresponding to cohomology classes α, β, one has

$$D(a_\alpha \wedge a_\beta) = (-1)^{|a_\alpha|}\, b_{\alpha\cup\beta},$$

where b_γ denotes the weight-2 generator corresponding to $\gamma \in H^*(M)$, with the convention $b_\gamma = 0$ if $\gamma = 0$ in $H^*(M)$. The differential vanishes on individual weight-1 generators a_α and on individual weight-2 generators b_γ, and extends by the Leibniz rule for a coderivation on the symmetric algebra.

4.2.2 The bigrading and the B_k-summand

The Chevalley–Eilenberg complex carries two gradings: the homological grading just described, and an auxiliary weight grading inherited from the weight grading of $\mathfrak{g}_M$. A monomial in $\mathrm{Sym}(\mathfrak{g}_M[1])$ has weight equal to the sum of the weights of its factors: each a_α contributes 1 and each b_γ contributes 2. By Theorem 4.1.7, the homology of the weight-k part of the Chevalley–Eilenberg complex is naturally isomorphic to $H_*(B_k(M);\mathbb{Q})$.

4.2.3 Application to $\mathbb{HP}^2$

For $M = \mathbb{HP}^2$ we have $n = 8$ and $H^*(\mathbb{HP}^2;\mathbb{Q}) = \mathbb{Q}[u]/(u^3)$ with $|u| = 4$. The generators of the Chevalley–Eilenberg complex are:

- weight-1 generators a_1, a_u, a_{u^2} in homological degrees $8, 4, 0$, corresponding to the cohomology classes $1, u, u^2 \in H^*(\mathbb{HP}^2;\mathbb{Q})$ respectively;
- weight-2 generators b_1, b_u, b_{u^2} in homological degrees $15, 11, 7$, corresponding to the same cohomology classes.

The differential is determined by the cup-product structure of $H^*(\mathbb{HP}^2;\mathbb{Q})$, which has

$$1\cdot 1 = 1, \quad 1\cdot u = u, \quad 1\cdot u^2 = u^2, \quad u\cdot u = u^2, \quad u\cdot u^2 = 0, \quad u^2\cdot u^2 = 0.$$

Since the weight-1 generators are in even homological degrees, the sign $(-1)^{|a_\alpha|}$ is uniformly $+1$, and we obtain:

$$D(a_\alpha \wedge a_\beta) = b_{\alpha\cdot\beta}, \qquad \text{for any pair of weight-1 generators.}$$

4.3 A computation: $H_*(B_2(\mathbb{HP}^2);\mathbb{Q})$ and $H_*(B_3(\mathbb{HP}^2);\mathbb{Q})$

We now compute the rational homology of $B_2(\mathbb{HP}^2)$ and $B_3(\mathbb{HP}^2)$ explicitly, working in the Chevalley–Eilenberg complex described in the previous section.

4.3.1 The weight-2 part: $H_*(B_2(\mathbb{HP}^2);\mathbb{Q})$

Theorem 4.3.1. *The rational homology of $B_2(\mathbb{HP}^2)$ is*

$$H_*(B_2(\mathbb{HP}^2);\mathbb{Q}) \;\cong\; \mathbb{Q}\oplus\mathbb{Q}[4]\oplus\mathbb{Q}[8],$$

with one generator each in homological degrees 0, 4, and 8.

Proof. The weight-2 part of the Chevalley–Eilenberg complex has the following generators:

Symbol	Homological degree	Description
a_1^2	16	product of two weight-1
a_1a_u	12	product of two weight-1
$a_1a_{u^2}$	8	product of two weight-1
a_u^2	8	product of two weight-1
$a_ua_{u^2}$	4	product of two weight-1
$a_{u^2}^2$	0	product of two weight-1
b_1	15	single weight-2
b_u	11	single weight-2
b_{u^2}	7	single weight-2

The differential D acts on the products of weight-1 generators and lands in the weight-2 generators, via the rule $D(a_\alpha a_\beta)=b_{\alpha\cdot\beta}$ from the cup product. Computing case by case using the multiplication table of $H^*(\mathbb{HP}^2;\mathbb{Q})$:

$$D(a_1^2)=b_1,\qquad D(a_1a_u)=b_u,\qquad D(a_1a_{u^2})=b_{u^2},$$
$$D(a_u^2)=b_{u^2},\qquad D(a_ua_{u^2})=0,\qquad D(a_{u^2}^2)=0.$$

The individual generators a_α and b_γ are themselves cycles since they sit in degree ≤ 1 of the symmetric algebra and the differential is a coderivation.

We compute the homology by homological degree.

Degree 0. The only generator is $a_{u^2}^2$, which is a cycle ($D(a_{u^2}^2)=0$). No boundary lands in this degree. $H_0=\mathbb{Q}$, generated by $a_{u^2}^2$.

Degree 4. The only generator is $a_ua_{u^2}$, a cycle. No boundary in this degree. $H_4=\mathbb{Q}$, generated by $a_ua_{u^2}$.

Degree 7. The only generator is b_{u^2}. It is a cycle. The boundary D from degree 8 contributes: $D(a_1a_{u^2})=b_{u^2}$ and $D(a_u^2)=b_{u^2}$. The image is the 1-dimensional span of b_{u^2}. The class b_{u^2} is therefore a boundary, and $H_7=0$.

Degree 8. The generators are $a_1a_{u^2}$ and a_u^2. The differential sends both to b_{u^2}, so the kernel is the 1-dimensional subspace generated by $a_1a_{u^2}-a_u^2$. No boundary lands in degree 8 from a higher-degree class, so $H_8=\mathbb{Q}$, generated by $[a_1a_{u^2}-a_u^2]$.

Degree 11. The only generator is b_u. It is a cycle. The boundary from degree 12: $D(a_1a_u)=b_u$. The image is the full span of b_u, so b_u is a boundary and $H_{11}=0$.

Degree 12. The only generator is a_1a_u, with $D(a_1a_u)=b_u\neq 0$. The kernel is zero and $H_{12}=0$.

Degree 15. The only generator is b_1. It is a cycle. The boundary from degree 16: $D(a_1^2) = b_1$. The image is the full span of b_1, so $H_{15} = 0$.

Degree 16. The only generator is a_1^2, with $D(a_1^2) = b_1 \neq 0$. The kernel is zero and $H_{16} = 0$.

Collecting the non-zero pieces, the homology is concentrated in degrees 0, 4, 8, with one generator each. □

Remark 4.3.2. As a check, the Euler characteristic of $B_2(\mathbb{HP}^2)$ predicted by the formula $\chi(F_2(M))/2!$ with $\chi(F_2(M)) = \chi(M)^2 - \chi(M) = 9-3 = 6$ is 3. The total Betti number sum from Theorem 4.3.1 is $1+1+1 = 3$, and the classes sit in even degrees so the alternating sum is also 3. The two computations agree.

Remark 4.3.3. The computation of $H^*(F_2(\mathbb{HP}^2);\mathbb{Q})$ via the Lambrechts–Stanley CDGA model, carried out at the end of Chapter 3, gives ranks $(1,2,2,1)$ in degrees $(0,4,8,12)$, with total dimension 6. Taking Σ_2-invariants under the action swapping the two configuration points yields ranks $(1,1,1,0)$, in agreement with Theorem 4.3.1.

4.3.2 The weight-3 part: $H_*(B_3(\mathbb{HP}^2);\mathbb{Q})$

Theorem 4.3.4. *The rational homology of* $B_3(\mathbb{HP}^2)$ *is*

$$H_*(B_3(\mathbb{HP}^2);\mathbb{Q}) \;\cong\; \mathbb{Q} \oplus \mathbb{Q}[4] \oplus \mathbb{Q}[8] \oplus \mathbb{Q}[15] \oplus \mathbb{Q}[19],$$

with one generator each in homological degrees 0*,* 4*,* 8*,* 15*, and* 19.

Proof. The weight-3 part of the Chevalley–Eilenberg complex consists of two types of generators: products $a_\alpha a_\beta a_\gamma$ of three weight-1 generators (contributing total weight 3), and products $a_\alpha b_\gamma$ of one weight-1 and one weight-2 generator (contributing total weight 3).

The triple products of weight-1 generators, indexed by multi-sets of size three from $\{a_1, a_u, a_{u^2}\}$, give the following ten elements:

Symbol	Homological degree
a_1^3	24
$a_1^2 a_u$	20
$a_1^2 a_{u^2}$	16
$a_1 a_u^2$	16
$a_1 a_u a_{u^2}$	12
a_u^3	12
$a_1 a_{u^2}^2$	8
$a_u^2 a_{u^2}$	8
$a_u a_{u^2}^2$	4
$a_{u^2}^3$	0

The mixed products of one weight-1 and one weight-2 give the nine elements:

Symbol	Homological degree
$a_1 b_1$	23
$a_1 b_u$	19
$a_u b_1$	19
$a_1 b_{u^2}$	15
$a_u b_u$	15
$a_{u^2} b_1$	15
$a_u b_{u^2}$	11
$a_{u^2} b_u$	11
$a_{u^2} b_{u^2}$	7

The differential acts as a coderivation. On a triple product $a_\alpha a_\beta a_\gamma$, the differential gives a sum over pairings of two factors, each pairing producing a weight-2 generator via the cup product. Since the weight-1 generators commute in the symmetric algebra (they sit in even homological degrees), the formula is

$$D(a_\alpha a_\beta a_\gamma) = a_\gamma \cdot b_{\alpha\cdot\beta} + a_\beta \cdot b_{\alpha\cdot\gamma} + a_\alpha \cdot b_{\beta\cdot\gamma},$$

with $b_\delta = 0$ if $\delta = 0$ in $H^*(\mathbb{HP}^2)$. On the mixed products $a_\alpha b_\gamma$, the differential vanishes, since $D(a_\alpha) = 0$ and $D(b_\gamma) = 0$.

The explicit differentials on the triple products of weight-1 generators are:

$$\begin{aligned}
D(a_1^3) &= 3\, a_1 b_1, \\
D(a_1^2 a_u) &= 2\, a_1 b_u + a_u b_1, \\
D(a_1^2 a_{u^2}) &= 2\, a_1 b_{u^2} + a_{u^2} b_1, \\
D(a_1 a_u^2) &= 2\, a_u b_u + a_1 b_{u^2}, \\
D(a_1 a_u a_{u^2}) &= a_{u^2} b_u + a_u b_{u^2}, \\
D(a_u^3) &= 3\, a_u b_{u^2}, \\
D(a_1 a_{u^2}^2) &= 2\, a_{u^2} b_{u^2}, \\
D(a_u^2 a_{u^2}) &= a_{u^2} b_{u^2}, \\
D(a_u a_{u^2}^2) &= 0, \\
D(a_{u^2}^3) &= 0.
\end{aligned}$$

We now compute the homology by homological degree.

Degree 0. The only generator is $a_{u^2}^3$, with $D = 0$. $H_0 = \mathbb{Q}$, generated by $a_{u^2}^3$.

Degree 4. The only generator is $a_u a_{u^2}^2$, with $D = 0$. $H_4 = \mathbb{Q}$, generated by $a_u a_{u^2}^2$.

Degree 7. The only generator is $a_{u^2} b_{u^2}$. It is a cycle. The image of D from degree 8: $D(a_1 a_{u^2}^2) = 2 a_{u^2} b_{u^2}$ and $D(a_u^2 a_{u^2}) = a_{u^2} b_{u^2}$. The image is the 1-dimensional span of $a_{u^2} b_{u^2}$. Hence $H_7 = 0$.

Degree 8. The generators are $a_1 a_{u^2}^2$ and $a_u^2 a_{u^2}$, both of which map non-trivially to $a_{u^2} b_{u^2}$ in degree 7, with scalar multiples 2 and 1 respectively. The kernel of D is the 1-dimensional subspace generated by $a_1 a_{u^2}^2 - 2 a_u^2 a_{u^2}$. No boundary lands in degree 8 from higher. $H_8 = \mathbb{Q}$, generated by $[a_1 a_{u^2}^2 - 2 a_u^2 a_{u^2}]$.

Degree 11. The generators are $a_u b_{u^2}$ and $a_{u^2} b_u$, both cycles. The image of D from degree 12: $D(a_1 a_u a_{u^2}) = a_{u^2} b_u + a_u b_{u^2}$ and $D(a_u^3) = 3 a_u b_{u^2}$. The image is the

2-dimensional span of $\{a_{u^2}b_u + a_u b_{u^2}, a_u b_{u^2}\}$, which equals the full 2-dimensional space. Hence $H_{11} = 0$.

Degree 12. The generators are $a_1 a_u a_{u^2}$ and a_u^3. Their images under D are linearly independent in degree 11. The kernel is zero, so $H_{12} = 0$.

Degree 15. The generators are $a_1 b_{u^2}$, $a_u b_u$, and $a_{u^2} b_1$. All cycles. The image of D from degree 16: $D(a_1^2 a_{u^2}) = 2a_1 b_{u^2} + a_{u^2} b_1$ and $D(a_1 a_u^2) = 2a_u b_u + a_1 b_{u^2}$. These are linearly independent vectors in the 3-dimensional space of cycles. The image is therefore 2-dimensional. $H_{15} = 3 - 2 = 1$.

A concrete generator of H_{15} can be found by completing the image to a basis: the cycles $a_1 b_{u^2}$ and $a_{u^2} b_1$ satisfy $2(a_1 b_{u^2}) + (a_{u^2} b_1) \in$ image and $(a_1 b_{u^2}) + 2(a_u b_u) \in$ image. The class of $a_u b_u - 2(a_{u^2} b_1)$ provides a generator of H_{15}.

Degree 16. The generators are $a_1^2 a_{u^2}$ and $a_1 a_u^2$. Their images under D are linearly independent, so the kernel is zero. $H_{16} = 0$.

Degree 19. The generators are $a_1 b_u$ and $a_u b_1$, both cycles. The image of D from degree 20: $D(a_1^2 a_u) = 2a_1 b_u + a_u b_1$, which is one specific vector in the 2-dimensional space. The image has rank 1. $H_{19} = 2 - 1 = 1$. A concrete generator: $a_u b_1$ itself (modulo the image, since $a_1 b_u$ and $a_u b_1$ are linked by the relation $2a_1 b_u + a_u b_1 \in$ image).

Degree 20. The only generator is $a_1^2 a_u$, with $D(a_1^2 a_u) \neq 0$. The kernel is zero. $H_{20} = 0$.

Degree 23. The only generator is $a_1 b_1$, a cycle. The image of D from degree 24: $D(a_1^3) = 3a_1 b_1$, full rank. $H_{23} = 0$.

Degree 24. The only generator is a_1^3, with $D(a_1^3) \neq 0$. $H_{24} = 0$.

Collecting the non-zero pieces, the homology is concentrated in degrees 0, 4, 8, 15, 19, with one generator each. □

Remark 4.3.5. The Euler characteristic of $B_3(\mathbb{HP}^2)$ predicted by $\chi(F_3(M))/3!$ with $\chi(F_3(M)) = \chi(M)(\chi(M) - 1)(\chi(M) - 2) = 3 \cdot 2 \cdot 1 = 6$ is 1. The alternating sum of Betti numbers from Theorem 4.3.4 is $1 + 1 + 1 - 1 - 1 = 1$, in agreement.

Remark 4.3.6. The appearance of odd-degree classes in $H_*(B_3(\mathbb{HP}^2);\mathbb{Q})$ at degrees 15 and 19 is new. These classes were absent from $H_*(B_2(\mathbb{HP}^2);\mathbb{Q})$, where the cohomology is concentrated in even degrees. The odd classes come from the (weight-1, weight-2) mixed terms $a_\alpha b_\gamma$ in the Chevalley–Eilenberg complex.

4.3.3 Consequence for the Eells–Kuiper family

We now apply the computation to the Eells–Kuiper family of Example 4.1.10.

Corollary 4.3.7. *For every member M of the Eells–Kuiper family of simply connected closed smooth 8-manifolds homotopy equivalent to $\mathbb{HP}^2$, the rational homology groups $H_*(B_2(M);\mathbb{Q})$ and $H_*(B_3(M);\mathbb{Q})$ have the same Betti numbers as those of $B_2(\mathbb{HP}^2)$ and $B_3(\mathbb{HP}^2)$ respectively.*

Proof. By Theorem 4.1.7, the rational homology of $B_k(M)$ for any M depends only on the rational cohomology ring of M together with the Poincaré duality structure encoded in $H_c^*(M;\mathbb{Q}_w)$. Every member of the Eells–Kuiper family has rational cohomology ring $\mathbb{Q}[u]/(u^3)$ with $|u| = 4$, and is closed orientable of dimension 8. The input to

Knudsen's formula is therefore identical for all members of the family, and so the output is identical. □

Remark 4.3.8. Corollary 5.5.3 should be read in the context of the constraints listed in Section 4.1.2. The Eells–Kuiper family of $\mathbb{HP}^2$-like manifolds is distinguished, as smooth manifolds, by the first Pontryagin class p_1, which is not a homotopy invariant. The configuration spaces $B_k(M)$ are nevertheless completely insensitive to this distinction at the level of rational homology, for every k. The Pontryagin class p_1 is therefore invisible to the entire collection of unordered configuration spaces of $\mathbb{HP}^2$-like manifolds, considered up to rational homology equivalence. The same phenomenon holds for the Wall family of Example 4.1.9 by the same argument.

Remark 4.3.9. The phenomenon of Remark 4.3.8 is consistent with, and a small refinement of, Idrissi's theorem of Chapter 3. Idrissi's theorem gives real homotopy invariance under just homotopy equivalence; our computation specializes to an invariance statement at the level of rational Betti numbers of unordered configuration spaces, for two specific values of k, for a specific manifold class. The point of the exercise is not to provide a new theorem beyond Idrissi's, but to make explicit the precise mechanism by which the failure of p_1 to be a homotopy invariant gets washed out at the level of configuration spaces: the input to Knudsen's formula simply does not see p_1.

A counterexample to Conjecture 4.1.1, if one exists, must therefore involve invariants of M beyond the rational cohomology ring together with the Poincaré duality structure.

5

Configuration spaces of manifolds like projective planes

5.1 Introduction and relation to the previous chapters

The first three chapters of this thesis investigated the rational and real homotopy type of *ordered* configuration spaces $F_k(M)$ of closed simply connected manifolds. The culminating result was Idrissi's theorem (Chapter 3): for every closed simply connected manifold M, the real homotopy type of $F_k(M)$ depends only on the real homotopy type of M.

In Chapter 4 we surveyed the state of the art on the analogous question for ordered configuration spaces under weaker hypotheses, and discussed in particular the Longoni–Salvatore counterexample which shows that homotopy invariance fails outside the simply connected setting.

The present chapter differs in three ways simultaneously:

1. we pass from *ordered* to *unordered* configuration spaces $B_k(M) = F_k(M)/\Sigma_k$;

2. we work over $\mathbb{Q}$ rather than $\mathbb{R}$ and at the level of *Betti numbers* rather than homotopy type;

3. we shift attention from invariance under homotopy equivalence to *homological stability* as the number of points k grows.

The chapter has three goals. First, we determine the complete rational Betti numbers of $B_k(\mathbb{HP}^2)$ for every $k \geq 0$ as a consequence of an existing theorem of Berceanu and Yameen [BY18]. Second, we observe that the same statement holds, with virtually no change, for all manifolds in the family of *rational homology projective planes*: the closed oriented manifolds with the rational cohomology of $\mathbb{CP}^2$, $\mathbb{HP}^2$, or $\mathbb{OP}^2$. This family was singled out by Eells and Kuiper [EK62] and contains infinitely many exotic representatives in dimensions 4, 8, and 16. Third, we produce by direct computation in the Chevalley–Eilenberg model two specific outputs that are not contained in the literature: explicit polynomial cocycle representatives for the six stable cohomology classes of $B_k(\mathbb{HP}^2)$ and of $B_k(\mathbb{OP}^2)$, and a structural observation (a modular separation of the four block components of the model) which simplifies the explicit calculations for $\mathbb{HP}^2$ and $\mathbb{OP}^2$ but not for $\mathbb{CP}^2$.

The technical part in this chapter is the characterisation by Berceanu and Yameen [BY18] of those closed orientable even-dimensional manifolds whose unordered configuration spaces have eventually constant rational cohomology. Their theorem identifies exactly two classes of manifolds with this property: rational homology spheres and rational homology projective planes. Since $\mathbb{HP}^2$ and $\mathbb{OP}^2$ both belong to the second class, the determination of all Betti numbers reduces to a finite check at low k and an explicit formula for the stable polynomial. The content of the chapter is therefore confined to the explicit polynomial representatives and the modular block structure.

The relation to the previous chapters can be summarised as follows. Chapters 1–3 concern *which manifolds have configuration spaces with the same homotopy type.* Chapter 4 extends this question to broader classes of manifolds. The present chapter, by contrast, concerns *which manifolds have configuration spaces whose Betti numbers stabilise completely.* Both questions interact with the same algebraic models (Lambrechts–Stanley, Félix–Thomas, Knudsen), and both have answers that single out the same elementary class of examples: spheres and projective planes.

5.2 From ordered to unordered

We begin by recording the basic relation between ordered and unordered configuration spaces.

Definition 5.2.1. Let M be a topological space and $k \geq 0$. The *unordered configuration space of k points in M* is the quotient

$$B_k(M) \;:=\; F_k(M)/\Sigma_k,$$

where Σ_k acts on $F_k(M)$ by permuting the coordinates.

When M is a manifold (without boundary), the symmetric group Σ_k acts freely on $F_k(M)$, so $B_k(M)$ is itself a manifold of the same dimension as $F_k(M)$, namely $k \dim M$. The projection $F_k(M) \to B_k(M)$ is a regular Σ_k-covering, which gives the following standard description of the rational cohomology of $B_k(M)$.

Proposition 5.2.2. *For every manifold M and every $k \geq 0$ there is a canonical isomorphism*

$$H^*(B_k(M);\mathbb{Q}) \;\cong\; H^*(F_k(M);\mathbb{Q})^{\Sigma_k},$$

where the right-hand side is the subspace of Σ_k-invariants for the action induced by the permutation of points.

Proof. Let

$$\pi\colon F_k(M) \longrightarrow B_k(M)$$

be the quotient map by the free action of the symmetric group Σ_k. Since the action is free, π is a regular covering with deck transformation group Σ_k.

Consider cohomology with rational coefficients. Since $\mathrm{char}(\mathbb{Q}) = 0$, the order $|\Sigma_k| = k!$ is invertible in $\mathbb{Q}$. Hence the standard transfer map

$$\mathrm{tr}\colon H^*(F_k(M);\mathbb{Q}) \longrightarrow H^*(B_k(M);\mathbb{Q})$$

is defined and satisfies
$$\operatorname{tr} \circ \pi^* = |\Sigma_k| \operatorname{id}_{H^*(B_k(M);\mathbb{Q})}.$$
The transfer map
$$\operatorname{tr}\colon H^*(F_k(M);\mathbb{Q}) \to H^*(B_k(M);\mathbb{Q})$$
is the averaging map associated to the finite covering $\pi\colon F_k(M) \to B_k(M)$; it is characterized by the identity
$$\pi^* \operatorname{tr}(\alpha) = \sum_{\sigma\in\Sigma_k} \sigma^*(\alpha).$$
It follows that π^* is injective.

Moreover, for every $\sigma \in \Sigma_k$ one has
$$\sigma^* \circ \pi^* = \pi^*,$$
so the image of π^* is contained in $H^*(F_k(M);\mathbb{Q})^{\Sigma_k}$.

Conversely, if $\alpha \in H^*(F_k(M);\mathbb{Q})^{\Sigma_k}$, define
$$\beta = \frac{1}{|\Sigma_k|} \operatorname{tr}(\alpha) \in H^*(B_k(M);\mathbb{Q}).$$
Using the identity
$$\pi^* \operatorname{tr}(\alpha) = \sum_{\sigma\in\Sigma_k} \sigma^*(\alpha),$$
and the Σ_k-invariance of α, we obtain
$$\pi^*(\beta) = \frac{1}{|\Sigma_k|} \sum_{\sigma\in\Sigma_k} \sigma^*(\alpha) = \alpha.$$
Thus every invariant class lies in the image of π^*.

Therefore π^* induces an isomorphism
$$H^*(B_k(M);\mathbb{Q}) \;\cong\; H^*(F_k(M);\mathbb{Q})^{\Sigma_k}.$$
□

In light of Proposition 5.2.2, every homotopy invariant of $F_k(M)$ that is functorial under the Σ_k-action descends to one of $B_k(M)$. In particular, Idrissi's theorem implies the following.

Corollary 5.2.3. *Let M and N be closed simply connected manifolds with the same real homotopy type. Then $B_k(M)$ and $B_k(N)$ have the same real homotopy type for every $k \geq 0$.*

Proof. By Idrissi's theorem the real homotopy types of $F_k(M)$ and $F_k(N)$ agree as Σ_k-equivariant CDGAs; the corollary follows by taking Σ_k-invariants. □

The further content of this chapter is independent of Idrissi's theorem: we shall work with explicit small models for $H^*(B_k(M);\mathbb{Q})$ obtained from a different source, namely the Chevalley–Eilenberg complex of a graded Lie algebra associated to M.

5.3 The Chevalley–Eilenberg model of Felix–Thomas and Knudsen

For a closed orientable even-dimensional manifold M^{2m} there is a simple CDGA model computing $H^*(B_k(M);\mathbb{Q})$. It was introduced by Félix and Thomas [FT00] for closed orientable manifolds and extended by Knudsen [Knu17] to arbitrary connected manifolds via factorisation homology. We use the formulation of [BY18], which is the most convenient for our purposes.

Definition 5.3.1. Let M^{2m} be a closed oriented manifold of even dimension. Set

$$V^* \;:=\; H^*(M;\mathbb{Q}), \qquad W^* \;:=\; H^*(M;\mathbb{Q})[2m-1],$$

where $[2m-1]$ denotes degree shift. Concretely, the elements of V^* have the degrees they carry in $H^*(M;\mathbb{Q})$ and length 1; the elements of W^* have degrees shifted up by $2m-1$ and length 2. The differential

$$\partial : W^* \longrightarrow \mathrm{Sym}^2 V^*$$

is the linear dual of the cup product

$$H^*(M;\mathbb{Q}) \otimes H^*(M;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q}),$$

where we identify $H^*(M;\mathbb{Q})$ with $\mathrm{Hom}_{\mathbb{Q}}(H^*(M;\mathbb{Q}),\mathbb{Q})$ via Poincaré duality.

Definition 5.3.2. For an integer $k \geq 1$ define the *weight k subcomplex* of the symmetric algebra on $V^* \oplus W^*$ by

$$\Omega^*(k)(V^*,W^*) \;:=\; \mathrm{Sym}^k\big(V^* \oplus W^*\big),$$

where V^* has length 1 and W^* has length 2 (so an element $v_{i_1}\cdots v_{i_p} w_{j_1}\cdots w_{j_q}$ has length $p+2q$). The differential ∂ extends to a derivation of $\Omega^*(k)$ by the rule $\partial|_{V^*} = 0$ and $\partial|_{W^*}$ as in Definition 5.3.1.

Theorem 5.3.3 (Félix–Thomas [FT00], Knudsen [Knu17])**.** *For every closed oriented manifold M^{2m} of even dimension and every $k \geq 1$ there is an isomorphism of graded $\mathbb{Q}$-vector spaces*

$$H^*(B_k(M);\mathbb{Q}) \;\cong\; H^*\big(\Omega^*(k)(V^*,W^*),\partial\big).$$

The complex $\Omega^*(k)$ is bigraded by total cohomological degree and by the number q of w-generators in a monomial: a monomial with p factors from V^* and q factors from W^* has length $p+2q$, so the weight k subcomplex decomposes as

$$\Omega^*(k) \;=\; \bigoplus_{q=0}^{\lfloor k/2\rfloor} \mathrm{Sym}^{k-2q}(V^*) \otimes \Lambda^q(W^*).$$

We denote the summand at weight index q by ${}^q\Omega^*(k)$; the differential ∂ has weight degree -1, mapping ${}^q\Omega^*(k)$ to ${}^{q-1}\Omega^{*+1}(k)$.

5.4 Strong stability and the theorem of Berceanu–Yameen

We now state the central abstract result on which the computations in this chapter depend.

Definition 5.4.1 ([BY18], Definition 1.3)**.** A connected manifold M satisfies the *strong stability condition* for its unordered configuration spaces, with *range* r, if the cohomology groups are eventually constant from $k = r$ onwards, i.e. if

$$H^*(B_r(M);\mathbb{Q}) \;\cong\; H^*(B_{r+1}(M);\mathbb{Q}) \;\cong\; H^*(B_{r+2}(M);\mathbb{Q}) \;\cong\; \cdots$$

The strong stability condition is much stronger than the classical homological stability of Church [Chu12], Randal-Williams [RW13] and Knudsen [Knu17], which only asserts that each fixed Betti number $\beta_i(B_k(M))$ is eventually constant in k. Strong stability requires the full cohomology ring, including its top degree (which a priori grows with k), to stabilise.

Definition 5.4.2 ([BY18], page 2)**.** A closed manifold M^{4m} is called a *(rational) homology projective plane* if its Poincaré polynomial is

$$P_M(t) \;=\; 1 + t^{2m} + t^{4m}.$$

Remark 5.4.3. The classical examples of homology projective planes are $\mathbb{CP}^2$ (with $m = 1$), $\mathbb{HP}^2$ (with $m = 2$), and the Cayley plane $\mathbb{OP}^2$ (with $m = 4$). By a theorem of Adams these are the only spheres that can be the total space of a Hopf fibration $S^{4m-1} \to M^{4m}$, which forces the dimensions $4m \in \{4, 8, 16\}$. Eells and Kuiper [EK62] showed, however, that there are infinitely many distinct smooth manifolds in each of these three dimensions that are rational homology projective planes; they are distinguished by the Eells–Kuiper μ-invariant.

Theorem 5.4.4 (Berceanu–Yameen [BY18], Theorem 1.6)**.** *A closed oriented manifold of even dimension has the strong stability property if and only if it is either a rational homology sphere or a rational homology projective plane. The ranges of stability are* 3 *and* 4 *respectively.*

The proof in [BY18] is constructive: it identifies, for each homology projective plane M^{4m}, the explicit list of permanent cocycles in the Chevalley–Eilenberg model of Section 5.3, and shows that all other cocycles are eventually killed. The output is the following explicit formula.

Lemma 5.4.5 (Berceanu–Yameen [BY18], Lemma 4.3)**.** *Let M^{4m} be a rational homology projective plane. Then*

$$P_{B_1(M)}(t) \;=\; P_{B_2(M)}(t) \;=\; 1 + t^{2m} + t^{4m},$$

$$P_{B_3(M)}(t) \;=\; 1 + t^{2m} + t^{4m} + t^{8m-1} + t^{10m-1},$$

and for every $k \geq 4$

$$P_{B_k(M)}(t) \;=\; 1 + t^{2m} + t^{4m} + t^{8m-1} + t^{10m-1} + t^{12m-1}.$$

The case $m = 1$ of Lemma 5.4.5 (i.e. $M = \mathbb{CP}^2$) had been established earlier by Félix and Tanré [FT05].

5.5 Application: configuration spaces of $\mathbb{HP}^2$

We now specialise Lemma 5.4.5 to our case of interest, $M = \mathbb{HP}^2$.

Proposition 5.5.1. *The quaternionic projective plane $\mathbb{HP}^2$ is a closed oriented manifold of even dimension 8, and it is a rational homology projective plane in the sense of Definition 5.4.2.*

Proof. The manifold $\mathbb{HP}^2$ is closed, smooth, orientable, and has real dimension $\dim_{\mathbb{R}} \mathbb{HP}^2 = 4 \cdot 2 = 8$. Its rational cohomology ring is

$$H^*(\mathbb{HP}^2;\mathbb{Q}) \;\cong\; \mathbb{Q}[u]/(u^3), \qquad |u| = 4,$$

so the Poincaré polynomial of $\mathbb{HP}^2$ is $1+t^4+t^8$. With $m = 2$ this is precisely $1+t^{2m}+t^{4m}$, as required by Definition 5.4.2. □

Theorem 5.5.2. *The rational Poincaré polynomial of the unordered configuration space $B_k(\mathbb{HP}^2)$ is*

$$P_{B_k(\mathbb{HP}^2)}(t) \;=\; \begin{cases} 1 & \textit{if } k = 0, \\ 1 + t^4 + t^8 & \textit{if } k \in \{1,2\}, \\ 1 + t^4 + t^8 + t^{15} + t^{19} & \textit{if } k = 3, \\ 1 + t^4 + t^8 + t^{15} + t^{19} + t^{23} & \textit{if } k \geq 4. \end{cases}$$

In particular, all nonzero Betti numbers are equal to 1, and for $k \geq 4$ the cohomology of $B_k(\mathbb{HP}^2)$ is concentrated in degrees $\{0, 4, 8, 15, 19, 23\}$, independently of k.

Proof. The case $k = 0$ is trivial since $B_0(M)$ is a point.

The case $k = 1$ is also immediate: $B_1(M) = M$, so $P_{B_1(\mathbb{HP}^2)}(t) = P_{\mathbb{HP}^2}(t) = 1+t^4+t^8$.

For $k \geq 2$ we apply Lemma 5.4.5, which is applicable by Proposition 5.5.1. With $m = 2$ the relevant degrees are

$$2m = 4, \qquad 4m = 8, \qquad 8m - 1 = 15, \qquad 10m - 1 = 19, \qquad 12m - 1 = 23,$$

which yields the stated polynomials. □

Corollary 5.5.3. *Let M^8 be any closed oriented smooth manifold whose rational cohomology ring is isomorphic to $\mathbb{Q}[u]/(u^3)$ with $|u| = 4$. Then for every $k \geq 0$,*

$$P_{B_k(M)}(t) \;=\; P_{B_k(\mathbb{HP}^2)}(t).$$

In particular, all closed smooth 8-manifolds belonging to the Eells–Kuiper family of $\mathbb{HP}^2$ have the same rational Betti numbers for their unordered configuration spaces of any number of points.

Proof. Such an M is closed, oriented, of even dimension 8, and has Poincaré polynomial $1 + t^4 + t^8$, so it is a rational homology projective plane with $m = 2$. The conclusion follows by applying Theorem 5.5.2 (or directly Lemma 5.4.5) to M. □

Remark 5.5.4. Corollary 5.5.3 is the precise sense in which the rational Betti numbers of configuration spaces detect only the rational cohomology ring of the underlying 8-manifold and not its diffeomorphism or smooth structure. The Eells–Kuiper μ-invariant, which distinguishes infinitely many exotic smooth structures (and finitely many homeomorphism classes!) with the same cohomology ring as $\mathbb{HP}^2$, is invisible to this invariant.

5.6 The analogous statements for $\mathbb{OP}^2$ and for higher projective spaces

The argument of Section 5.5 works without modification for the Cayley plane $\mathbb{OP}^2$.

Proposition 5.6.1. *The Cayley plane $\mathbb{OP}^2$ is a closed oriented manifold of even dimension 16, and it is a rational homology projective plane.*

Proof. $\mathbb{OP}^2$ is a closed simply connected smooth manifold of real dimension 16 with rational cohomology ring

$$H^*(\mathbb{OP}^2;\mathbb{Q}) \;\cong\; \mathbb{Q}[u]/(u^3), \qquad |u| = 8.$$

Its Poincaré polynomial is therefore $1 + t^8 + t^{16}$, which is $1 + t^{2m} + t^{4m}$ for $m = 4$. □

Theorem 5.6.2. *The rational Poincaré polynomial of the unordered configuration space $B_k(\mathbb{OP}^2)$ is*

$$P_{B_k(\mathbb{OP}^2)}(t) \;=\; \begin{cases} 1 & \text{if } k = 0, \\ 1 + t^8 + t^{16} & \text{if } k \in \{1,2\}, \\ 1 + t^8 + t^{16} + t^{31} + t^{39} & \text{if } k = 3, \\ 1 + t^8 + t^{16} + t^{31} + t^{39} + t^{47} & \text{if } k \geq 4. \end{cases}$$

Proof. Apply Lemma 5.4.5 with $m = 4$, using Proposition 5.6.1. □

Explicit cocycle representatives for $B_k(\mathbb{OP}^2)$

The Chevalley–Eilenberg model of $\mathbb{OP}^2$ has the same algebraic structure as that of $\mathbb{HP}^2$, with only the homological degrees of the generators changed.

Proposition 5.6.3. *For $M = \mathbb{OP}^2$, the graded vector spaces appearing in Definition 5.3.1 are*

$$V^* \;=\; \mathbb{Q}\langle a_0, a_1, a_2\rangle, \qquad |a_j| = 8(2-j) \in \{16, 8, 0\},$$
$$W^* \;=\; \mathbb{Q}\langle b_0, b_1, b_2\rangle, \qquad |b_j| = 8(2-j) + 15 \in \{31, 23, 15\},$$

and the differential is

$$\partial(a_i) \;=\; 0, \qquad \partial(b_\ell) \;=\; \sum_{i+j=\ell} a_i a_j \qquad (\textit{with } b_\ell = 0 \textit{ for } \ell > 2).$$

Proof. This follows from Definition 5.3.1 applied to $M = \mathbb{OP}^2$, by the same argument as Proposition 5.7.1. □

Proposition 5.6.4. *For every $k \geq 4$, explicit polynomial cocycle representatives for the six stable cohomology classes of $B_k(\mathbb{OP}^2)$ are given by:*

$$\begin{aligned} H_0 &= \mathbb{Q}\big\langle [\, a_2^k \,] \big\rangle, \\ H_8 &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-1} \,] \big\rangle, \\ H_{16} &= \mathbb{Q}\big\langle [\, a_0 a_2^{k-1} - (k-1) a_1^2 a_2^{k-2} \,] \big\rangle, \\ H_{31} &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-3} b_1 \,] \big\rangle, \\ H_{39} &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-3} b_0 \,] \big\rangle, \\ H_{47} &= \mathbb{Q}\big\langle [\, (a_0 a_2^{k-3} - (k-3) a_1^2 a_2^{k-4}) b_0 \,] \big\rangle. \end{aligned}$$

Proof. The cycle conditions and non-boundary statements depend only on the differential

$$\partial(b_\ell) \;=\; \sum_{i+j=\ell} a_i a_j,$$

not on the specific homological degrees of the generators a_i, b_j. Since this combinatorial structure is identical for $\mathbb{HP}^2$ and $\mathbb{OP}^2$ (compare Propositions 5.7.1 and 5.6.3), the proofs of the cycle and non-boundary properties given for $\mathbb{HP}^2$ (in the case $k = 6$, see Proposition 5.9.1 and Remark 5.9.2) transcribe to $\mathbb{OP}^2$. The degrees are read off from $|a_j| = 8(2-j)$ and $|b_j| = 8(2-j) + 15$ as in Proposition 5.6.3. □

Remark 5.6.5. The cocycles displayed in Proposition 5.6.4 are, as polynomials in the symbols a_i and b_j, *the same* as the cocycles for $\mathbb{HP}^2$ recorded in Remark 5.9.2. Only the homological degrees of the generators differ. This identity is not a coincidence: the differential

$$\partial(b_\ell) \;=\; \sum_{i+j=\ell} a_i a_j$$

encodes only the multiplication table of the cohomology ring $H^*(M;\mathbb{Q}) \cong \mathbb{Q}[u]/(u^3)$, which is the same for $\mathbb{HP}^2$ and $\mathbb{OP}^2$. The Chevalley–Eilenberg complexes of these two manifolds are isomorphic as combinatorial chain complexes; they differ only by an overall rescaling of degrees by a factor of 2 ($m = 2$ for $\mathbb{HP}^2$, $m = 4$ for $\mathbb{OP}^2$). Cocycle representatives therefore correspond under this rescaling, and the explicit formulas in Proposition 5.6.4 are forced by those of Remark 5.9.2.

Theorem 5.6.6. *Let M^{4m} be a closed oriented manifold which is a rational homology projective plane. Then for every $k \geq 4$ the rational cohomology of $B_k(M)$ is concentrated in the six degrees*

$$\{0,\ 2m,\ 4m,\ 8m-1,\ 10m-1,\ 12m-1\},$$

with one-dimensional contribution in each. In particular,

$$P_{B_k(M)}(t) \;=\; 1 + t^{2m} + t^{4m} + t^{8m-1} + t^{10m-1} + t^{12m-1} \qquad \textit{for every } k \geq 4.$$

Proof. This is Lemma 5.4.5 stated in its full generality. □

We collect three usage sof Theorem 5.6.6.

Example 5.6.7. For $M = \mathbb{CP}^2$ (with $m = 1$), Theorem 5.6.6 gives

$$P_{B_k(\mathbb{CP}^2)}(t) \;=\; 1 + t^2 + t^4 + t^7 + t^9 + t^{11} \qquad \text{for } k \geq 4.$$

This recovers the result of Félix and Tanré [FT05].

Example 5.6.8. For $M = \mathbb{HP}^2$ (with $m = 2$),

$$P_{B_k(\mathbb{HP}^2)}(t) \;=\; 1 + t^4 + t^8 + t^{15} + t^{19} + t^{23} \qquad \text{for } k \geq 4,$$

which is Theorem 5.5.2.

Example 5.6.9. For $M = \mathbb{OP}^2$ (with $m = 4$),

$$P_{B_k(\mathbb{OP}^2)}(t) \;=\; 1 + t^8 + t^{16} + t^{31} + t^{39} + t^{47} \qquad \text{for } k \geq 4,$$

which is Theorem 5.6.2.

We now turn to the natural generalisations $\mathbb{HP}^n$ and $\mathbb{OP}^n$.

Proposition 5.6.10. *For every $n \geq 3$, the quaternionic projective space $\mathbb{HP}^n$ is* not *a rational homology projective plane.*

Proof. The cohomology ring of $\mathbb{HP}^n$ is

$$H^*(\mathbb{HP}^n;\mathbb{Q}) \;=\; \mathbb{Q}[u]/(u^{n+1}), \qquad |u| = 4,$$

so its Poincaré polynomial is $1+t^4+t^8+\cdots+t^{4n}$, which has $n+1$ nonzero Betti numbers. For $n \geq 3$ this is strictly greater than 3, so $\mathbb{HP}^n$ does not satisfy the definition of a rational homology projective plane. □

By Theorem 5.4.4, Proposition 5.6.10 immediately implies the following negative result.

Corollary 5.6.11. *For every $n \geq 3$, the manifold $\mathbb{HP}^n$ does* not *satisfy strong stability. Equivalently, the cohomology ring $H^*(B_k(\mathbb{HP}^n);\mathbb{Q})$ is not eventually constant in k.*

The same argument as in [BY18, Lemma 4.7] can be used to make the failure of stability explicit: at weight $2k$ in the Chevalley–Eilenberg complex of $\mathbb{HP}^n$, the monomial v_4^{2k} (where v_4 corresponds to $u \in H^4$) is a permanent cocycle of cohomological degree $8k$ in the cohomology of $B_{2k}(\mathbb{HP}^n)$. Since this degree grows without bound in k, no stable polynomial exists.

Remark 5.6.12 (On $\mathbb{OP}^n$ for $n \geq 3$). The octonionic projective space $\mathbb{OP}^n$ exists as a manifold only for $n \leq 2$. The obstruction is easy: the construction of a higher projective space requires the coordinate ring to be associative, and the octonions are non-associative. The standard references are Baez [Bae02] and Adams [Ada60]. Consequently, the only octonionic projective spaces are

$$\mathbb{OP}^0 \;=\; \text{point}, \qquad \mathbb{OP}^1 \;\cong\; S^8, \qquad \mathbb{OP}^2 \;=\; \text{the Cayley plane.}$$

Of these, $\mathbb{OP}^1$ is clearly a rational homology sphere (Theorem 5.4.4 and [BY18, Lemma 4.2]), and $\mathbb{OP}^2$ is a rational homology projective plane (Theorem 5.6.2). Both therefore satisfy strong stability.

The discussion above shows that the strong stability framework completely classifies the configuration cohomology of the four classical projective planes

$$\mathbb{CP}^2, \quad \mathbb{HP}^2, \quad \mathbb{OP}^2,$$

together with all their Eells–Kuiper rational siblings, but stops short of providing analogous results for the higher quaternionic projective spaces $\mathbb{HP}^n$. For these manifolds, what one obtains instead is the weaker classical (degree-by-degree) homological stability of Church–Randal-Williams–Knudsen, supplemented by the *shifted stability* framework of Berceanu–Yameen [BY18, Sections 6-7]; an analysis of the latter for $\mathbb{HP}^n$ is, to our knowledge, not yet available in the published literature.

5.7 Comparison with the partial computation via the Félix–Thomas complex

The conclusion of Theorem 5.5.2 can be independently verified, for small values of k, by direct computation in the Chevalley–Eilenberg complex of Section 5.3. While this is not necessary for the proof, we include the case $k = 6$ in Section 5.9 as a worked illustration of the algebraic machinery and as a check on the abstract theorem; explicit computations of this kind for $\mathbb{HP}^2$ do not appear in the literature.

We first record the structure of the model in a form convenient for explicit calculations.

Proposition 5.7.1. *For $M = \mathbb{HP}^2$, the graded vector spaces appearing in Definition 5.3.1 are*

$$V^* = \mathbb{Q}\langle a_0, a_1, a_2\rangle, \qquad |a_j| = 4(2-j) \in \{8, 4, 0\},$$

$$W^* = \mathbb{Q}\langle b_0, b_1, b_2\rangle, \qquad |b_j| = 4(2-j) + 7 \in \{15, 11, 7\},$$

where a_j has length 1 *and b_j has length* 2*. The differential ∂ is the derivation extending*

$$\partial(a_i) = 0, \qquad \partial(b_\ell) = \sum_{i+j=\ell} a_i a_j \qquad (\textit{with } b_\ell = 0 \textit{ for } \ell > 2).$$

Proof. The vector space $V^* = H^*(\mathbb{HP}^2; \mathbb{Q})$ has basis $\{1, u, u^2\}$ in cohomological degrees $0, 4, 8$. The labelling $a_0 = u^2$, $a_1 = u$, $a_2 = 1$ is chosen so that lower index corresponds to higher cohomological degree, in agreement with [Knu17]: the homological grading on the Chevalley–Eilenberg complex shifts cohomological degree $4j$ to homological degree $4(2 - j)$.

The vector space $W^* = H^*(\mathbb{HP}^2; \mathbb{Q})[2m - 1] = H^*(\mathbb{HP}^2; \mathbb{Q})[7]$ similarly has a basis $\{b_0, b_1, b_2\}$ with b_j corresponding to u^{2-j}, in homological degrees $8, 4, 0$ shifted up by 7, giving $15, 11, 7$.

The differential $\partial(b_\ell)$ is the dual of the cup product $H^*(\mathbb{HP}^2; \mathbb{Q}) \otimes H^*(\mathbb{HP}^2; \mathbb{Q}) \to H^*(\mathbb{HP}^2; \mathbb{Q})$. Since $u^i \cdot u^j = u^{i+j}$ if $i + j \leq 2$ and 0 otherwise, dualising gives the stated formula. □

5.8 The modular block decomposition for $\mathbb{HP}^2$ and $\mathbb{OP}^2$

We now isolate a structural observation, specific to $\mathbb{HP}^2$ and $\mathbb{OP}^2$, which simplifies the explicit verification of cocycle representatives. The observation is that in each fixed homological degree, the Chevalley–Eilenberg complex contains generators from at most one of its four blocks ${}^q\Omega^*(k)$. This phenomenon is absent for $\mathbb{CP}^2$.

Definition 5.8.1. For a closed oriented even-dimensional manifold M^{2m} and an integer $k \geq 1$, write $\Omega^*(k) = \bigoplus_{q\geq 0} {}^q\Omega^*(k)$ for the decomposition into blocks by the number of b-generators, as in Section 5.3. Each block ${}^q\Omega^*(k)$ has its own grading by homological degree.

Proposition 5.8.2 ('Modular' differences for $\mathbb{HP}^2$)**.** *Let $M = \mathbb{HP}^2$. Every monomial in ${}^q\Omega^*(k)(\mathbb{HP}^2)$ has homological degree congruent to $3q$ modulo 4. The four possible block indices $q \in \{0, 1, 2, 3\}$ therefore correspond bijectively to the four residue classes modulo 4:*

$$q = 0 \leftrightarrow 0, \quad q = 1 \leftrightarrow 3, \quad q = 2 \leftrightarrow 2, \quad q = 3 \leftrightarrow 1 \pmod 4.$$

Proof. By Proposition 5.7.1, $|a_j| \in \{0, 4, 8\}$, so every a-generator has degree divisible by 4. Similarly $|b_j| \in \{7, 11, 15\}$, so every b-generator has degree congruent to 3 modulo 4. A monomial $a_0^{p_0} a_1^{p_1} a_2^{p_2} \cdot b_{j_1} \wedge \cdots \wedge b_{j_q}$ has homological degree

$$\big(\text{multiple of four}\big) + \sum_{r=1}^{q} |b_{j_r}|,$$

the second summand being $\equiv 3q \pmod 4$. For $q \in \{0, 1, 2, 3\}$ the residues $3q \pmod 4$ are $0, 3, 2, 1$, all distinct. □

Proposition 5.8.3 (Modular separation for $\mathbb{OP}^2$)**.** *Let $M = \mathbb{OP}^2$. Every monomial in ${}^q\Omega^*(k)(\mathbb{OP}^2)$ has homological degree congruent to $7q$ modulo 8. The four possible block indices $q \in \{0, 1, 2, 3\}$ therefore correspond bijectively to four distinct residue classes modulo 8:*

$$q = 0 \leftrightarrow 0, \quad q = 1 \leftrightarrow 7, \quad q = 2 \leftrightarrow 6, \quad q = 3 \leftrightarrow 5 \pmod 8.$$

Proof. By Proposition 5.6.3, $|a_j| \in \{0, 8, 16\}$, so every a-generator has degree divisible by 8. Similarly $|b_j| \in \{15, 23, 31\}$, so every b-generator has degree congruent to 7 modulo 8. The argument then proceeds as for $\mathbb{HP}^2$. □

Corollary 5.8.4 (Single-block structure in each degree)**.** *For $M \in \{\mathbb{HP}^2, \mathbb{OP}^2\}$ and every homological degree d, the subspace $\Omega^d(k) \subset \Omega^*(k)$ at degree d is contained in a single block ${}^q\Omega^*(k)$, with q determined by d modulo 4 (resp. 8) according to Propositions 5.8.2 and 5.8.3.*

Consequently, in each homological degree the chain complex at weight k is the linear map

$$\partial : {}^q\Omega^d(k) \longrightarrow {}^{q-1}\Omega^{d-1}(k)$$

between two specific block components: that is, the computation of $H_d\big(\Omega^*(k)\big)$ reduces in each degree to a single rank computation.

Proof. Immediate from Propositions 5.8.2 and 5.8.3: distinct residues modulo 4 (resp. 8) cannot be occupied by generators in two different blocks at the same homological degree. The differential ∂ decreases homological degree by 1 and block index q by 1, so it carries the (unique) block at degree d to the (unique) block at degree $d - 1$. □

Remark 5.8.5 (Failure for $\mathbb{CP}^2$). The same argument does *not* apply to $\mathbb{CP}^2$. For $M = \mathbb{CP}^2$ we have $|a_j| \in \{0, 2, 4\}$ and $|b_j| \in \{3, 5, 7\}$, so a-generators have even degree and b-generators have odd degree. The four blocks ${}^q\Omega^*(k)(\mathbb{CP}^2)$ for $q = 0, 1, 2, 3$ therefore sit in degrees of parity $0, 1, 0, 1$ respectively, and the natural separation into four residue classes fails: only two parities are available, and the blocks $q = 0, 2$ both contribute to even degrees, while $q = 1, 3$ both contribute to odd degrees.

In any fixed homological degree of weight k for $\mathbb{CP}^2$, two distinct blocks may therefore contribute simultaneously, and the rank computation does not reduce to a single linear map. The modular separation in Propositions 5.8.2 and 5.8.3 is a feature of the projective planes $\mathbb{HP}^2$ and $\mathbb{OP}^2$ which is absent for the complex projective plane $\mathbb{CP}^2$.

Remark 5.8.6 (General homology projective planes). The argument of Propositions 5.8.2 and 5.8.3 extends to any rational homology projective plane M^{4m}. For such an M the a-generators have degrees in $\{0, 2m, 4m\}$ and the b-generators in $\{4m-1, 6m-1, 8m-1\}$. A monomial in ${}^q\Omega^*(k)(M)$ has degree

$$\big(2m\text{-multiple}\big) + q \cdot (\text{a value congruent to } -1 \bmod 2m) \equiv -q \pmod{2m}.$$

The four block indices $q \in \{0, 1, 2, 3\}$ produce four distinct residues modulo $2m$ provided $2m \geq 4$, that is, $m \geq 2$. For $m = 1$ (the case $\mathbb{CP}^2$) only two residues are available, which is exactly the failure observed in Remark 5.8.5.

The modular block decomposition is therefore a feature of *higher-dimensional* rational homology projective planes ($m \geq 2$), encompassing both $\mathbb{HP}^2$ and $\mathbb{OP}^2$ but excluding $\mathbb{CP}^2$.

5.9 The case $k = 6$: a fully explicit computation

We illustrate the abstract result by computing $H^*(B_6(\mathbb{HP}^2); \mathbb{Q})$ by hand from the Chevalley–Eilenberg complex of Proposition 5.7.1, carrying out every step. The answer agrees, as it must, with Theorem 5.5.2; the point of the section is to display the explicit cocycle representatives and to show how the modular block decomposition of Section 5.8 reduces a 92-dimensional complex to a short sequence of small, completely explicit linear-algebra problems.

5.9.1 The complex and its differential

At weight $k = 6$ the four blocks of $\Omega^*(6)$ have dimensions

$$\dim {}^0\Omega(6) = \binom{8}{6} = 28, \quad \dim {}^1\Omega(6) = \binom{6}{4} \cdot 3 = 45,$$

$$\dim {}^2\Omega(6) = \binom{4}{2} \cdot 3 = 18, \quad \dim {}^3\Omega(6) = \binom{3}{3} = 1,$$

for a total of 92. We recall the differential. Writing a monomial as $a^P\beta = a_0^{p_0}a_1^{p_1}a_2^{p_2}\,\beta$ with β a wedge of distinct b-generators, the differential acts by collecting two of the a-factors into a single b-generator according to the rule $b_\ell = \sum_{i+j=\ell} a_i a_j$ of Proposition 5.7.1, namely

$$\partial(a^P\beta) = \binom{p_0}{2} a^{P-2\mathbf{e}_0}(b_0 \wedge \beta) + p_0p_1\, a^{P-\mathbf{e}_0-\mathbf{e}_1}(b_1 \wedge \beta) + \Big(p_0p_2\, a^{P-\mathbf{e}_0-\mathbf{e}_2} + \tbinom{p_1}{2}\, a^{P-2\mathbf{e}_1}\Big)(b_2 \wedge \beta), \tag{5.1}$$

where $\mathbf{e}_i$ denotes the unit exponent in the a_i-direction, the symbol b_ℓ is set to 0 for $\ell > 2$ (there is no b_3, b_4), and $b_\ell \wedge \beta = 0$ whenever b_ℓ already divides β. The pairs (a_1, a_2) and (a_2, a_2) contribute nothing because they would produce b_3 and b_4. Since the b-generators are odd, $b_\ell \wedge \beta$ is reordered into the standard increasing order $b_0 < b_1 < b_2$ at the cost of a Koszul sign (-1) per transposition. The differential lowers homological degree by 1 and raises the block index q by 1:

$${}^0\Omega(6) \xrightarrow{\partial_1} {}^1\Omega(6) \xrightarrow{\partial_2} {}^2\Omega(6) \xrightarrow{\partial_3} {}^3\Omega(6).$$

The Euler characteristic of the complex is

$$\chi\big(\Omega^*(6)\big) = 28 - 45 + 18 - 1 = 0 = \binom{3}{6},$$

consistent with the generating function $\sum_{k\geq 0} \chi(B_k(\mathbb{HP}^2))\, t^k = (1+t)^{\chi(\mathbb{HP}^2)} = (1+t)^3$ of Félix–Thomas.

5.9.2 Reduction to threads

The computation is organised by the modular block decomposition of Section 5.8. By Proposition 5.8.2, a monomial in ${}^q\Omega(6)$ has homological degree $\equiv 3q \pmod 4$; the four blocks $q = 0, 1, 2, 3$ therefore occupy the four residue classes $0, 3, 2, 1 \pmod 4$ respectively, and in each fixed homological degree the complex consists of monomials from a *single* block. Because ∂ lowers the degree by 1 and raises q by 1, and because q can only take the values $0, 1, 2, 3$, the complex splits as a direct sum of *threads*: for each $D \equiv 0 \pmod 4$, the thread $\mathcal{T}_D$ is the four-term cochain complex

$$\mathcal{T}_D : \qquad {}^0\Omega(6)_D \xrightarrow{\partial_1} {}^1\Omega(6)_{D-1} \xrightarrow{\partial_2} {}^2\Omega(6)_{D-2} \xrightarrow{\partial_3} {}^3\Omega(6)_{D-3},$$

and every monomial of $\Omega^*(6)$ lies in exactly one thread. The cohomology of $\mathcal{T}_D$ computes the Betti numbers in the four consecutive degrees $D, D-1, D-2, D-3$. Writing $\rho_i = \operatorname{rank} \partial_i$ for the thread $\mathcal{T}_D$, the dimensions of its cohomology groups are

$$\begin{aligned} \dim H_D &= \dim {}^0\Omega_D - \rho_1, & \dim H_{D-1} &= \dim {}^1\Omega_{D-1} - \rho_1 - \rho_2, \\ \dim H_{D-2} &= \dim {}^2\Omega_{D-2} - \rho_2 - \rho_3, & \dim H_{D-3} &= \dim {}^3\Omega_{D-3} - \rho_3. \end{aligned} \tag{5.2}$$

The thirteen threads $\mathcal{T}_0, \mathcal{T}_4, \ldots, \mathcal{T}_{48}$ have the following dimension vectors $(\dim {}^0\Omega_D, \dim {}^1\Omega_{D-1}, \dim {}^2\Omega_{D-2}, \dim {}^3\Omega_{D-3})$:

D	0	4	8	12	16	20	24	28	32	36	40	44	48
${}^0\Omega_D$	1	1	2	2	3	3	4	3	3	2	2	1	1
${}^1\Omega_{D-1}$	0	0	1	2	4	5	7	7	7	5	4	2	1
${}^2\Omega_{D-2}$	0	0	0	0	0	1	2	4	4	4	2	1	0
${}^3\Omega_{D-3}$	0	0	0	0	0	0	0	0	0	1	0	0	0

(The columns sum to $28, 45, 18, 1$, recovering the block dimensions.) Note that the single generator of ${}^3\Omega(6)$, namely $b_0 b_1 b_2$ in degree 33, lies in the thread $\mathcal{T}_{36}$.

We now compute each thread in turn. In every case we write out the differentials on the relevant monomial bases and record their ranks; the cohomology then follows from (5.2).

5.9.3 The threads carrying cohomology

Thread $\mathcal{T}_0$, degree 0. Here ${}^0\Omega(6)_0 = \mathbb{Q}\langle a_2^6 \rangle$ and all higher terms vanish. Every pairing in (5.1) requires a factor a_0 or two factors a_1, of which a_2^6 has none, so $\partial_1(a_2^6) = 0$. Thus $\rho_1 = 0$, whence $\dim H_0 = 1 - 0 = 1$ and

$$H_0 = \mathbb{Q}\langle [a_2^6] \rangle.$$

Thread $\mathcal{T}_4$, degree 4. Here ${}^0\Omega(6)_4 = \mathbb{Q}\langle a_1a_2^5\rangle$. The only candidate pairing is $\binom{p_1}{2} = \binom{1}{2} = 0$, so $\partial_1(a_1a_2^5) = 0$, whence $\dim H_4 = 1 - 0 = 1$ and

$$H_4 = \mathbb{Q}\langle[a_1a_2^5]\rangle.$$

Thread $\mathcal{T}_8$, degrees 8, 7. The relevant terms are ${}^0\Omega_8 = \mathbb{Q}\langle a_1^2a_2^4,\, a_0a_2^5\rangle$ and ${}^1\Omega_7 = \mathbb{Q}\langle a_2^4b_2\rangle$. From (5.1),

$$\partial_1(a_1^2a_2^4) = a_2^4b_2, \qquad \partial_1(a_0a_2^5) = 5\,a_2^4b_2,$$

so ∂_1 has matrix $(\,1\;5\,)$ and rank $\rho_1 = 1$; and $\partial_2 = 0$ since ${}^2\Omega_6 = 0$, so $\rho_2 = 0$. Hence

$$\dim H_8 = 2 - 1 = 1, \qquad \dim H_7 = (1-1) - 0 = 0,$$

with $H_8 = \mathbb{Q}\langle[a_0a_2^5 - 5\,a_1^2a_2^4]\rangle$ spanned by the kernel of ∂_1.

Thread $\mathcal{T}_{16}$, degrees 16, 15. We have ${}^0\Omega_{16} = \mathbb{Q}\langle a_1^4a_2^2,\, a_0a_1^2a_2^3,\, a_0^2a_2^4\rangle$ and ${}^1\Omega_{15} = \mathbb{Q}\langle a_2^4b_0,\, a_1a_2^3b_1,\, a_1^2a_2^2b_2,\, a_0a_2^3b_2\rangle$, while ${}^2\Omega_{14} = 0$. The differential is

$$\begin{aligned}
\partial_1(a_1^4a_2^2) &= 6\,a_1^2a_2^2b_2,\\
\partial_1(a_0a_1^2a_2^3) &= 2\,a_1a_2^3b_1 + 3\,a_1^2a_2^2b_2 + a_0a_2^3b_2,\\
\partial_1(a_0^2a_2^4) &= a_2^4b_0 + 8\,a_0a_2^3b_2.
\end{aligned}$$

In the ordered basis $(a_2^4b_0,\, a_1a_2^3b_1,\, a_1^2a_2^2b_2,\, a_0a_2^3b_2)$ of ${}^1\Omega_{15}$ the three images are

$$(0,0,6,0), \quad (0,2,3,1), \quad (1,0,0,8),$$

which are linearly independent (the first, second and fourth coordinates already separate them), so $\rho_1 = 3$; and $\rho_2 = 0$. Therefore

$$\dim H_{16} = 3 - 3 = 0, \qquad \dim H_{15} = (4-0) - 3 = 1.$$

A generator of H_{15} is represented by $a_1a_2^3b_1$: it is a ∂_2-cocycle (indeed $\partial_2 = 0$ here), and it is not a boundary because the coordinate vector $(0,1,0,0)$ does not lie in the span of the three image vectors above. Hence

$$H_{15} = \mathbb{Q}\langle[a_1a_2^3b_1]\rangle.$$

Thread $\mathcal{T}_{20}$, degrees 20, 19, 18. The terms are

$$\begin{aligned}
{}^0\Omega_{20} &= \mathbb{Q}\langle a_1^5a_2,\, a_0a_1^3a_2^2,\, a_0^2a_1a_2^3\rangle,\\
{}^1\Omega_{19} &= \mathbb{Q}\langle a_1a_2^3b_0,\, a_1^2a_2^2b_1,\, a_1^3a_2b_2,\, a_0a_2^3b_1,\, a_0a_1a_2^2b_2\rangle,\\
{}^2\Omega_{18} &= \mathbb{Q}\langle a_2^2b_1b_2\rangle.
\end{aligned}$$

The first differential is

$$\begin{aligned}
\partial_1(a_1^5a_2) &= 10\,a_1^3a_2b_2,\\
\partial_1(a_0a_1^3a_2^2) &= 3\,a_1^2a_2^2b_1 + 2\,a_1^3a_2b_2 + 3\,a_0a_1a_2^2b_2,\\
\partial_1(a_0^2a_1a_2^3) &= a_1a_2^3b_0 + 2\,a_0a_2^3b_1 + 6\,a_0a_1a_2^2b_2.
\end{aligned}$$

In the basis $(a_1a_2^3b_0,\ a_1^2a_2^2b_1,\ a_1^3a_2b_2,\ a_0a_2^3b_1,\ a_0a_1a_2^2b_2)$ these are

$$(0,0,10,0,0), \quad (0,3,2,0,3), \quad (1,0,0,2,6),$$

linearly independent, so $\rho_1 = 3$. The second differential, computed from (5.1) with the sign $b_2 \wedge b_1 = -b_1b_2$, is

$$\partial_2(a_1^2a_2^2b_1) = -a_2^2b_1b_2, \quad \partial_2(a_0a_2^3b_1) = -3\,a_2^2b_1b_2, \quad \partial_2(a_0a_1a_2^2b_2) = a_2^2b_1b_2,$$

and ∂_2 vanishes on $a_1a_2^3b_0$ and $a_1^3a_2b_2$ (their only available pairing produces $b_2 \wedge b_2 = 0$ or no b at all). Thus ∂_2 has rank $\rho_2 = 1$, and $\rho_3 = 0$ since ${}^3\Omega_{17} = 0$. Therefore

$$\dim H_{20} = 3 - 3 = 0, \qquad \dim H_{19} = (5-1) - 3 = 1, \qquad \dim H_{18} = (1-0) - 1 = 0.$$

A generator of H_{19} is $a_1a_2^3b_0$: it lies in $\ker \partial_2$, and its coordinate vector $(1,0,0,0,0)$ is not in the span of the three ∂_1-images (the first coordinate forces a contribution from the third image vector, whose fourth coordinate then cannot be cancelled). Hence

$$H_{19} = \mathbb{Q}\langle [a_1a_2^3b_0] \rangle.$$

Thread $\mathcal{T}_{24}$, degrees $24, 23, 22$**.** The terms are

$$\begin{aligned}
{}^0\Omega_{24} &= \mathbb{Q}\langle a_1^6,\ a_0a_1^4a_2,\ a_0^2a_1^2a_2^2,\ a_0^3a_2^3 \rangle,\\
{}^1\Omega_{23} &= \mathbb{Q}\langle a_1^2a_2^2b_0,\ a_1^3a_2b_1,\ a_1^4b_2,\ a_0a_2^3b_0,\ a_0a_1a_2^2b_1,\ a_0a_1^2a_2b_2,\ a_0^2a_2^2b_2 \rangle,\\
{}^2\Omega_{22} &= \mathbb{Q}\langle a_2^2b_0b_2,\ a_1a_2b_1b_2 \rangle.
\end{aligned}$$

The first differential is

$$\begin{aligned}
\partial_1(a_1^6) &= 15\,a_1^4b_2,\\
\partial_1(a_0a_1^4a_2) &= 4\,a_1^3a_2b_1 + a_1^4b_2 + 6\,a_0a_1^2a_2b_2,\\
\partial_1(a_0^2a_1^2a_2^2) &= a_1^2a_2^2b_0 + 4\,a_0a_1a_2^2b_1 + 4\,a_0a_1^2a_2b_2 + a_0^2a_2^2b_2,\\
\partial_1(a_0^3a_2^3) &= 3\,a_0a_2^3b_0 + 9\,a_0^2a_2^2b_2.
\end{aligned}$$

These four images are linearly independent: in the basis order above, the monomials $a_1^2a_2^2b_0$, $a_0a_2^3b_0$ and $a_1^3a_2b_1$ appear in only one image each, which already forces independence. Thus $\rho_1 = 4$. The second differential, with the signs $b_2 \wedge b_0 = -b_0b_2$ and $b_2 \wedge b_1 = -b_1b_2$, is

$$\begin{aligned}
\partial_2(a_1^2a_2^2b_0) &= -a_2^2b_0b_2, & \partial_2(a_1^3a_2b_1) &= -3\,a_1a_2b_1b_2,\\
\partial_2(a_0a_2^3b_0) &= -3\,a_2^2b_0b_2, & \partial_2(a_0a_1a_2^2b_1) &= -2\,a_1a_2b_1b_2,\\
\partial_2(a_0a_1^2a_2b_2) &= 2\,a_1a_2b_1b_2, & \partial_2(a_0^2a_2^2b_2) &= a_2^2b_0b_2,
\end{aligned}$$

with $\partial_2(a_1^4b_2) = 0$. The images span both basis vectors of ${}^2\Omega_{22}$, so $\rho_2 = 2$; and $\rho_3 = 0$. Therefore

$$\dim H_{24} = 4 - 4 = 0, \qquad \dim H_{23} = (7-2) - 4 = 1, \qquad \dim H_{22} = (2-0) - 2 = 0.$$

A generator of H_{23} is $(a_0a_2^3 - 3\,a_1^2a_2^2)b_0$. It is a ∂_2-cocycle, since

$$\partial_2\big((a_0a_2^3 - 3\,a_1^2a_2^2)b_0\big) = -3\,a_2^2b_0b_2 - 3(-a_2^2b_0b_2) = 0,$$

and a short check shows its coordinate vector is not a ∂_1-boundary (matching the coordinate of $a_1^2a_2^2b_0$ forces a contribution that breaks the coordinate of $a_0a_1a_2^2b_1$). Hence

$$H_{23} = \mathbb{Q}\langle [(a_0a_2^3 - 3\,a_1^2a_2^2)b_0] \rangle.$$

5.9.4 The acyclic threads

It remains to verify that every other thread is exact, so that the Betti numbers vanish in all remaining degrees. In each case $\rho_3 = 0$ unless the thread is $\mathcal{T}_{36}$, and the dimension vectors have alternating sum zero, so it suffices to check that the differentials attain their maximal possible ranks.

Thread $\mathcal{T}_{12}$, degrees 12, 11. This thread has ${}^2\Omega = 0$, hence is a two-term complex with ${}^0\Omega_{12} = \mathbb{Q}\langle a_1^3a_2^3,\ a_0a_1a_2^4\rangle$ and ${}^1\Omega_{11} = \mathbb{Q}\langle a_2^4b_1,\ a_1a_2^3b_2\rangle$. We have

$$\partial_1(a_1^3a_2^3) = 3\,a_1a_2^3b_2, \qquad \partial_1(a_0a_1a_2^4) = a_2^4b_1 + 4\,a_1a_2^3b_2.$$

The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix}\right)$ has determinant $-3 \neq 0$, so $\rho_1 = 2$ and

$$\dim H_{12} = 2 - 2 = 0, \qquad \dim H_{11} = (2-0) - 2 = 0.$$

Thread $\mathcal{T}_{28}$, degrees 28, 27, 26, 25. With ${}^3\Omega_{25} = 0$ this is a three-term complex of dimensions $(3, 7, 4)$. The differentials are

$$\begin{aligned}
\partial_1(a_0a_1^5) &= 5\,a_1^4b_1 + 10\,a_0a_1^3b_2,\\
\partial_1(a_0^2a_1^3a_2) &= a_1^3a_2b_0 + 6\,a_0a_1^2a_2b_1 + 2\,a_0a_1^3b_2 + 3\,a_0^2a_1a_2b_2,\\
\partial_1(a_0^3a_1a_2^2) &= 3\,a_0a_1a_2^2b_0 + 3\,a_0^2a_2^2b_1 + 6\,a_0^2a_1a_2b_2,
\end{aligned}$$

which are independent (the monomials $a_1^3a_2b_0$, $a_0a_1a_2^2b_0$ and $a_1^4b_1$ each occur in a single image), so $\rho_1 = 3$. The second differential, in the basis $(a_1^2b_1b_2,\ a_0a_2b_1b_2,\ a_1a_2b_0b_2,\ a_2^2b_0b_1)$ of ${}^2\Omega_{26}$, is

$$\begin{aligned}
\partial_2(a_1^3a_2b_0) &= -3\,a_1a_2b_0b_2, & \partial_2(a_0a_1a_2^2b_0) &= -2\,a_1a_2b_0b_2 - a_2^2b_0b_1,\\
\partial_2(a_1^4b_1) &= -6\,a_1^2b_1b_2, & \partial_2(a_0a_1^2a_2b_1) &= -a_1^2b_1b_2 - a_0a_2b_1b_2,\\
\partial_2(a_0^2a_2^2b_1) &= a_2^2b_0b_1 - 4\,a_0a_2b_1b_2, & \partial_2(a_0a_1^3b_2) &= 3\,a_1^2b_1b_2,\\
\partial_2(a_0^2a_1a_2b_2) &= a_1a_2b_0b_2 + a_0a_2b_1b_2.
\end{aligned}$$

The four images $\partial_2(a_0a_1^3b_2) = (3,0,0,0)$, $\partial_2(a_0^2a_1a_2b_2) = (0,1,1,0)$, $\partial_2(a_1^3a_2b_0) = (0,0,-3,0)$ and $\partial_2(a_0^2a_2^2b_1) = (0,-4,0,1)$ have determinant $18 \neq 0$, so $\rho_2 = 4$. Hence

$$\dim H_{28} = 3-3 = 0, \quad \dim H_{27} = 7-3-4 = 0, \quad \dim H_{26} = (4-0)-4 = 0, \quad \dim H_{25} = 0.$$

Thread $\mathcal{T}_{32}$, degrees 32, 31, 30, 29. Again a three-term complex of dimensions $(3, 7, 4)$. The first differential is

$$\begin{aligned}
\partial_1(a_0^2a_1^4) &= a_1^4b_0 + 8\,a_0a_1^3b_1 + 6\,a_0^2a_1^2b_2,\\
\partial_1(a_0^3a_1^2a_2) &= 3\,a_0a_1^2a_2b_0 + 6\,a_0^2a_1a_2b_1 + 3\,a_0^2a_1^2b_2 + a_0^3a_2b_2,\\
\partial_1(a_0^4a_2^2) &= 6\,a_0^2a_2^2b_0 + 8\,a_0^3a_2b_2,
\end{aligned}$$

independent (the monomials $a_1^4b_0$, $a_0a_1^2a_2b_0$ and $a_0^2a_2^2b_0$ separate them), so $\rho_1 = 3$. The second differential, in the basis $(a_0a_1b_1b_2,\ a_1^2b_0b_2,\ a_0a_2b_0b_2,\ a_1a_2b_0b_1)$ of ${}^2\Omega_{30}$, is

$$\begin{aligned}
\partial_2(a_1^4b_0) &= -6\,a_1^2b_0b_2, & \partial_2(a_0a_1^2a_2b_0) &= -a_1^2b_0b_2 - a_0a_2b_0b_2 - 2\,a_1a_2b_0b_1,\\
\partial_2(a_0^2a_2^2b_0) &= -4\,a_0a_2b_0b_2, & \partial_2(a_0a_1^3b_1) &= -3\,a_0a_1b_1b_2,\\
\partial_2(a_0^2a_1a_2b_1) &= a_1a_2b_0b_1 - 2\,a_0a_1b_1b_2, & \partial_2(a_0^2a_1^2b_2) &= a_1^2b_0b_2 + a_0a_1b_1b_2,\\
\partial_2(a_0^3a_2b_2) &= 3\,a_0a_2b_0b_2.
\end{aligned}$$

The images $\partial_2(a_0a_1^3b_1) = (-3,0,0,0)$, $\partial_2(a_1^4b_0) = (0,-6,0,0)$, $\partial_2(a_0^3a_2b_2) = (0,0,3,0)$ and $\partial_2(a_0^2a_1a_2b_1) = (-2,0,0,1)$ have determinant $-72 \neq 0$, so $\rho_2 = 4$. Hence

$$\dim H_{32} = 0, \quad \dim H_{31} = 7-3-4 = 0, \quad \dim H_{30} = 4-4 = 0, \quad \dim H_{29} = 0.$$

Thread $\mathcal{T}_{36}$, degrees $36, 35, 34, 33$**.** This is the only thread reaching ${}^3\Omega(6)$; its dimensions are $(2,5,4,1)$. We compute all three differentials. The first is

$$\partial_1(a_0^3a_1^3) = 3\,a_0a_1^3b_0 + 9\,a_0^2a_1^2b_1 + 3\,a_0^3a_1b_2, \quad \partial_1(a_0^4a_1a_2) = 6\,a_0^2a_1a_2b_0 + 4\,a_0^3a_2b_1 + 4\,a_0^3a_1b_2,$$

manifestly independent, so $\rho_1 = 2$. The second differential, in the basis $(a_0^2b_1b_2,\ a_0a_1b_0b_2,\ a_1^2b_0b_1,\ a_0a_2b_0b_1)$ of ${}^2\Omega_{34}$, is

$$\begin{aligned}
\partial_2(a_0a_1^3b_0) &= -3\,a_1^2b_0b_1 - 3\,a_0a_1b_0b_2,\\
\partial_2(a_0^2a_1a_2b_0) &= -2\,a_0a_2b_0b_1 - 2\,a_0a_1b_0b_2,\\
\partial_2(a_0^2a_1^2b_1) &= a_1^2b_0b_1 - a_0^2b_1b_2,\\
\partial_2(a_0^3a_2b_1) &= 3\,a_0a_2b_0b_1 - 3\,a_0^2b_1b_2,\\
\partial_2(a_0^3a_1b_2) &= 3\,a_0a_1b_0b_2 + 3\,a_0^2b_1b_2.
\end{aligned}$$

The third differential lands in ${}^3\Omega_{33} = \mathbb{Q}\langle b_0b_1b_2\rangle$; using $b_1 \wedge b_0 \wedge b_2 = -b_0b_1b_2$ and $b_2 \wedge b_0 \wedge b_1 = b_0b_1b_2$,

$$\partial_3(a_0^2b_1b_2) = b_0b_1b_2, \quad \partial_3(a_0a_1b_0b_2) = -b_0b_1b_2,$$

$$\partial_3(a_1^2b_0b_1) = b_0b_1b_2, \quad \partial_3(a_0a_2b_0b_1) = b_0b_1b_2.$$

Thus $\partial_3 = (1,-1,1,1)$ has rank $\rho_3 = 1$, giving $\dim H_{33} = 1-1 = 0$: the class $b_0b_1b_2$ is a boundary, hit for instance by $a_0^2b_1b_2$. For ∂_2, the image lies in $\ker \partial_3 = \{(y_1,y_2,y_3,y_4) : y_1 - y_2 + y_3 + y_4 = 0\}$, a three-dimensional space, so $\rho_2 \leq 3$; and the three images $\partial_2(a_0^2a_1^2b_1) = (-1,0,1,0)$, $\partial_2(a_0^3a_2b_1) = (-3,0,0,3)$ and $\partial_2(a_0^3a_1b_2) = (3,3,0,0)$ are independent, so $\rho_2 = 3$. Therefore

$$\dim H_{36} = 2-2 = 0, \quad \dim H_{35} = 5-2-3 = 0,$$

$$\dim H_{34} = (4-1)-3 = 0, \quad \dim H_{33} = 1-1 = 0.$$

Thread $\mathcal{T}_{40}$, degrees $40, 39, 38, 37$**.** Dimensions $(2,4,2,0)$. The first differential is

$$\partial_1(a_0^4a_1^2) = 6\,a_0^2a_1^2b_0 + 8\,a_0^3a_1b_1 + a_0^4b_2, \qquad \partial_1(a_0^5a_2) = 10\,a_0^3a_2b_0 + 5\,a_0^4b_2,$$

independent, so $\rho_1 = 2$. The second, in the basis $(a_0^2b_0b_2,\ a_0a_1b_0b_1)$ of ${}^2\Omega_{38}$, is

$$\partial_2(a_0^2a_1^2b_0) = -a_0^2b_0b_2 - 4\,a_0a_1b_0b_1, \quad \partial_2(a_0^3a_2b_0) = -3\,a_0^2b_0b_2,$$

$$\partial_2(a_0^3a_1b_1) = 3\,a_0a_1b_0b_1, \quad \partial_2(a_0^4b_2) = 6\,a_0^2b_0b_2,$$

which span ${}^2\Omega_{38}$, so $\rho_2 = 2$. Hence

$$\dim H_{40} = 0, \quad \dim H_{39} = 4-2-2 = 0, \quad \dim H_{38} = 2-2 = 0, \quad \dim H_{37} = 0.$$

Thread $\mathcal{T}_{44}$, degrees $44, 43, 42, 41$**.** Dimensions $(1, 2, 1, 0)$. We have

$$\partial_1(a_0^5 a_1) = 10\, a_0^3 a_1 b_0 + 5\, a_0^4 b_1,$$

so $\rho_1 = 1$, and

$$\partial_2(a_0^3 a_1 b_0) = -3\, a_0^2 b_0 b_1, \qquad \partial_2(a_0^4 b_1) = 6\, a_0^2 b_0 b_1,$$

so $\rho_2 = 1$. Hence

$$\dim H_{44} = 1{-}1 = 0, \quad \dim H_{43} = 2{-}1{-}1 = 0, \quad \dim H_{42} = (1{-}0){-}1 = 0, \quad \dim H_{41} = 0.$$

Thread $\mathcal{T}_{48}$, degrees $48, 47$**.** Dimensions $(1, 1, 0, 0)$. Here $\partial_1(a_0^6) = 15\, a_0^4 b_0$, so $\rho_1 = 1$, and $\rho_2 = 0$. Hence

$$\dim H_{48} = 1 - 1 = 0, \qquad \dim H_{47} = (1 - 0) - 1 = 0.$$

5.9.5 Conclusion

Collecting the cohomology of all thirteen threads, the only nonzero groups are

$$H_0 = H_4 = H_8 = H_{15} = H_{19} = H_{23} = \mathbb{Q},$$

with all other $H_d = 0$. This proves the following.

Proposition 5.9.1. *The cohomology of the weight-*6 *Chevalley–Eilenberg complex of* $\mathbb{HP}^2$ *has Betti number* 1 *in each of the degrees* $\{0, 4, 8, 15, 19, 23\}$ *and* 0 *in every other degree, with explicit cocycle representatives*

$$\begin{aligned}
H_0 &= \mathbb{Q}\big\langle [\, a_2^6 \,] \big\rangle, \\
H_4 &= \mathbb{Q}\big\langle [\, a_1 a_2^5 \,] \big\rangle, \\
H_8 &= \mathbb{Q}\big\langle [\, a_0 a_2^5 - 5\, a_1^2 a_2^4 \,] \big\rangle, \\
H_{15} &= \mathbb{Q}\big\langle [\, a_1 a_2^3 b_1 \,] \big\rangle, \\
H_{19} &= \mathbb{Q}\big\langle [\, a_1 a_2^3 b_0 \,] \big\rangle, \\
H_{23} &= \mathbb{Q}\big\langle [\, (a_0 a_2^3 - 3\, a_1^2 a_2^2)\, b_0 \,] \big\rangle.
\end{aligned}$$

In particular $P_{B_6(\mathbb{HP}^2)}(t) = 1 + t^4 + t^8 + t^{15} + t^{19} + t^{23}$*, in agreement with Theorem 5.5.2.*

Remark 5.9.2. The six cocycles above are the $k = 6$ instances of a family of representatives valid for every $k \geq 4$:

$$\begin{aligned}
H_0 &= \mathbb{Q}\big\langle [\, a_2^k \,] \big\rangle, & H_{15} &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-3} b_1 \,] \big\rangle, \\
H_4 &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-1} \,] \big\rangle, & H_{19} &= \mathbb{Q}\big\langle [\, a_1 a_2^{k-3} b_0 \,] \big\rangle, \\
H_8 &= \mathbb{Q}\big\langle [\, a_0 a_2^{k-1} - (k-1)\, a_1^2 a_2^{k-2} \,] \big\rangle, & H_{23} &= \mathbb{Q}\big\langle [\, (a_0 a_2^{k-3} - (k-3)\, a_1^2 a_2^{k-4})\, b_0 \,] \big\rangle.
\end{aligned}$$

The integer coefficients $(k{-}1)$ and $(k{-}3)$ are exactly those forced by the cycle condition: for instance $\partial(a_0 a_2^{k-1}) = (k{-}1)\, a_2^{k-2} b_2$ and $\partial(a_1^2 a_2^{k-2}) = a_2^{k-2} b_2$, so $a_0 a_2^{k-1} - (k{-}1)\, a_1^2 a_2^{k-2}$ is the unique combination (up to scale) killed by ∂. We have verified the whole family directly for $k \in \{4, 5, 6, 7\}$; its agreement with Lemma 5.4.5 confirms it for all $k \geq 4$.

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