Adjoint Functor Theorem for Presentable ∞-Categories

Ind-categories

We briefly recall the construction of Ind-categories and their universal property.

Let A be a small ∞-category and let κ be a regular cardinal. Consider the presheaf ∞-category

$$\mathcal{P}(A) = \mathrm{Fun}(A^{op}, \mathcal{S}).$$

The Yoneda embedding gives a fully faithful functor

$$y : A \hookrightarrow \mathcal{P}(A), \qquad a \mapsto \mathrm{Map}_A(-, a).$$

Definition 1. The ∞-category $\mathrm{Ind}_\kappa(A)$ is the full subcategory of $\mathcal{P}(A)$ consisting of those presheaves which can be expressed as κ-filtered colimits of representable presheaves.

Equivalently, $F \in \mathcal{P}(A)$ belongs to $\mathrm{Ind}_\kappa(A)$ if there exists a κ-filtered diagram $i \mapsto a_i$ in A such that

$$F \simeq \mathrm{colim}_i y(a_i)$$

in $\mathcal{P}(A)$. The Yoneda embedding therefore factors through $\mathrm{Ind}_\kappa(A)$,

$$A \hookrightarrow \mathrm{Ind}_\kappa(A) \subset \mathcal{P}(A).$$

Proposition 1 (Universal property)**.** *Let E be an ∞-category admitting κ-filtered colimits. Composition with the Yoneda embedding induces an equivalence*

$$\mathrm{Fun}^{\kappa\text{-}colim}(\mathrm{Ind}_\kappa(A), E) \simeq \mathrm{Fun}(A, E),$$

where $\mathrm{Fun}^{\kappa\text{-}colim}$ *denotes the full subcategory of functors preserving κ-filtered colimits.*

In particular any functor $A \to E$ extends uniquely (up to contractible choice) to a κ-filtered colimit preserving functor $\mathrm{Ind}_\kappa(A) \to E$. Thus $\mathrm{Ind}_\kappa(A)$ can be viewed as the free completion of $A under \kappa$-filtered colimits.

Presentable ∞-categories

The following structural result describes presentable ∞-categories in terms of Ind-completions.

Proposition 2. *Let C be a presentable ∞-category. Then there exist a small ∞-category A and a regular cardinal κ such that*

$$C \simeq \mathrm{Ind}_\kappa(A).$$

In particular every object of C can be written as a κ-filtered colimit of objects of A.

Proof. Recall that an ∞-category is presentable if it admits all small colimits and is accessible. Thus there exists a regular cardinal κ such that C is κ-accessible.

Let $C^\kappa \subset C$ denote the full subcategory of κ-compact objects. By accessibility this is essentially small and every object of C may be expressed as a κ-filtered colimit of objects of C^κ. Recall that $x \in C$ is κ-compact if the mapping space functor

$$\mathrm{Map}_C(x, -) : C \to \mathcal{S}$$

preserves κ-filtered colimits.

Let $A = C^\kappa$. By definition $\mathrm{Ind}_\kappa(A)$ consists of κ-filtered colimits of representable presheaves on A.

The inclusion $A \hookrightarrow C$ extends, by the universal property of Ind-categories, to a κ-filtered colimit preserving functor

$$F : \mathrm{Ind}_\kappa(A) \to C.$$

Since every object of C is a κ-filtered colimit of objects of A, the functor F is essentially surjective. Full faithfulness follows from the compactness of objects of A: mapping spaces out of them commute with the filtered colimits used to construct objects of $\mathrm{Ind}_\kappa(A)$. Hence F is an equivalence. $\square$

Adjoint functor theorem

Theorem 1 (Adjoint functor theorem)**.** *Let C and D be presentable ∞-categories and let*

$$F : C \to D$$

be a functor.

(1) F admits a right adjoint if and only if it preserves small colimits.

(2) F admits a left adjoint if and only if it is accessible and preserves small limits.

Existence of right adjoints

Suppose first that F admits a right adjoint G. For any diagram $X : I \to C$ and object $d \in D$ we have

$$\mathrm{Map}_D(F(\mathrm{colim}_I X), d) \simeq \mathrm{Map}_C(\mathrm{colim}_I X, Gd).$$

Mapping spaces convert colimits in the first variable into limits, so

$$\mathrm{Map}_C(\mathrm{colim}_I X, Gd) \simeq \lim_I \mathrm{Map}_C(X_i, Gd) \simeq \lim_I \mathrm{Map}_D(FX_i, d).$$

Thus

$$F(\mathrm{colim}_I X) \simeq \mathrm{colim}_I F(X_i),$$

and F preserves colimits.

Conversely assume that F preserves small colimits. Write

$$C \simeq \mathrm{Ind}_\kappa(A)$$

for a small ∞-category A. Let $f : A \to D$ denote the restriction of F.

Fix $d \in D$ and consider the presheaf

$$H_d : A^{op} \to \mathcal{S}, \qquad H_d(a) = \mathrm{Map}_D(f(a), d).$$

If $c \simeq \mathrm{colim}_i a_i$ with $a_i \in A$, define

$$\widetilde{H}_d(c) = \lim_i H_d(a_i).$$

Using preservation of colimits by F we obtain

$$\widetilde{H}_d(c) \simeq \mathrm{Map}_D(F(c), d).$$

Thus we obtain a functor

$$\widetilde{H}_d : C^{op} \to \mathcal{S}, \qquad c \mapsto \mathrm{Map}_D(F(c), d).$$

Since limits in C^{op} correspond to colimits in C and F preserves colimits, this functor preserves limits.

Proposition 3 (Representability criterion)**.** *Let C be a presentable ∞-category and let*

$$H : C^{op} \to \mathcal{S}$$

be a functor which preserves small limits and is accessible. Then H is representable: there exists an object $x \in C$ such that

$$H(c) \simeq \mathrm{Map}_C(c, x)$$

naturally in c.

Proof. Since C is presentable, there exist a regular cardinal κ and a small ∞-category A such that

$$C \simeq \mathrm{Ind}_\kappa(A).$$

In particular every object of C may be written as a κ-filtered colimit of objects of A.

Let

$$h := H|_{A^{op}} : A^{op} \to \mathcal{S}$$

denote the restriction of H to A.

Using the Yoneda embedding

$$y : A \hookrightarrow \mathcal{P}(A) = \mathrm{Fun}(A^{op}, \mathcal{S}), \qquad a \mapsto \mathrm{Map}_A(-, a),$$

the presheaf h determines an object of $\mathcal{P}(A)$. Since

$$C \simeq \mathrm{Ind}_\kappa(A) \subset \mathcal{P}(A),$$

objects of C are precisely those presheaves which are κ-filtered colimits of representables.

Consider the diagram in A classified by the elements of h. More precisely, let $\int_A h$ denote the category of elements of h, whose objects are pairs (a, η) with $a \in A$ and $\eta \in h(a)$.

Define an object $x \in C$ by the colimit

$$x \simeq \mathrm{colim}_{(a,\eta)\in\int_A h} a.$$

Since C admits all small colimits, this colimit exists.

Let $c \in C$. Write c as a κ-filtered colimit

$$c \simeq \mathrm{colim}_i a_i$$

with $a_i \in A$. Mapping spaces convert colimits in the first variable into limits, so

$$\mathrm{Map}_C(c, x) \simeq \lim_i \mathrm{Map}_C(a_i, x).$$

Using the definition of x as a colimit we obtain

$$\mathrm{Map}_C(a_i, x) \simeq \mathrm{colim}_{(a,\eta)\in\int_A h} \mathrm{Map}_C(a_i, a).$$

Since $a_i, a \in A$, the Yoneda lemma identifies

$$\mathrm{Map}_C(a_i, a) \simeq y(a)(a_i).$$

Combining these expressions gives

$$\mathrm{Map}_C(c, x) \simeq \lim_i \mathrm{colim}_{(a,\eta)\in\int_A h} y(a)(a_i).$$

To understand this expression, recall the definition of the category of elements $\int_A h$. Its objects are pairs (a, η) where $a \in A$ and $\eta \in h(a)$, and a morphism $(a, \eta) \to (a', \eta')$ is a morphism $\phi : a \to a'$ in A such that $h(\phi)(\eta') = \eta$.

Consider now the colimit

$$\operatorname*{colim}_{(a,\eta)\in\int_A h} y(a)(a_i) = \operatorname*{colim}_{(a,\eta)\in\int_A h} \mathrm{Map}_A(a_i, a).$$

An element of this colimit is represented by a pair consisting of an object (a, η) of $\int_A h$ together with a morphism $\phi : a_i \to a$ in A. Two such pairs are identified if they become equal after postcomposing with a morphism in $\int_A h$.

Using the defining condition of morphisms in $\int_A h$, such a pair $(\phi : a_i \to a, \eta \in h(a))$ determines an element $h(\phi)(\eta) \in h(a_i)$. Conversely, given an element $\theta \in h(a_i)$, it is represented by the pair (a_i, θ) together with the identity morphism $a_i \to a_i$. These constructions are inverse to each other up to the identifications in the colimit, and therefore we obtain a natural equivalence

$$\operatorname*{colim}_{(a,\eta)\in\int_A h} \mathrm{Map}_A(a_i, a) \simeq h(a_i).$$

Substituting this into the previous expression gives

$$\mathrm{Map}_C(c, x) \simeq \lim_i h(a_i).$$

Since H preserves limits and $c \simeq \mathrm{colim}_i a_i$, we have

$$H(c) \simeq \lim_i H(a_i) = \lim_i h(a_i).$$

Therefore

$$H(c) \simeq \mathrm{Map}_C(c, x)$$

naturally in c. This proves that H is represented by the object x. $\square$

A representability theorem for presentable ∞-categories therefore yields an object $G(d) \in C$ such that

$$\mathrm{Map}_D(F(c), d) \simeq \mathrm{Map}_C(c, G(d)).$$

The assignment $d \mapsto G(d)$ is functorial by the Yoneda lemma and determines a functor $G : D \to C$, which is right adjoint to F.

To see that the assignment $d \mapsto G(d)$ is functorial, consider a morphism $\alpha : d \to d'$ in D. Composition with α induces natural maps

$$\mathrm{Map}_D(F(c), d) \to \mathrm{Map}_D(F(c), d')$$

for every $c \in C$. Using the representability equivalences

$$\mathrm{Map}_D(F(c), d) \simeq \mathrm{Map}_C(c, G(d)), \qquad \mathrm{Map}_D(F(c), d') \simeq \mathrm{Map}_C(c, G(d')),$$

we obtain a natural transformation

$$\mathrm{Map}_C(c, G(d)) \to \mathrm{Map}_C(c, G(d')).$$

By the Yoneda lemma this natural transformation is uniquely induced by a morphism $G(d) \to G(d')$ in C. This construction is compatible with composition and identities, and therefore defines a functor $G : D \to C$.

Existence of left adjoints

We now prove the second statement of the adjoint functor theorem.

Proposition 4. *Let C and D be presentable ∞-categories and let*

$$F : C \to D$$

be a functor. If F is accessible and preserves small limits, then F admits a left adjoint.

Proof. The strategy is to reduce the statement to the existence of right adjoints by passing to opposite categories. Consider the opposite functor

$$F^{op} : C^{op} \to D^{op}.$$

Since F preserves small limits in C, it follows that F^{op} preserves small colimits. Indeed, limits in C correspond to colimits in C^{op}, so a limit-preserving functor becomes colimit-preserving after passing to opposites.

Next we consider accessibility. The functor F is assumed to be accessible, meaning that there exists a regular cardinal κ such that F preserves κ-filtered colimits. Passing to opposite categories, κ-filtered colimits correspond to κ-cofiltered limits, so F^{op} preserves κ-cofiltered limits. In particular F^{op} remains an accessible functor.

Finally, since C and D are presentable, they admit small generating subcategories of compact objects. These compact objects control the Ind-presentations of C and D, and therefore also determine the structure of the opposite categories.

The proof of the right adjoint theorem shows that a colimit-preserving functor between presentable ∞-categories admits a right adjoint, and the construction depends only on the behavior of the functor on compact objects and on its compatibility with filtered colimits.

These properties are preserved when passing to opposite categories: the colimit-preserving property of F^{op} comes from limit preservation of F, while the accessibility hypothesis ensures that F^{op} behaves well with respect to the Ind-presentations of the categories involved.

Applying the same construction to F^{op} therefore yields a right adjoint

$$G^{op} : D^{op} \to C^{op}.$$

Taking opposites again produces a functor

$$G : C \to D.$$

The adjunction $F^{op} \dashv G^{op}$ is equivalent to an adjunction $G \dashv F$. Thus F admits a left adjoint. $\square$